Iterated elementary embeddings and the model theory of infinitary logic

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Abstract

We use iterations of elementary embeddings derived from the nonstationary ideal on ω_1 to provide a uniform proof of some classical results connecting the number of models of cardinality $\aleph_1$ in various infinitary logics to the number of syntactic types over the empty set. We introduce the notion of an analytically presented abstract elementary class (AEC) which allows the formulation and proof of generalizations of these results to refer to Galois types rather than syntactic types. We the equivalence of this descriptive set theoretic condition on countable models of an AEC with a logical condition on all models in the class. We further apply the iterated embeddings method to provide the first absoluteness condition for categoricity in $\aleph_1$ for AEC's (rather than syntactically given classes).

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This paper combines methods of axiomatic and descriptive set theory to study problems in model theory. In particular, we use iterated generic elementary embeddings to analyze the number of models in $\aleph_1$ in various infinitary logics and for Abstract Elementary Classes (AEC). The technique here provides a uniform method for approaching and extending theorems that Keisler et al. proved in the 1970's relating the existence of uncountable models realizing many types to the existence of many models in $\aleph_1$ (Theorem 2.4). To formalize this uniformity we introduce the notion of an analytically presented AEC and show that is a further disguise for a well-known notion (Theorem 4.3). This allows us to extend the Keisler-style results relating the number of types in $\aleph_0$ to the number of models in $\aleph_1$ from syntactic types to Galois types (Theorem 5.6). Finally, we show categoricity in $\aleph_1$ is absolute for analytically presented AEC that satisfy the amalgamation property in $\aleph_0$ and are almost Galois-ω-stable (Theorem 6.3).

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The arguments presented here are very much in the spirit of [9, 10], in which these embeddings were used to prove forcing-absoluteness results. Those papers focused on the large cardinal context. Here we work primarily in ZFC, though we note some cases where our results can be extended assuming the existence of large cardinals.

We refer the reader to [1] for model-theoretic definitions such as *Abstract Elementary Class* and for background on the notions used here. For example, Theorem 0.2 is stated for atomic models of first order theories. The equivalence between this context and models of a complete sentence in $L_{\omega_1,\omega}$ is explained in Chapter 6 of [1]. Abstract Elementary Classes form a general context unifying many of the properties of such infinitary logics as $L_{\omega_1,\omega}$, $L_{\omega_1,\omega}(Q)$, and $L_{\omega_1,\omega}(aa)$.

A fundamental result in the study of $\aleph_1$-categoricity for Abstract Elementary Classes is the following theorem of Shelah (see [1], Theorem 17.11).

Theorem 0.1 (Shelah)**.** *Suppose that* $\mathbf{K}$ *is an Abstract Elementary Class such that*

- *The Lówenheim-Skolem number,* $\mathrm{LS}(\mathbf{K})$*, is* $\aleph_0$*;*
- $\mathbf{K}$ *is* $\aleph_0$*-categorical;*
- *amalgamation fails for countable models in* $\mathbf{K}$[1]*.*

Suppose also that $2^{\aleph_0} < 2^{\aleph_1}$*. Then there are* $2^{\aleph_1}$ *non-isomorphic models of cardinality* $\aleph_1$ *in* $\mathbf{K}$*.*

Theorem 0.1 is one of the two fundamental tools to develop the stability theory of $L_{\omega_1,\omega}$. The second is the following theorem of Keisler (see [1], Theorem 18.15).

Theorem 0.2 (Keisler)**.** *Suppose that* $\mathbf{K}$ *is the class of atomic models of a complete first order theory, and that uncountably many types over the empty set are realized in some uncountable model in* $\mathbf{K}$*. Then there are* $2^{\aleph_1}$ *non-isomorphic models of cardinality* $\aleph_1$ *in* $\mathbf{K}$*.*

The notion of ω-stability for sentences in $L_{\omega_1,\omega}$ is a bit subtle and is more easily formulated for the associated class $\mathbf{K}$ of atomic models of a first theory. For *countable* $A \subseteq M \in \mathbf{K}$, $S_{at}(A)$ denotes the set of first order types over A realized in atomic models[2]. $\mathbf{K}$ is ω-stable if for each countable $M \in \mathbf{K}$, $|S_{at}(M)| = \aleph_0$ [3].

Combining these two theorems, Shelah showed (under the assumption $2^{\aleph_0} < 2^{\aleph_1}$) that a complete sentence of $L_{\omega_1,\omega}$ which has less that $2^{\aleph_1}$ models in $\aleph_1$ has the amalgamation property in $\aleph_0$ and is ω-stable. Crucially, Shelah's argument

[1]Unlike first order logic, this is a strictly stronger statement than 'amalgamation fails over subsets of models of $\mathbf{K}$.'

[2]This definition does not extend to uncountable A, see page 138 of [1]

[3]This requirement that M is a model is essential; Example 3.17 of [1], covers of the multiplicative group of $\mathbb{C}$, is ω-stable but there are countable atomic A with $|S_{at}(A)| = 2^{\aleph_0}$

relies on the assumption $2^{\aleph_0} < 2^{\aleph_1}$ in two ways. It first uses a variation of the Devlin-Shelah weak diamond principle [6] for Theorem 0.1. Then using amalgamation, extending Keisler's theorem from types over the empty set to types over a countable model is a straightforward counting argument, as it is in this paper. In Section 4 we work on analogs of this analysis for AEC for which the class of countable models is analytic.

Using the iterated ultrapower approach we give a new proof of an extension of Theorem 0.2 to the logic $L_{\omega_1,\omega}(\mathrm{aa})$ (as claimed in [20]). Again, it suffices to consider the case where amalgamation holds. Theorem 0.3 follows from Theorem 2.4 below.

Theorem 0.3. *Suppose that* $\mathbf{K}$ *is the class of models of some fixed sentence of* $L_{\omega_1,\omega}(\mathrm{aa})$*, and that, for some countable fragment* F *of* $L_{\omega_1,\omega}(\mathrm{aa})$*-sentences, uncountably many* F*-types are realized over some countable model in* $\mathbf{K}$*. Suppose also that* $2^{\aleph_0} < 2^{\aleph_1}$*. Then there are* $2^{\aleph_1}$ *non-isomorphic models of cardinality* $\aleph_1$ *in* $\mathbf{K}$*.*

We introduce the notion of an analytically presented AEC (the natural descriptive set theoretic definition) the countable models (and elementary submodel relation) and prove:

Theorem 0.4. *If* $\mathbf{K}$ *is an AEC in a countable language with countable Löwenheim-Skolem number, then* $\mathbf{K}$ *can be analytically presented iff and only if its restriction to* $\aleph_0$ *is the restriction to* $\aleph_0$ *of a* $PC\Gamma(\aleph_0, \aleph_0)$*-AEC.*

We can prove the following partial extension of Keisler's Theorem for analytically presented Abstract Elementary Classes. Hypothesis (3) below corresponds to one of the cases given by Burgess's theorem for analytic equivalence relations (see [13], Theorem 9.1.5). Theorem 0.5 follows from Theorem 5.6 below.

Theorem 0.5. *Suppose that* $\mathbf{K}$ *is an Abstract Elementary Class such that*

1. *the set of reals coding countable structures in* $\mathbf{K}$ *and the corresponding strong submodel relation* $\prec_{\mathbf{K}}$ *are both analytic (we say analytically presented);*
2. $\mathbf{K}$ *satisfies amalgamation for countable models;*
3. *there is a countable model in* $\mathbf{K}$ *over which there is a perfect set of reals coding inequivalent Galois types.*

Suppose also that $2^{\aleph_0} < 2^{\aleph_1}$*. Then there are* $2^{\aleph_1}$ *non-isomorphic models of cardinality* $\aleph_1$ *in* $\mathbf{K}$*.*

Though the approach here can very likely be applied more generally, we restrict our attention in this paper to the contexts of Theorems 0.3 and 0.5.

Finally, we turn our attention to absoluteness and prove:

Theorem 0.6. *Let* $\mathbf{K}$ *be an analytically presented almost Galois* ω*-stable AEC satisfying amalgamation in* $\aleph_0$*, and having an uncountable model. Then the* $\aleph_1$*-categoricity of* $\mathbf{K}$ *is equivalent to a* Π^1_2*-sentence, and therefore absolute.*

In Section 1 we lay out the method of iterated ultrapowers of models of set theory; Section 2 applies this method to classes defined syntactically in various infinitary logics. Section 3 discusses the descriptive set theory of analytic equivalence relation; Section 4 adapts these methods to study 'analytically presented' AEC. Section 5 extends the Keisler theorem to relate the number of Galois types to the number of models in this context. In Section 6 we address the issue of absoluteness of $\aleph_1$-categoricity for AEC. Finally Section 7 raises some further problems.

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1 Iterations

The main technical tool in this paper is the iterated generic elementary embedding induced by the nonstationary ideal on ω_1, which we will denote by NS_{ω_1}. We are using this as a device to reproduce Keisler's constructions for expanding a countable model of set theory in such a way that sets in the original model get new members in the extension if and only if they are uncountable from the point of view of the original model. Though this will not be relevant here, we note that this these iterated embeddings and their relatives play a fundamental role in Woodin's $\mathbb{P}_{\max}$ forcing [33]. Most of this section is a condensed version of Section 1 of [26].

The iterations constructed here could be developed using the construction of carefully specified extensions of models of set theory. See [21, 16, 8] for background on these methods. We illustrate this technique in [5].

Recall that NS_{ω_1} is closed under countable unions. Moreover, Fodor's Lemma (see, for instance, [18]) says that for any stationary $A \subseteq \omega_1$, if $f\colon A \to \omega_1$ is regressive (i.e., $f(\alpha) < \alpha$ for all $\alpha \in A$), then f is constant on a stationary set. Forcing with the Boolean algebra $(\mathcal{P}(\omega_1)/\mathrm{NS}_{\omega_1})^M$ over a ZFC model M gives rise to an M-normal ultrafilter U on ω_1^M (i.e., every regressive function on ω_1^M in M is constant on a set in U). Given such M and U, we can form the generic ultrapower $\mathrm{Ult}(M,U)$, which consists of all functions in M with domain ω_1^M, where for any two such functions f, g, and any relation R in $\{=,\in\}$, fRg in $\mathrm{Ult}(M,U)$ if and only if $\{\alpha < \omega_1^M \mid f(\alpha)Rg(\alpha)\} \in U$. By convention, we identify the well-founded part of the ultrapower $\mathrm{Ult}(M,U)$ with its Mostowski collapse. The corresponding elementary embedding $j\colon M \to \mathrm{Ult}(M,U)$ (where each element of M is mapped to the equivalence class of its corresponding constant function on ω_1^M) has critical point (i.e., first ordinal moved) ω_1^M (see Fact 1.4 and the discussion before). We say that such an embedding is *derived by forcing with* $(\mathcal{P}(\omega_1)/\mathrm{NS}_{\omega_1})^M$ *over* M. Fodor's Lemma implies that the identity function represents the ordinal ω_1^M in the ultrapower. It follows then by the definition of $\mathrm{Ult}(M,U)$ that for each $A \in \mathcal{P}(\omega_1)^M$, $A \in U$ if and only if $\omega_1^M \in j(A)$. Each ordinal $\gamma \in \omega_2^M$ is represented in $\mathrm{Ult}(M,U)$ by a function of the form $f(\alpha) = o.t.(g[\alpha])$, where $g\colon \omega_1 \to \gamma$ is a surjection (and $o.t.$ stands for "ordertype"), so the ordinals of $\mathrm{Ult}(M,U)$ always contain an isomorphic copy of

ω_2^M (which is less than or equal to $j(\omega_1^M)$, since each such f has range contained in ω_1^M) as an initial segment. We call such a function f a *canonical function* for γ. While it is possible to have well-founded ultrapowers of the form $\mathrm{Ult}(M, U)$ (at least assuming the existence of large cardinals), this does not always happen (see Lemma 1.10, for instance).

Since we want to deal with structures whose existence can be proved in ZFC, we define a useful fragment of ZFC.

1.1 Definition. The fragment ZFC$^\circ$ is the theory ZFC $-$ Powerset $-$ Replacement + "$\mathcal{P}(\mathcal{P}(\omega_1))$ exists" plus the following scheme, which is a strengthening of ω_1-Replacement: every (possibly proper class) tree of height ω_1 definable from set parameters has a maximal branch (i.e., a branch with no proper extensions; in the cases we are concerned with, this just means a branch of length ω_1).

The theory ZFC$^\circ$ holds in every structure of the form $H(\kappa)$ or V_κ, where κ is a regular cardinal greater than $2^{2^{\aleph_1}}$ (recall that $H(\kappa)$ is the collection of sets whose transitive closures have cardinality less than κ).

1.2 Remark. While one can prove stronger preservation results for ZFC$^\circ$, we note the following, which suffices for the applications in this paper. Suppose that θ is a regular cardinal and P is a partial order in $H(\theta)$ such that the following hold in any forcing extension by P :

- θ is a regular cardinal greater than $2^{2^{\aleph_1}}$;
- every element of the $H(\theta)$ of the forcing extension is the realization of a P-name in $H(\theta)$ of the ground model.

Then any forcing extension of $H(\theta)$ (of V) by P is a model of ZFC$^\circ$. Therefore, if X is a countable elementary submodel of $H(\theta)$ with P as a member, then any forcing extension of the transitive collapse of X satisfies ZFC$^\circ$. The conditions above on P and θ are satisfied if $2^{2^{|P|}} < \theta$. If P is c.c.c. then $2^{(|P|^{\aleph_1})} < \theta$ suffices.

For us, the importance of ZFC$^\circ$ is that it proves Fact 1.3 below, which implies that M is elementarily embedded in $\mathrm{Ult}(M, U)$ whenever M is a model of ZFC$^\circ$ and U is an M-ultrafilter on ω_1^M.[4] The proof of the fact is a direct application of the ω_1-Replacement-like scheme in ZFC$^\circ$.

1.3 Fact (ZFC$^\circ$). Let n be an integer. Suppose that ϕ is a formula with $n+1$ many free variables and $f_0, \ldots, f_{n-1}$ are functions with domain ω_1. Then there is a function g with domain ω_1 such that for all $\alpha < \omega_1$,

$$\exists x \phi(x, f_0(\alpha), \ldots, f_{n-1}(\alpha)) \Rightarrow \phi(g(\alpha), f_0(\alpha), \ldots, f_{n-1}(\alpha)).$$

[4]An M-ultrafilter on ω_1 is a maximal proper filter contained in $\mathcal{P}(\omega_1)^M$; in the cases we are interested in, the filter is not an element of M.

We let $j[x]$ denote $\{j(y) \mid y \in x\}$. One direction of Fact 1.4 below follows from the fact that every partition in M of ω_1^M into ω many pieces must have one piece in the ultrafilter U, so, if x is countable then every function from ω_1 to x in M (i.e., every representative of a member of $j(x)$) must be constant on a set in U and so must represent a member of $j[x]$). For the other direction, note that if x is uncountable then any injection from ω_1 to x represents an element of $j(x) \setminus j[x]$ in the ultrapower $\mathrm{Ult}(V, U)$.

1.4 Fact. Suppose that M is a model of ZFC°, and that $j\colon M \to \mathrm{Ult}(M, U)$ is an elementary embedding derived from forcing over M with $(\mathcal{P}(\omega_1)/\mathrm{NS}_{\omega_1})^M$. Then for all $x \in M$, $j(x) = j[x]$ if and only if x is countable in M.

If M is a countable model of ZFC° then there exist M-generic filters for the partial order $(\mathcal{P}(\omega_1)/\mathrm{NS}_{\omega_1})^M$. Furthermore, if $j\colon M \to N$ is an ultrapower embedding of this form (where N may be ill-founded), then $\mathcal{P}(\mathcal{P}(\omega_1))^N$ is countable (recall that the ultrapower uses only functions from M), and there exist N-generic filters for $(\mathcal{P}(\omega_1)/\mathrm{NS}_{\omega_1})^N$. We can continue choosing generic filters in this way for up to ω_1 many stages, defining a commuting family of elementary embeddings and using this family to take direct limits at limit stages.

We use the following formal definition.

1.5 Definition. Let M be a model of ZFC° and let γ be an ordinal less than or equal to ω_1. An *iteration* of M of length γ consists of models M_α ($\alpha \leq \gamma$), sets G_α ($\alpha < \gamma$) and a commuting family of elementary embeddings $j_{\alpha\beta}\colon M_\alpha \to M_\beta$ ($\alpha \leq \beta \leq \gamma$) such that

- $M_0 = M$,
- each G_α is an M_α-generic filter for $(\mathcal{P}(\omega_1)/\mathrm{NS}_{\omega_1})^{M_\alpha}$,
- each $j_{\alpha\alpha}$ is the identity mapping,
- each $j_{\alpha(\alpha+1)}$ is the ultrapower embedding induced by G_α,
- for each limit ordinal $\beta \leq \gamma$, M_β is the direct limit of the system

  $$\{M_\alpha, j_{\alpha\delta} : \alpha \leq \delta < \beta\},$$

  and for each $\alpha < \beta$, $j_{\alpha\beta}$ is the induced embedding.

The models M_α in Definition 1.5 are called *iterates* of M. When the individual parts of an iteration are not important, we sometimes call the elementary embedding $j_{0\gamma}$ corresponding to an iteration an iteration itself. For instance, if we mention an iteration $j\colon M \to M^*$, we mean that j is the embedding $j_{0\gamma}$ corresponding to some iteration

$$\langle M_\alpha, G_\beta, j_{\alpha\delta} : \alpha \leq \delta \leq \gamma, \beta < \gamma\rangle$$

of M, and that M^* is the final model of this iteration.

1.6 Remark. We emphasize that for any countable model M of ZFC$^\circ$ there are $2^{\aleph_0}$ many M-generic ultrafilters for $(\mathcal{P}(\omega_1)/\mathrm{NS}_{\omega_1})^M$. It follows that there are $2^{\aleph_1}$ many iterations of M of length ω_1.

1.7 Remark. As noted above, the ordinals of $\mathrm{Ult}(M, U)$ always contain an isomorphic copy of ω_2^M as an initial segment, whenever M is a countable (well-founded or illfounded) model of ZFC$^\circ$ and U is an M-normal ultrafilter. It follows from this that whenever

$$\langle M_\alpha, G_\beta, j_{\alpha\delta} : \alpha \leq \delta \leq \omega_1, \beta < \omega_1 \rangle$$

is an iteration of M, $\omega_1^{M_{\omega_1}}$ contains a closed copy of ω_1 corresponding to the members of the set $\{\omega_1^{M_\alpha} : \alpha < \omega_1\}$. This set is called the *critical sequence* of the iteration.

Fact 1.8 below says that the final model of an iteration of length ω_1 is correct about uncountability. It is an immediate consequence of Fact 1.4 and the definition of iterations. This gives another proof of Corollary B on page 138 of [20]. Corollary A on page 137 can also be proved by considering ideals on other cardinals. The last sentence of Fact 1.8 follows from the remarks at the end of the second paragraph of this section. The second author observed that the absoluteness of the existence of a model in $\aleph_1$ of an arbitrary sentence is $L_{\omega_1,\omega}$ (i.e., Theorem 2.1) follows easily from Fact 1.8; it is shown in [7] that this argument can be carried out using Corollary A of [20].

1.8 Fact. Suppose that M is a model of ZFC$^\circ$, and that M_{ω_1} is the final model of an iteration of M of length ω_1. Then for all $x \in M_{\omega_1}$, $M_{\omega_1} \models$ "x is uncountable" if and only if $\{y \mid M_{\omega_1} \models x \in y\}$ is uncountable. Furthermore, ω_2^M is a proper initial segment of $\omega_1^{M_{\omega_1}}$.

Fact 1.9 records the fact that one can easily make M_{ω_1} correct about stationarity for subsets of its ω_1 (again, this is due to Woodin [33]). Note that the notion of stationarity makes sense for any uncountable set (so in particular, for $\omega_1^{M_{\omega_1}}$ as below, even if it is ill-founded) : $Y \subseteq [X]^{\aleph_0}$ is stationary if and only if every for every function $F \colon X^{<\omega} \to X$ there is a nonempty element of Y closed under F.

1.9 Fact. Suppose that M is a model of ZFC$^\circ$, $\{B_\xi : \xi < \omega_1\}$ is a partition of ω_1 into stationary sets and

$$\langle M_\alpha, G_\beta, j_{\alpha,\gamma} : \alpha \leq \gamma \leq \omega_1, \beta < \omega_1 \rangle \tag{1}$$

is an iteration of M of length ω_1. Suppose that for every $\alpha < \omega_1$ and every $A \in (\mathcal{P}(\omega_1) \setminus NS_{\omega_1})^{M_\alpha}$ there is a $\xi < \omega_1$ such that, for all $\beta \in \omega_1 \setminus \alpha$,

$$\beta \in B_\xi \Rightarrow j_{\alpha,\beta}(A) \in G_\beta.$$

Then for all $A \in \mathcal{P}(\omega_1)^{M_{\omega_1}}$, $M_{\omega_1} \models$ "A is stationary" if and only if A is stationary.

Lemma 1.10 gives a construction for building generic ultrapowers whose ω_1's are illfounded, though, as remarked above, they must be well-founded up to at least the ω_2 of the ground model. We will use the lemma in the proof of Lemma 2.5. Given a function $f\colon \omega_1 \to \omega_1$, we let I_f be the normal ideal on ω_1 generated by sets of the form

$$\{\beta < \omega_1 \mid g(\beta) \geq f(\beta)\},$$

where g is a canonical function for an ordinal less than ω_2. Whenever $\gamma < \gamma' < \omega_2$, g is a canonical function for γ and γ' is a canonical function for γ', it follows that $\{\beta < \omega_1 \mid g(\beta) < g'(\beta)\}$ contains a club. It follows (using the regularity of ω_2) that for each $S \in \mathcal{P}(\omega_1)$, $S \in I_f$ if and only if $\{\beta \in S \mid f(\beta) \geq g(\beta)\}$ is nonstationary for some canonical function g for an element of ω_2. If $\langle \sigma_\beta : \beta < \omega_1\rangle$ is a $\diamondsuit$-sequence and $\pi\colon \omega_1 \to \omega_1 \times \omega_1$ is a bijection, then $\omega_1 \notin I_f$, where $f\colon \omega_1 \to \omega_1$ is the function defined by letting $h(\beta)$ be $o.t.(\pi[\beta])+1$ whenever $\pi[\beta]$ is a wellordering (and 0 otherwise). We note that $\diamondsuit$ is forced by the partial order which adds a subset of ω_1 by countable initial segments, and that this partial order does not add subsets of ω. Some hypothesis beyond ZFC° is needed for Lemma 1.10, as it is false for models in which the nonstationary ideal on ω_1 is saturated.

Lemma 1.10. *Suppose that M is a countable transitive model of* ZFC°*, and that $f^*\colon \omega_1^M \to \omega_1^M$ is a function in M such that $\omega_1 \notin I_{f^*}$. Then there is an M-normal ultrafilter U such that the well-founded ordinals of* $\mathrm{Ult}(M,U)$ *are exactly ω_2^M.*

Proof. Applying the usual construction of an M-normal ultrafilter, it suffices to show that if

- S is a subset of ω_1^M in M,
- $f\colon S \to \omega_1^M$,
- $S \notin I_f$,
- $\{T_\alpha : \alpha \in \omega_1^M\}$ is a collection of stationary subsets of S in ω_1 whose diagonal union is S,

then there exist $\alpha < \omega_1^M$, $S' \subseteq T_\alpha$ and $f'\colon S' \to \omega_1$ in M such that

- for all $\beta \in S'$, $f'(\beta) < f(\beta)$,
- $S' \notin I_{f'}$.

This implication gives a recipe for building an M-normal filter with the property that every function in M from ω_1^M to the ordinals either represents an ordinal below ω_2^M or dominates on a set in the filter another function which does not represent an ordinal below ω_2^M. The recipe uses an enumeration $\{h_n : n \in \omega\}$ of $(\omega_1^{\omega_1})^M$. In each step, starting with $f = f^*$ and $S = \omega_1$, it applies the implication above to $\min\{f, h_n\}$ (for the next n, considered in order) if $S \notin I_{\min f, h_n}$, and to f otherwise.

To see that the implication holds, fix f and S as given. Since I_f is normal and $S \notin I_f$, there is an α such that $S \cap T_\alpha \notin I_f$. Let S_0 be the set of $\beta \in S \cap T_\alpha$ for which $f(\beta)$ is a successor ordinal. If S_0 is not in I_f, then let $S' = S_0$ and let $f'(\beta) = f(\beta) - 1$ for $\beta \in S'$. Then since adding 1 to the values of any canonical function for any $\gamma < \omega_2$ gives a canonical function for $\gamma + 1$, we have that $S' \notin I_{f'}$.

If $S_0 \in I_f$, there is an I_f-positive $S_1 \subseteq S \cap T_\alpha$ such that $f(\beta)$ is a limit ordinal for all $\beta \in S_1$. Let $f_n \colon S_1 \to \omega_1$ $(n \in \omega)$ be functions such that for each $\beta \in S_1$, $\langle f_n(\beta) : n < \omega \rangle$ is an increasing sequence with supremum $f(\beta)$. It suffices to see that $S_1 \notin I_{f_n}$ for some $n \in \omega$. Supposing towards a contradiction that $S_1 \in I_{f_n}$ for each $n \in \omega$, fix, for each n a canonical function g_n (for some ordinal $\gamma_n < \omega_2^M$) such that $\{\beta \in S_1 \mid f_n(\beta) \geq g_n(\beta)\}$ is nonstationary. Let γ be an element of ω_2^M greater than all the g_n's, and fix a canonical function g for γ. Then for each $n \in \omega$ the set $\{\beta \in S_1 \mid f_n(\beta) > g(\beta)\}$ is nonstationary, which means that the set $\{\beta \in S_1 \mid f(\beta) > g(\beta)\}$ is nonstationary, which means that $S_1 \in I_f$, giving a contradiction. $\square$

2 $L_{\omega_1,\omega}(\mathrm{aa})$

Briefly, the logic $L_{\omega_1,\omega}$ is the extension of first order logic where one allows conjunctions and disjunctions of countable sets of formulas so that only finitely many free variables appear in the union of the set of formulas. Each formula in $L_{\omega_1,\omega}$ has a *rank*, the number (less than ω_1) of steps it takes to construct the formula from atomic formulas (see the appendix to [2]). More explicitly, we may think of sentences of $L_{\omega_1,\omega}$ as well-founded trees of height of at most ω; then the rank of a sentence is just the rank of the corresponding tree in the sense of Section 3. An ill-founded model of ZFC$^\circ$ can contain objects which it thinks are sentences of $L_{\omega_1,\omega}$ which are really not, i.e., if the rank of the sentence as computed in the model is an ill-founded ordinal of the model. On the other hand, if a (real) sentence ϕ of $L_{\omega_1,\omega}$ exists in an ω-model M of ZFC$^\circ$, then M computes the rank correctly, and is therefore well-founded at least up the rank of ϕ. Furthermore, M correctly verifies whether the models that it sees satisfy ϕ. In both cases, the computation of the rank and the verification of the truth value, M runs exactly the same process that is carried out in V.

The logic $L_{\omega_1,\omega}(\mathrm{aa})$ extends $L_{\omega_1,\omega}$ by adding the quantifier aa, where $\mathrm{aa}x \in [X]^{\aleph_0}\, \phi$ means "for stationarily many countable $x \subseteq X$, ϕ holds", i.e., for any function $f \colon X^{<\omega} \to X$, there is a countable $x \subseteq X$ closed under f such that x satisfies ϕ. Note that "there exist uncountably many $x \in X$ such that ϕ holds" can be expressed using aa. If M is a model of ZFC$^\circ$ as in conclusion of Fact 1.9, i.e., such that for all $A \in \mathcal{P}(\omega_1)^{M_{\omega_1}}$, $M_{\omega_1} \models$ "A is stationary" if and only if A is stationary, then if X is a set in M of cardinality $\aleph_1$ (in M) and Y is a subset of $[X]^{\aleph_0}$ in M, then $M_{\omega_1} \models$ "Y is stationary" if and only if Y is stationary

The second parts of the equivalences in the following theorems are $\underset{\sim}{\Sigma}^1_1$, and therefore absolute. The forward directions simply involve taking the transitive collapse of a countable elementary submodel of suitable initial segment of the universe. The reverse directions involve building iterations as in the previous section (using Fact 1.9 for correctness about stationarity). Since the final models of these iterations are well-founded up to at least the ω_2 of the corresponding original models, they verify correctly truth for ϕ and for members of the set F for the models that they see.

Theorem 2.1. *Given a sentence ϕ of $L_{\omega_1,\omega}(\mathrm{aa})$, the existence of a model of ϕ of size $\aleph_1$ is equivalent to the existence of a countable model of* ZFC° *containing $\{\phi, \omega\}$ which thinks there is a model of ϕ of size $\aleph_1$.*

Theorem 2.2. *Given a countable fragment F of $L_{\omega_1,\omega}(\mathrm{aa})$, the existence of a model of size $\aleph_1$ satisfying $\aleph_1$-many F-types is equivalent to the existence of a countable model of* ZFC° *containing $F \cup \{F, \omega\}$ which thinks there is a model of size $\aleph_1$ satisfying $\aleph_1$-many F-types.*

We prove in Theorem 2.4 below that the second part of the equivalence in the previous theorem implies that there are $2^{\aleph_1}$ many models of size $\aleph_1$, pairwise satisfying only countably many F-types in common. First we present an easier argument for getting $\aleph_1$ many such models.

Suppose that M is an ω-model of ZFC° and $\bar{x} = \langle x_\alpha : \alpha < \omega_1^M \rangle$ is a sequence of distinct subsets of ω in M. Then given any iteration of M as above, $\bar{x}$ will be an initial segment of $j_{0,\omega_1}(\bar{x}) = \langle x_\alpha : \alpha < \omega_1^{M_{\omega_1}} \rangle$, and $x_\alpha \notin M_\beta$ whenever $\alpha \geq \omega_1^{M_\beta}$ (by the remarks before Fact 1.4).

Furthermore, if A is any countable set of reals not in M, one can easily build an iteration of M such that $A \cap M_{\omega_1} = \emptyset$. Now let F be a countable fragment of $L_{\omega_1,\omega}(\mathrm{aa})$, and let M be a ω-model of ZFC° in which F is countable, which thinks there exists a model N of size $\aleph_1$ realizing uncountably many F-types. Then there are uncountably many iterations $\{j^\xi : \xi < \omega_1\}$ of M producing models $\{M^\xi_{\omega_1} : \xi < \omega_1\}$ such that the models $M^\xi_{\omega_1}$ pairwise have only the reals from M in common, and thus the models $j^\xi(N)$ pairwise realize just countably many F-types in common.

To get $2^{\aleph_1}$ many uncountable iterates pairwise having just countably many reals in common, we use Theorem 2.3 below. Note that one can force $\mathrm{MA}_{\aleph_1}$ (the restriction of Martin's Axiom which asserts the existence of a filter meeting any $\aleph_1$ many maximal antichains from a c.c.c. partial order) to hold over any countable model of ZFC°. By "distinct iterations" we mean literally iterations that are not the same set, formally speaking. In particular, this means (using the notation from Theorem 2.3) that there is some β such that $G_\beta \neq G'_\beta$. When β is minimal with this property, $M_\beta = M'_\beta$ and there is a set $A \in \mathcal{P}(\omega_1)^{M_\beta}$ such that $A \in G_\beta$ and $\omega_1^{M_\beta} \setminus A \in G'_\beta$, since G_β and G'_β are distinct M_β-ultrafilters.

Theorem 2.3 (Larson [25])**.** *If M is a countable model of* ZFC° + $\mathrm{MA}_{\aleph_1}$ *and*

$$\langle M_\alpha, G_\beta, j_{\alpha,\gamma} : \alpha \leq \gamma \leq \omega_1, \beta < \omega_1 \rangle$$

and

$$\langle M'_\alpha, G'_\beta, j'_{\alpha,\gamma} : \alpha \leq \gamma \leq \omega_1, \beta < \omega_1 \rangle$$

are two distinct iterations of M, then

$$\mathcal{P}(\omega)^{M_{\omega_1}} \cap \mathcal{P}(\omega)^{M'_{\omega_1}} = \mathcal{P}(\omega)^{M_\beta},$$

where β is least such that $G_\beta \neq G'_\beta$.

For the reader's convenience, we sketch the proof of the version of Theorem 2.3 for iterations of length 1 (which appears in [11]). Suppose that M is a countable model of $\mathrm{ZFC}^\circ + \mathrm{MA}_{\aleph_1}$ and let G and G' be two distinct M-generic filters for $(\mathcal{P}(\omega_1)/\mathrm{NS}_{\omega_1})^M$. Then there exist disjoint sets A, A' in $(\mathcal{P}(\omega_1 \setminus NS_{\omega_1})^M$ such that $A \in G$ and $A' \in G'$. Let $N = \mathrm{Ult}(M, G)$ and $N' = \mathrm{Ult}(M, G')$, and fix $x \in \mathcal{P}(\omega)^N \setminus M$ and $x' \in \mathcal{P}(\omega)^{N'} \setminus M$. Then there exist functions $f\colon A \to \mathcal{P}(\omega)^M$ and $f'\colon A' \to \mathcal{P}(\omega)^M$ representing x in N and x' in N' respectively. Applying Fodor's Lemma we see that, since x and x' are not in M, there exist $B \subseteq A$ and $B' \subseteq A'$ in G and G' respectively on which f and f' (respectively) are injective. Applying Fodor's Lemma again we can thin B and B' to sets C and C' on which the ranges of f and f' are disjoint and contain only infinite, co-infinite sets, by subtracting nonstationary sets. Finally, it is a consequence of $\mathrm{MA}_{\aleph_1}$ (see [19], for instance) that for any two disjoint sets of infinite, co-infinite subsets of ω, there is a subset of ω which intersects each member of the first set infinitely, and no member of the second set infinitely. Thus if M satisfies $\mathrm{MA}_{\aleph_1}$ there is such a $z \subseteq \omega$ in M with respect to the ranges of $f \restriction C$ and $f \restriction C'$, which means that $x \cap z$ is infinite and $x' \cap z$ is not.

Using this, one gets the following version of Keisler's theorem (see Fact 18.15 of [1]), for $L_{\omega_1,\omega}(\mathrm{aa})$.

Theorem 2.4. *Let F be a countable fragment of $L_{\omega_1,\omega}(\mathrm{aa})$. If there exists a model of cardinality $\aleph_1$ realizing uncountably many F-types, there exists a $2^{\aleph_1}$-sized family of such models, each of cardinality $\aleph_1$ and pairwise realizing just countably many F-types in common.*

Proof. Let N be a model of cardinality $\aleph_1$ realizing uncountably many F-types, let X be a countable elementary submodel of $H((2^{(2^{\aleph_1})^+})^+)$ containing $\{N\}$ and the transitive closure of $\{F\}$. Let M be the transitive collapse of X, and let N_0 be the image of N under this collapse. Let M' be a forcing extension of M satisfying Martin's Axiom via a c.c.c. partial order of cardinality $(2^{\aleph_1})^+$. Then, like M, M' is a wellfounded model of ZFC° (see Remark 1.2). By choosing a pair of distinct generic ultrafilters for each model we can build a tree of iterates of M' giving rise to $2^{\aleph_1}$ many distinct iterations of M' of length ω_1 (as in Remark 1.6). Since F-types can be coded by reals using an enumeration of F in M, the images of N_0 under these iterations will pairwise realize just countably many F-types in common, by Theorem 2.3. $\square$

If one assumes in addition that $2^{\aleph_0} < 2^{\aleph_1}$, then, as in Theorem 18.16 of [1], one gets that if there exists a model of cardinality $\aleph_1$ realizing uncountably many types over some countable subset, then there exists a $2^{\aleph_1}$-sized family of nonisomorphic models. That is, if there is an uncountable model N with a countable subset A over which uncountably many types are realized, then there are models N_f ($f \in 2^{\aleph_1}$) all containing the same countable set A and all realizing different sets of types over A, so that any isomorphisms of any two N_{f_1} and N_{f_2} into a third N_{f_3} must map A pointwise to different sets (which is impossible if $2^{\aleph_1} > 2^{\aleph_0}$).

We conclude this section by showing that a strengthening of Lemma 5.1.8 (non-definability of well-order in $L(Q)$) of [1] can be proved using Lemma 1.10.

Lemma 2.5. *Suppose that ϕ is a sentence of $L_{\omega_1,\omega}(\mathrm{aa})$ in a language with a binary predicate $<$, and suppose that there is a model M of ϕ for which the order-type of $(M,<)$ is ω_1. Then there is a model M' of ϕ of cardinality $\aleph_1$ such that $(M',<)$ embeds $\mathbb{Q}$. Furthermore, if θ is a regular cardinal greater than $2^{2^{\aleph_1}}$ and Z is a hereditarily countable set, then M' can be taken to be an element of a model N of* ZFC° *such that*

- *$(M',<)$ is isomorphic to ω_1^N, and*
- *for all $z_1,\dots,z_n$ in Z and every $(n+1)$-ary first order formula ψ in the language of set theory,*
$$H(\theta) \models \psi(M, z_1,\dots,z_n)$$
if and only if
$$N \models \psi(M', z_1,\dots,z_n).$$

Proof. Let θ' be a regular cardinal greater than θ and $2^{2^{2^{\aleph_0}}}$ and let X be a countable elementary submodel of $H(\theta')$ with $\theta, M, Z \in X$. Let N_1 be the transitive collapse of X and let $\pi\colon X \to N_1$ be the collapsing map. Let $N_0 = \pi(H(\theta))$ and let $M_0 = \pi(M)$.

Let N_2 be a forcing extension of N_1 (via a σ-closed partial order of cardinality $2^{\aleph_0}$) satisfying $\diamondsuit$ (recall from the paragraph before Lemma 1.10 that $\diamondsuit$ implies the hypothesis of that lemma). Then N_2 is a model of ZFC° (see Remark 1.2). Applying Lemma 1.10 (for the first step of the iteration) and Fact 1.9 (for the rest), we can find an iteration $j\colon N_2 \to N_3$ of length ω_1 such that the well-founded ordinals of N_3 are exactly $\omega_2^{N_2}$. Letting $N = j(N_0)$ (which is $j(\pi(H(\theta)))$, we have that $\omega_1^N = \omega_1^{N_3}$ Letting $M' = j(M_0)$ (which is $j(\pi(M))$), we have that $(M',<)$ is isomorphic to ω_1^N, which embeds $\mathbb{Q}$ as it is illfounded. Finally,
$$(j \circ \pi)\restriction(X \cap H(\theta))\colon X \cap H(\theta) \to N$$
is an elementary embedding which sends M to M' and fixes every element of Z. □

3 Analytic equivalence relations

In this section we prove two lemmas about analytic equivalence relations on the reals in ω-models of set theory. The second of these, Lemma 3.3, will be applied in Section 5 to an equivalence relation corresponding to the notion of Galois type. After writing this section we noticed that for our purposes one could replace Lemma 3.3 with the classical fact that every partial function from ω^ω to itself with analytic graph has a Borel extension (Theorem 3.2 below). While we have not found this fact stated in the literature, it is an easy consequence of the First Separation Theorem, via the argument for Theorem 4.5.2 in [31] or Exercise 35.13 of [23] (our proof uses the fact that the Borel sets of reals are exactly the analytic, co-analytic ones). We retain our original argument for completeness, and note at the end of this section how one might use the Borel extension fact instead.

In this paper, a *tree* is a set of finite sequences closed under initial segments. If $T \subseteq X^{<\omega}$ is a tree, for some set X, then $[T]$ is the set of $x \in X^\omega$ such that $x \restriction n \in T$ for all $n \in \omega$. If $T \subseteq (X \times Y)^{<\omega}$, for some sets X and Y, then the *projection* of T, $p[T]$ is the set of $f \in X^\omega$ such that for some $g \in Y^\omega$, $(f, g) \in [T]$ (this definition involves a standard identification of pairs of sequences with sequences of pairs). For any positive $n \in \omega$, a subset of $(\omega^\omega)^n$ is *analytic* if it has the form $p[T]$ for some tree $T \subseteq (\omega^n \times \omega)^{<\omega}$.

Recall that for a tree $T \subseteq X^{<\omega}$ for some set X, the ranking function $rank_T \colon T \to Ord \cup \{\infty\}$ is defined in such a way that for all $t \in T$, $rank_T(t)$ is the smallest ordinal α such that $\alpha > rank_T(s)$ for all proper extensions s of t in T, and $rank_T(t) = \infty$ if no such α exists (which happens if and only if $rank_T(s) = \infty$ for some proper extension s of t). We write $rank(T)$ for $rank_T(\langle\rangle)$. Then $rank(T) = \infty$ if and only if T has an infinite branch.

Now suppose that M is an ω-model of ZFC°, and $T \subseteq X^{<\omega}$ is a tree in M, for some X in M. If $rank(T)^M = \infty$, then there is an infinite branch through T in M. If $rank(T)^M$ is in the well-founded part of M, then there is no infinite branch through T (in V). It follows easily from the definition of $rank(T)$ that if $rank(T)^M$ is an ill-founded ordinal of M, then T has an infinite branch in V but no infinite branch in M. This happens, for instance, in the case where t is an illfounded ordinal of M and T is the tree of descending sequences from t.

Given sets X, Y, a tree $T \subseteq (X \times Y)^{<\omega}$ and $s^* \in X^{<\omega}$, T_{s^*} is the set of $(s, t) \in T$ such that s is compatible with s^* (i.e., one of them extends the other).

Lemma 3.1. *Suppose that M is a (possibly ill-founded) ω-model of* ZFC°*, and that $T \subseteq (X \times Y)^{<\omega}$ is a tree in M, for some sets X and Y. Suppose that x is the unique element of $p[T]$. Then $x \in M$.*

Proof. Since $p[T]$ is nonempty, $rank(T)^M$ cannot be in the well-founded part of M. If $rank(T)^M = \infty$, then $[T] \cap M$ is nonempty, which means that $p[T] \cap M$ is nonempty. Suppose then that $rank(T)^M$ is an ill-founded ordinal of M. Then, starting with with $\langle\rangle$, M can find all the initial segments of x by the following process. Suppose that $s \in X^{<\omega}$ is an initial segment of x. Then $rank(T_s)^M$ is an ill-founded ordinal of M. Since s is an initial segment of the unique element of $p[T]$, the unique integer n such that $s^\frown\langle n\rangle$ is an initial segment of x is also the unique integer n such that

$$\sup\{rank_T^M(s^\frown\langle n\rangle, t) : (s^\frown\langle n\rangle, t) \in T\}$$

is greater than

$$\sup\{rank_T^M(s^\frown\langle m\rangle, t) : (s^\frown\langle m\rangle, t) \in T\}$$

for all $m \in \omega \setminus \{n\}$, since the former set contains ill-founded ordinals of M and the latter contains only well-founded ordinals. $\square$

The proof just given cannot in general give an element of $[T]$ in M. Consider, for instance a tree of the form $\{(x{\restriction}n, t) : n \in \omega, |t| = n, t \in T\}$, for x an element of $\omega^\omega \cap M$ and T a tree in M whose rank is an illfounded ordinal of M.

The proof of Lemma 3.1 gives the following classical result, which, as noted above, can also be used to prove Lemma 3.3.

Theorem 3.2. *If $f\colon \omega^\omega \to \omega^\omega$ is partial function which is analytic as a subset of $\omega^\omega \times \omega^\omega$, then f extends to a Borel partial function $f'\colon \omega^\omega \to \omega^\omega$.*

Proof. Let $T \subseteq (\omega \times \omega \times \omega)^{<\omega}$ be a tree projecting to the graph of f. For each $y \in \omega^\omega$, let T_y be the tree consisting of those pairs (b, c) for which $(y{\restriction}|b|, b, c) \in T$. Then T_y exists in any model of ZFC° containing T and y, and, it projects to $\{f(y)\}$ if y is in the domain of f, and to $\emptyset$ otherwise. The corresponding search for x (i.e., $f(y)$) outlined in the proof of Lemma 3.1 (using T_y in place of T, and in each step finding the unique n such that $\sup\{rank_{T_y}^M(s^\frown\langle n\rangle, t) : (s^\frown\langle n\rangle, t) \in T_y\}$ is greater than $\sup\{rank_{T_y}^M(s^\frown\langle m\rangle, t) : (s^\frown\langle m\rangle, t) \in T_y\}$ for all $m \in \omega \setminus \{n\}$, if such an n exists) returns the same value x in any model M containing y and T if it returns a value in any such model. The set of y for which a value x is returned is then analytic and co-analytic, and thus Borel, and the corresponding function is likewise Borel. $\square$

In the proof above, domain of f' may include some y's not in the domain of f, i.e., where an x not in $p[T_y]$ is found. In these cases the values $rank_{T_y}^M(s)$ in the construction from Lemma 3.1 are all necessarily well-founded.

Now suppose that E is an analytic equivalence relation on an analytic set $X \subseteq \omega^\omega$. By the Burgess Trichotomy Theorem (Theorem 9.1.5 of [13]), either E has at most $\aleph_1$ many equivalence classes, or

there is a perfect set P consisting of E-inequivalent members of X. The following lemma shows that in this second case, if M is an ω-model containing codes for E and P, and $x \in \omega^\omega \cap M$ is E-equivalent to a member of P, then this member of P is also in M. The lemma follows from Lemma 3.1 plus the fact that the set of members of P which are E-equivalent to x is an analytic set with a unique member.

Lemma 3.3. *Suppose that M is a (possibly ill-founded) ω-model of* ZFC°*, and E is an analytic equivalence relation on ω^ω which is the projection of a tree T on $\omega \times \omega \times \omega$ in M. Suppose that P is a perfect set of E-inequivalent members of ω^ω such that $P = [S]$ for a tree $S \subseteq \omega^{<\omega}$ in M. Let $x \in M \cap \omega^\omega$ be such that xEy for some $y \in P$. Then $y \in M$.*

As noted above, the classical fact that every partial function from ω^ω to itself with analytic graph has a Borel extension can be used in place of Lemma

3.3 in Section 5. We briefly sketch the argument for this. Suppose that E is an analytic equivalence relation on ω^ω, and A is an analytic set of E-inequivalent reals. Then the set of pairs (x, y) from ω^ω for which xEy and $y \in A$ is a partial function with analytic graph. Let f be a Borel extension of this function. Then if M is an ω-model of ZFC$^\circ$ containing a suitable code for f, and $x \in \omega^\omega \cap M$ is E-equivalent to a member of A, then $f(x)$ is this member, and $f(x) \in M$.

4 Analytically Presented Classes

In this section, we single out a class of AEC's that can be treated by the methods of descriptive set theory. We work with an abstract elementary class $\mathbf{K}$ in a countable vocabulary τ with Löwenheim number $\aleph_0$. As in [13] we code τ-structures on ω by functions $f\colon \omega \to 2$, where f is the characteristic function of the (suitably coded by pairing functions) of the relational predicates and (graphs of) function symbols of τ. In this way the set of codes for τ-structures is a closed subset of 2^ω. For any given $L_{\omega_1,\omega}(\tau)$-sentence ϕ, the set of codes for models of ϕ is Borel, and, conversely, any set of countable τ-structures (invariant under isomorphism) for which the corresponding set of codes is Borel is the class of models of some $L_{\omega_1,\omega}(\tau)$-sentence. (These facts are Lemma 11.3.3 and Theorem 11.3.6 of [13].)

Definition 4.1. *Let* $\mathbf{K}$ *be an abstract elementary class in a countable language with countable Löwenheim-Skolem number. We say that* $\mathbf{K}$ *is* analytically presented *if the set of countable models in* $\mathbf{K}$*, and the corresponding strong submodel relation* $\prec_{\mathbf{K}}$*, are both analytic.*

This requirement is not as *ad hoc* as it might seem. Shelah's presentation theorem (Theorem 4.15 of [1]) asserts that any AEC of τ-structures with countable Löwenheim-Skolem number can be presented as the reducts to τ of models of a first order theory in a countable language τ' which omit a family of at most $2^{\aleph_0}$-types, and the class of pairs of elementary submodels has a definition of the same form. In [1] these are called $PC\Gamma(\aleph_0, \aleph_0)$ classes when the collection of omitted types is countable.[5] Keisler writes PC_δ *over* $L_{\omega_1,\omega}$ for this notion to emphasize that it can also be described as the class of τ-structures satisfying reducts to τ of a countable conjunction (thus a single sentence) of $L_{\omega_1,\omega}(\tau')$-sentences. (Note that for Keisler's class we have to omit only countably many types by Chang's trick as in Theorem 6.1.8 of [1].)

Example 4.2. Sentences ϕ in $L_{\omega_1,\omega}$ define $PC\Gamma(\aleph_0, \aleph_0)$-presented AEC with $\prec_{\mathbf{K}}$ taken as elementary substructure in the smallest fragment containing ϕ. Sentences ϕ in $L_{\omega_1,\omega}(Q)$ are more problematic, as being an elementary submodel in the smallest fragment containing ϕ is not preserved under unions (a union of countable sets may become uncountable). And of course many $L_{\omega_1,\omega}(Q)$ sentences have Löwenheim-Skolem number $\aleph_1$. But if we restrict to $\mathbf{K}$ the class of models of a sentence ϕ where the Q-quantifier is only used negatively and

[5]Shelah writes $PC_{\aleph_0}$ or $PC(\aleph_0, \aleph_0)$, suppressing the type omission.

we use $\leq^*$ (i.e., small sets can't grow; see Notation 6.4.6 of [1]) then $(\mathbf{K}, \leq^*)$ is $PC\Gamma(\aleph_0, \aleph_0)$. Examples of such classes include Zilber's pseudoexponentiation, Shelah's counterexample to absoluteness of $\aleph_1$-categoricity in $L(Q)$ (Theorem 17.7 of [1]), and Example 5.3 below.

We now show that 'analytically presented' is another *nom de plume* for $PC\Gamma(\aleph_0, \aleph_0)$.

Theorem 4.3. *An abstract elementary class* $\mathbf{K}$ *is analytically presented if and only if its restriction to countable models is the restriction to countable models of a* $PC\Gamma(\aleph_0, \aleph_0)$ *class.*

The proof of Theorem 4.3 starts here and ends with the proof of Lemma 4.6. A straightforward induction (Lemma 11.3.3 of [13]) shows that any $L_{\omega_1,\omega}$-definable set of countable models is invariant Borel (a Borel class whose membership is preserved by any permutation of the universe). Any $PC\Gamma(\aleph_0, \aleph_0)$-presented AEC is analytically presented, as omission of a countable family of types in τ' is Borel, and taking the reduct to τ makes the class of countable models analytic. (*Mutatis mutandis* we show the analogous result for pairs of countable models (M, N) with $M \prec_{\mathbf{K}} N$.)

The converse is more complicated and we proceed by two lemmas. We first show that if an AEC $\mathbf{K}$ of τ-structures is analytically presented then the countable models of $\mathbf{K}$ are the countable models of a $PC\Gamma(\aleph_0, \aleph_0)$ class. Lemma 4.4 is the restriction of Theorem 4.3 for countable models. This result is reported as folklore[6] in Theorem 1.3.1(a) of [32]).) We haven't found a published proof so we give more details below which will motivate the proof for uncountable models in Lemma 4.6.

For notational simplicity in this proof, we assume τ *contains a single binary relation* R. As in [13], membership in a class of τ-structures X that is analytically definable can be coded as: there is a tree T_X (contained in $2^{<\omega} \times \omega^{<\omega}$) such that $M = (\omega, R) \in X$ if and only if for some $f \in \omega^\omega$, $(g_R, f) \in [T_X]$ is a path through T_X, where $g_R \in 2^\omega$ codes the characteristic function of R. If U is an $m+1$ ary relation symbol, $U(M, \boldsymbol{a})$ denotes the set of elements b of M such that $M \models U(b, \boldsymbol{a})$.

Lemma 4.4. *The countable* τ*-models of an analytically presented class can be represented as reducts to* τ *of a sentence in* $L_{\omega_1,\omega}(\tau')$ *for appropriate* $\tau' \supseteq \tau$*. (i.e. as noted above, the countable models of a* $PC\Gamma(\aleph_0, \aleph_0)$*.)*

Moreover the class of countable pairs (M, N) *such that* $M \prec_{\mathbf{K}} N$ *is also a* $PC\Gamma(\aleph_0, \aleph_0)$*-class.*

Proof. Extend τ to τ' by adding unary functions s, f, g, a constant symbol 0 and for each n, a $2n$-ary relation symbol S_n. Let θ_0 be an $L_{\omega_1,\omega}(\tau')$ sentence such that if M is a model of θ_0:

[6] Lopez-Escobar [27] describes Scott's role in understanding the connection between invariant-Borel and $L_{\omega_1,\omega}$-definability but the analytic set version doesn't appear there.

1. Every element of M is equal to a unique expression of the form $s^n(0)$, for some $n \in \omega$.

   Notation: For a finite sequence σ (of length n) of natural numbers, we will write $\hat{\sigma}$ to denote the sequence $s^{\sigma(0)}(0) \ldots s^{\sigma(n-1)}(0)$ of elements of M. When convenient we will write $\mathbf{n}$ for $s^n(0)$.

2. g and f map M into M.

3. g is the characteristic function of R via a pairing function.

4. $S_n(\hat{\sigma}, \hat{\sigma'})$ if and only if $(\sigma, \sigma') \in T_X$.

5. For every n, $S_n(g_R \restriction n, f \restriction n)$.

Now, checking through the definitions one sees that (M, R) is in X if and only if (M, R) can be expanded to a model of θ_0. Namely, if $(\omega, R) \in X$, choose g_R as just before the statement of Lemma 4.4; interpret 0 as 0 and s as the successor function on M. Choose f with $(g_R, f) \in T_X$. Interpret S_n by condition 4. Conversely, suppose

$(\omega, R, s, 0, g, f) \models \theta$. Identifying $s^n(0)$ with n, if g and f are maps from M to N, we can identify f and g with maps from ω to ω. Suppose under this indentification, $(g, f) \in [T_X]$. So g is a code for a relation R^* with $(\omega, R^*) \in X$. But (ω, R^*) is isomorphic to (ω, R) and X is invariant so $(\omega, R) \in X = p[T]$.

To complete the proof of Lemma 4.4 we need to show that

$$Y = \{(M, N, R) : M \prec_{\mathbf{K}} N, |M| = |N| = \aleph_0\}$$

is also defined as a $PC\Gamma(\aleph_0, \aleph_0)$-class. A pair of models (M, W, R) (W denotes the submodel, R is the relation) is coded by a characteristic function $p_{R,W} \in [2^{<\omega} \times 2^{<\omega}]$. In this case we begin with a vocabulary $\hat{\tau}$ obtained by adding a unary predicate W for the smaller model to τ. We expand $\hat{\tau}$ to $\hat{\tau}'$ as we expanded τ to τ'. Again, there is a tree T_Y (contained in $2^{<\omega} \times 2^{<\omega} \times \omega^{<\omega}$) such that $M = (\omega, W, R) \in Y$ if and only if for some $h \in \omega^\omega$, $(p_{R,W}, h) \in [T_Y]$ (is a path through T_Y). The argument is exactly as before in this larger vocabulary; add now a set of $2n$-ary predicates W_n to code the tree T_Y. Let θ_1 be the $\hat{\tau}'$ sentence expressing this. $\square_{4.4}$

4.5 Remark. We could just prove the second part of Lemma 4.4 by applying the first part to the class of models (M, W, R). But we want to prepare for the next argument.

Finally we show that there is a further vocabulary $\tilde{\tau}$ which contains uniformly definable analogs of the extra predicates in $\hat{\tau}$ and a $\tilde{\tau}$-sentence $\tilde{\theta}_0$ such that the τ-reducts of $\tilde{\theta}_0$ (in all cardinalities) are exactly the members of $\mathbf{K}$. Moreover if we add a unary predicate W to $\tilde{\tau}$ (as we extended $\hat{\tau}$ to $\hat{\tau}'$) to get $\tilde{\tau}'$, there is a sentence $\tilde{\theta}_1$ such that the $\tilde{\tau}'$-structure $(M, N, \ldots)$ satisfies $\tilde{\theta}_1$ if and only if its $\tau \cup \{W\}$-reduct satisfies $M \prec_{\mathbf{K}} N$. For this extension to uncountable models think of each model as a direct limit[7] of finitely generated (and hence countable) submodels and use the idea of the proof of Lemma 4.4 to verify that these finitely generated submodels reduct to members of $\mathbf{K}$ and that the submodel relation is $\prec_{\mathbf{K}}$. We need to rewrite and extend the argument rather than merely quote Lemma 4.4, because we appeal to the analyticity on every (at least λ of them) finitely generated (hence countable) τ-substructure of a model M with cardinality λ. Thus we introduce parameterized versions of the functions in Lemma 4.4. This argument is inspired by the proof of Shelah's presentation theorem (Theorem 4.15 of [1]): we use the functions $t_{\boldsymbol{a}}$ to artificially create infinite finitely generated substructures. For example, the $p_{R,W}$ of Lemma 4.4 becomes the parameterized family of functions $p_{\mathbf{c},\mathbf{d}}$ to represent a model pair $(U_{\mathbf{c}}, U_{\mathbf{c},\mathbf{d}})$ consisting of the finitely generated substructures indexed by $\mathbf{c}$ and $\mathbf{cd}$ respectively.

Lemma 4.6. *All τ-models of an analytically presented AEC $\mathbf{K}$ can be represented as reducts to τ of a sentence $\tilde{\theta}_0$ in $L_{\omega_1,\omega}(\tilde{\tau})$ for appropriate $\tilde{\tau} \supseteq \tau$.*

Moreover, if $\tilde{M}$ is a $\tilde{\tau}$-substructure of $\tilde{N}$ and both $\tilde{M}$ and $\tilde{N}$ satisfy $\tilde{\theta}_0$ then $\tilde{M}\restriction\tau \prec_{\mathbf{K}} \tilde{N}\restriction\tau$.

Further, the class of pairs of τ-structures (M, N) such that $M \prec_{\mathbf{K}} N$ is the class of reducts to $\tau \cup \{W\}$ of models of $\tilde{\theta}_1$, where $\tilde{\theta}_1$ is $\tilde{\theta}_0 \wedge \tilde{\theta}_0\restriction W$.

Proof. The countable models of $\mathbf{K}$ are τ-structures coded by a tree T_X as in the paragraph before Lemma 4.4. Extend τ to $\tilde{\tau}$ by adding a unary predicate N, constant symbol 0 and unary function symbol s, for each m an $m+1$-ary relation symbol $U^m(x, \mathbf{y})$, $m+1$-ary function symbols $t^m(x, \mathbf{x}), f^m(x, \mathbf{x}), g^m(x, \mathbf{x})$, $1 + k + \ell$-ary functions $p^m(x, \mathbf{x}, \mathbf{y})$, $h^m(x, \mathbf{x}, \mathbf{y})$, $W(x, \mathbf{x}, \mathbf{y})$ and $m+1$-ary relations $U^m(x, \mathbf{x})$, $1 + k + \ell$-ary relations $W^{k,\ell}(x, \mathbf{x}, \mathbf{y})$ and for each n, $2n$-ary relation symbols S_n and W_n. (For ease of reading below, we often omit the superscripts on $U^m, f^m, g^m, t^m \ldots$; the reader should infer that the length of the parameter sequence determines the suppressed superscript.)

Let $\tilde{\theta}_0$ be an $L_{\omega_1,\omega}(\tau')$ sentence such that if M is a model of $\tilde{\theta}_0$:

1. Every element of $N(M)$ is equal to a unique expression of the form $s^n(0)$.
2. Every element of M is equal to an expression of the form $t^n(0, \boldsymbol{a})$, for some $n \in \omega$ and $\boldsymbol{a} \in M$ with length m. $U_{\boldsymbol{a}} = U(M, \boldsymbol{a}) = \{(t(\boldsymbol{a}))^i(0) : i < \omega\}$. The map $t_{\boldsymbol{a}} : \mathbf{n} \mapsto s^n_m(0, \boldsymbol{a})$ is a bijection.
3. Each $U_{\boldsymbol{a}}$ is the universe of τ-structure.

   Notation: For a finite sequence σ (of length n) of natural numbers, we will write $\hat{\sigma}$ to denote the sequence $s^{\sigma(0)}(0) \ldots s^{\sigma(n-1)}(0)$ of elements of M. When convenient we will write $\mathbf{n}$ for $s^n(0)$.

   For a finite sequence σ (of length n) of natural numbers, we will write $\hat{\sigma}_{\boldsymbol{a}}$ to denote the sequence $(t(\boldsymbol{a}))^{\sigma(0)}(0) \ldots (t(\boldsymbol{a}))^{\sigma(n-1)}(0)$ of elements of M.

[7] Recall that although AEC's are defined in terms of unions of chains, any AEC is closed under $\prec_{\mathbf{K}}$-direct limits.

For each of the parameterized functions we abbreviate, e.g. $\lambda x g(x, \boldsymbol{a})$ by $g_{\boldsymbol{a}} \colon U_{\boldsymbol{a}} \mapsto N(M)$.

4. For any disjoint sequences $\mathbf{c}, \mathbf{d}$ of length k and ℓ, $W_{\mathbf{c},\mathbf{d}} = W^{k+\ell}(M, \mathbf{c}, \mathbf{d}) = U^k_{\mathbf{c}}$.
5. If $\boldsymbol{a}$ has length k, $g_{\boldsymbol{a}}$ and $f_{\boldsymbol{a}}$ map $U^k(M, \boldsymbol{a})$ into ω.
6. For any disjoint sequences $\mathbf{c}, \mathbf{d}$ of length k and ℓ, $p_{\mathbf{c},\mathbf{d}}$ and $h_{\mathbf{c},\mathbf{d}}$ map $U^{k+\ell}(M, \mathbf{c}, \mathbf{d})$ into ω.
7. $g_{\boldsymbol{a}}$ is the characteristic function of $R \restriction U_{\boldsymbol{a}}$ via a pairing function.
8. For any disjoint sequences $\mathbf{c}, \mathbf{d}$ of length k and ℓ, $p_{\mathbf{c},\mathbf{d}}$ is the characteristic function of the model pair $(W^{k+\ell}_{\mathbf{c}}, U^{k+\ell}_{\mathbf{cd}})$ and the relation $R \restriction U^{k+\ell}_{\mathbf{cd}}$ via a pairing function.
9. We code $U_{\boldsymbol{a}} \in \mathbf{K}$:
   (a) $S_n(\hat{\sigma}, \hat{\sigma'})$ if and only if $(\sigma, \sigma') \in T_X$.
   (b) For every n, $S_n(g_{\boldsymbol{a}} \circ t_{\boldsymbol{a}} \restriction n, f_{\boldsymbol{a}} \circ t_{\boldsymbol{a}} \restriction n)$.
10. For $\boldsymbol{a} \subset \boldsymbol{a}'$ we code $U_{\boldsymbol{a}} \prec_{\mathbf{K}} U_{\boldsymbol{a}'}$:
    (a) $W_n(\hat{\sigma}, \hat{\sigma'})$ if and only if $(\sigma, \sigma') \in T_Y$.
    (b) If $\mathbf{c} \subset \mathbf{d}$, for every n, $W_n(p_{\mathbf{c},\mathbf{d}} \circ t_{\mathbf{cd}} \restriction n, h_{\mathbf{c},\mathbf{d}} \circ t_{\mathbf{cd}} \restriction n)$.

Now (*): $M \models \tilde{\theta}_0$ if and only if M is a direct limit of finitely generated $\tilde{\tau}$-substructures, which are in $\mathbf{K}$ by clause 9. The direct limit is with respect to the subsequence ($\lhd$) ordering of the finite indexing sequences and $\boldsymbol{a} \lhd \boldsymbol{a}'$ implies $U_{\boldsymbol{a}} \prec_{\mathbf{K}} U_{\boldsymbol{a}'}$ by clause 10. To see (*) note: If M is a direct limit then M is in $\mathbf{K}$ since $\mathbf{K}$ is closed under direct limits. To write M as a direct limit that witnesses $\tilde{\theta}_0$, choose the $U_{\boldsymbol{a}}$ by induction on $|\boldsymbol{a}|$. Demand that each $U_{\boldsymbol{a}} \prec_{\mathbf{K}} M$ is enumerated by $t^n_{\boldsymbol{a}}(0)$, and contains the $U_{\mathbf{b}}$ for each $\mathbf{b} \lhd \boldsymbol{a}$ and $|\mathbf{b}| < |\boldsymbol{a}|$.

Now we consider the moreover clause.

First we have $M' \restriction \tau$ is a direct limit of finitely generated partial $\tilde{\tau}$-structures $U_{\boldsymbol{a}}$ and $N' \restriction \tilde{\tau}$ is a $\prec_{\mathbf{K}}$-direct limit of $U_{\boldsymbol{a}}$ where $U_{\boldsymbol{a}}$ in the sense of M' equals $U_{\boldsymbol{a}}$ in the sense of N' for $\boldsymbol{a} \in M$ because M' is a $\tilde{\tau}$-substructure of N'. Each $U_{\boldsymbol{a}} \restriction \tau \prec_{\mathbf{K}} N' \restriction \tau$ so, since AEC's are closed under direct limits, the direct limit $M' \restriction \tau$ is a strong submodel of $N' \restriction \tau$. For the 'further' clause, just be careful in carrying out the expansion of N to a $\tilde{\tau}$ structure in the previous paragraph, that if $\boldsymbol{a} \in M'$, $U_{\boldsymbol{a}} \subseteq M$. The claim about $\tilde{\theta}_1$ is now evident. $\square_{4.6}$

This completes the proof of Theorem 4.3. We have the following corollary.

4.7 Corollary. If the countable models of an AEC $\mathbf{K}$ with Löwenheim number $\aleph_0$ can be represented as a $PC\Gamma(\aleph_0, \aleph_0)$ class then the class has a $PC\Gamma(\aleph_0, \aleph_0)$-representation.

Proof. By one direction of Theorem 4.3, the given representation of the countable models of **K** implies it is analytically presented. By the other direction, the entire class has a $PC\Gamma(\aleph_0, \aleph_0)$-representation. $\square_{4.7}$

To see the effect of this corollary, suppose one has a sentence in $L_{\omega_1,\omega}(Q)$ which has countable models that form an AEC. The translations to order structures of Keisler (e.g. [20], Theorem 5.1.8 of [1]) give us the hypotheses of the Corollary. But the class defined is not closed under unions of uncountable chains. So this is not the proper axiomatization; the more complicated parameterization in Lemma 4.6 is needed.

5 Galois types

In this section we review the notion of a Galois type and specify how to apply descriptive set theoretic techniques to study Galois types in analytically presented AEC. We expound the use of Burgess's theorem to provide a trichotomy of stability classes for analytically presented AEC. In particular we explain the relevance of work by Hyttinen-Kesala and Kueker on the one hand and [5] on the other to describing the connections between 'almost Galois ω-stable' and 'Galois ω-stable'.Then we prove a partial generalization of Keisler's theorem that many types imply many models to analytically presented classes. The generalization does not include Theorem 2.4 because $L(aa)$-classes are not analytically presented.

Following [1] we define for **K** a reflexive and symmetric relation $\sim_0$ on the set of triples of the form (M, a, N), where M and N are countable structures in **K** with $M \prec_{\mathbf{K}} N$, and $a \in N \setminus M$. We say that $(M_0, a_0, N_0) \sim_0 (M_1, a_1, N_1)$ if $M_0 = M_1$ and there exist a structure $N \in \mathbf{K}$ and strong embeddings $f_0 \colon N_0 \to N$ and $f_1 \colon N_1 \to N$ such that $f_0 \restriction M_0 = f_1 \restriction M_1$ and $f_0(a_0) = f_1(a_1)$. We let $\sim$ be the transitive closure of $\sim_0$. The equivalence classes of $\sim$ are called *Galois types*.

If an abstract elementary class is given syntactically the Galois types over a countable M refine the syntactic types; in general there may be more Galois types than syntactic types (e.g. [3]).

There is a natural coding of triples (M, a, N) as above by elements of 2^ω, where, for instance, M is taken to be a structure whose domain is the even elements of ω, N has domain ω, and a is an odd integer. For analytically presented AEC, the set B consisting of those $x \in 2^\omega$ coding such a triple is an analytic set. We let E be the equivalence relation on B where xEy if and only if x and y code (respectively) triples (M_0, a_0, N_0) and (M_1, a_1, N_1) for which there exists an isomorphism $\pi : N_0 \to N_0'$ (for some $N_0' \in \mathbf{K}$) such that $(M_1, \pi(a_0), N_0') \sim (M_1, a_1, N_1)$ (the need for π and N_0' arises from the fact that the definition of $\sim$ requires the first models in each triple to be literally the same; this way of defining E allows us to ignore the details of the coding). Then E is analytic. Given a countable $M \in \mathbf{K}$, we let E_M be the equivalence relation

E restricted to the set B_M consisting of codes for triples whose first element is isomorphic to M. Then E_M is also analytic.

Given a real $x \in B$ and a $\mathbf{K}$-structure N^*, we say that N^* *realizes the Galois type coded by* x if there is a triple (M, a, N) coded by x such that $N \prec_{\mathbf{K}} N^*$. If in addition M_0 is countable and $M_0 \prec_{\mathbf{K}} N^*$, we say that N^* *realizes the Galois type coded by* x *over* M_0 if there is a triple (M_0, a, N) coded by x such that $N \prec_{\mathbf{K}} N^*$.

By Burgess's Trichotomy, for each countable $M \in \mathbf{K}$ there are either at most $\aleph_1$ many E_M-equivalence classes, or a perfect set of E_M-inequivalent reals[8] For the syntactic types discussed in the earlier sections the intermediate possibility of $\aleph_1$-types without there being a perfect set of types is impossible, as for each countable fragment of $(L_{\omega_1,\omega},\ L_{\omega_1,\omega}(Q),\ L_{\omega_1,\omega}(aa))$ the set of types is Borel (See 4.4.13 in [28].) Note that this intermediate possibility is obscured in the presence of the CH if this notion is described in terms of the number of classes.

But even for analytically presented AEC all three parts of the trichotomy can occur (see Example 5.3 below) and Theorem 0.2 does not generalize in full. Following [30], we use the following definitions.

5.1 Definition. The abstract elementary class $(\mathbf{K}, \prec)$ is said to be *Galois* ω*-stable* if for every countable $M \in \mathbf{K}$, E_M has countably many equivalence classes, and *almost Galois* ω*-stable* if for each countable $M \in \mathbf{K}$, E_M does not have a perfect set of equivalence classes.[9]

The analog for Galois types of the first order theorem that ω-stability implies stability in all powers fails except under very restrictive conditions. Baldwin and Kolesnikov [3] exhibit complete sentences that are ω-Galois stable but not Galois stable in $\aleph_1$.

Example 5.2. *Consider the abstract elementary class* $(\mathbf{K}, \prec)$ *where* $\mathbf{K}$ *is the class of well-order types of length* $\leq \omega_1$ *and* $\prec$ *is initial segment.* $(\mathbf{K}, \prec)$ *has amalgamation and joint embedding in* $\aleph_0$*, is almost Galois* ω*-stable, but not Galois* ω*-stable despite being* $\aleph_1$*-categorical.*

In view of Example 5.2, there is no hope of a direct generalization of Theorem 0.2 to arbitrary Abstract Elementary Classes. The existence of almost Galois ω-stable but not Galois ω-stable classes is one obstruction. This example seems extreme as there are no models beyond $\aleph_1$ and no nice syntactic description of the class. In particular it is not analytically presented. But, we can find apparently more tractable examples of almost ω-Galois stability (without ω-Galois stability).

A linear order L is 1-transitive (equivalently, groupable, i.e admits a compatible group structure) if for any a, b in L, there is an automorphism of L taking a to b. The class of groupable linear orders has exactly $\aleph_1$ countable models.

[8] This is basically folklore for $PC\Gamma(\aleph_0, \aleph_0)$ since as we noticed in Section 4, such classes are easily seen to be analytically presented so Burgess applies.

[9] We make the definition this way to avoid the awkwardness that if almost Galois ω-stable is defined as having only $\aleph_1$ classes, then under CH every AEC is almost Galois ω-stable. It is not clear to us which notion is more natural for larger κ.

(See Corollary 8.6 of [29].) The following example is a variant by Jarden of a somewhat less natural version in Chapter 1 of [30].

Example 5.3. *Let* $(\mathbf{K}, \prec)$ *be the class of partially ordered sets such that each connected component is a countable* 1*-transitive linear order with* $M \prec N$ *if* $M \subseteq N$ *and no component is extended. Since there are only* $\aleph_1$*-isomorphism types of components this class is almost Galois* ω*-stable. This AEC is analytically presented and definable as a reduct of a class in* $L(Q)$*. But it has* $2^{\aleph_1}$ *models in* $\aleph_1$ *and* $2^{\aleph_0}$ *models in* $\aleph_0$*.*

We sketch an argument (told to us by Kesälä) that implies that every almost ω-Galois stable sentence of $L_{\omega_1,\omega}$ with the amalgamation property and jep is ω-Galois stable. Hyttinen and Kesälä [15] introduced the important notions: finite character and weak Galois type. An AEC $\mathbf{K}$ has *finite character* if for $M \subseteq N$ with $M, N \in \mathbf{K}$: if for every finite $\boldsymbol{a} \in M$ there is a $\mathbf{K}$-embedding of M into N fixing $\boldsymbol{a}$, then $M \prec_{\mathbf{K}} N$. The key point is that any sentence of $L_{\omega_1,\omega}$ has finite character and any such AEC is very close to $L_{\omega_1,\omega}$. Generally speaking, sentences of $L_{\omega_1,\omega}(Q)$ do not have finite character. Two points have the same *weak Galois type* over a model M if they have the same Galois type over every finite subset of M.

It follows easily from work of Kueker [24] and Hyttinen-Kesälä [15] that *for countable models* of an AEC with finite character satisfying the amalgamation and joint embedding properties, almost Galois ω-stability implies Galois ω-stability. Here is the argument. Hyttinen and Kesala call an AEC satisfying these conditions *weakly Galois* ω*-stable* if there are only countably many weak types over each countable model. For such classes, Hyttinen and Kesala show, if two elements have the same weak Galois type over a *countable* model M they have the same Galois type over M. Kueker proves (Corollary 4.9 of [24]) that for finitary AEC (with ap) points a and b have the same weak-Galois type over a countable model M if and only if $\mathrm{tp}_{\infty,\omega}(a/M) = \mathrm{tp}_{\infty,\omega}(a/M)$[10]. Thus for countable models of such sentences, syntactic ω-stability implies Galois ω-stability. Since we noted above that almost Galois ω-stability implies syntactic ω-stability (If there were a model M with uncountably many syntactic types, it would have a perfect set of syntactic types and thus there would be a perfect set of Galois types over M.), we get the following.

5.4 Fact. If a sentence in $L_{\omega_1,\omega}$-sentence, satisfying amalgamation and joint embedding, is almost Galois ω-stable then it is Galois ω-stable.

Baldwin, Larson, and Shelah [5] have shown a related fact, which we apply below:

Theorem 5.5.

We deal here with the case that there is a perfect set of E_M-inequivalent reals, for some M (i.e., the case where almost Galois ω-stability fails). This perfect

[10] Note this type is evaluated in a fixed Galois-saturated monster model.

set plays roughly the role that the $2^{\aleph_0}$ syntactic types played in Theorem 2.4. Since a Galois type is not a real but a set of reals, we cannot reproduce the same argument from an uncountable set of Galois types, but rather use this perfect set to identify a sufficiently large set of Galois types with reals.

The following generalization of Keisler's Theorem 2.4 gives a uniform proof of the results for various logics. We do not assume that $\mathbf{K}$ satisfies amalgamation or the joint embedding property. However, one would typically use amalgamation to obtain hypothesis (4) of the theorem.

Theorem 5.6. *Suppose that*

1. $\mathbf{K}$ *is an analytically presented abstract elementary class;*
2. N *is a* $\mathbf{K}$*-structure of cardinality* $\aleph_1$*, and* N_0 *is countable, with* $N_0 \prec_{\mathbf{K}} N$*;*
3. P *is a perfect set of* E_{N_0}*-inequivalent members of* B_{N_0}*;*
4. N *realizes Galois types coded by uncountably many members of* P *over* N_0*.*

Then there exists a family $\{N^\alpha : \alpha \in 2^{\aleph_1}\}$ *of* $\mathbf{K}$*-structures of cardinality* $\aleph_1$ *such that*

- *for each* $\alpha \in 2^{\aleph_1}$*,* $N_0 \prec_{\mathbf{K}} N^\alpha$*;*
- *for each* $\alpha \in 2^{\aleph_1}$*,* N^α *realizes Galois types coded by uncountably many members of* P *over* N_0*.*
- *for each distinct pair* α*,* α' *from* $2^{\aleph_1}$*, the set of* $x \in P$ *for which both* N^α *and* $N^{\alpha'}$ *realize the Galois type coded by* x *over* N_0 *is countable.*

Proof. Fix a regular $\kappa > 2^{2^{\aleph_1}}$, and let Y be a countable elementary submodel of $H(\kappa)$ with $\mathbf{K} \cap H(\aleph_1)$, N_0, N and P in Y. Let M_0 be a c.c.c. forcing extension of M^* satisfying $\mathrm{MA}_{\aleph_1}$. By the elementarity of the collapsing map on Y, there exists in M^* a continuous increasing chain $\langle N^*_\alpha : \alpha < \omega_1^{M^*}\rangle$ such that, for each $\alpha \in \omega_1^{M^*}$, N^*_α is countable in M^* and $N^*_\alpha \prec_{\mathbf{K}} N^*$. For each $\alpha \in \omega_1^{M^*}$, let X_α be the set of reals of $M_0 \cap P$ coding triples which are $\sim$-equivalent to triples (N_0, a, N') with $N' \prec_{\mathbf{K}} N^*_\alpha$. Let $X = \bigcup_{\alpha\in\omega_1^{M^*}} X_\alpha$. Then $X \in M_0$, since for each α, X_α is Σ^1_1 in any real coding N^*_α (such reals exist in M^* since N^*_α is countable there), and M_0, being well-founded, computes Σ^1_1-truth correctly. The set X is uncountable in M^* by the elementarity of the collapsing map, and therefore also uncountable in M_0, as $\omega_1^{M_0} = \omega_1^{M^*}$. By Theorem 2.3, there are $2^{\aleph_1}$ many iterates $\{M^\alpha : \alpha \in 2^{\aleph_1}\}$ of M_0 pairwise having just countably many reals in common.

Let M^α be such an iterate via an iteration j^α, and let N^α be the corresponding image of N^*. Then in M^α, N^α realizes the Galois types of uncountably many members of $j(P)$ over N_0. Since $j(P) = [j^\alpha(S)]^{M^\alpha} = [S]^{M^\alpha} = [S] \cap M^\alpha$, $j(P) \subseteq P$. Furthermore, M^α is correct about uncountability, so N^α realizes (in V) the Galois types of uncountably many members of P over N_0. For each countable $N' \prec_{\mathbf{K}} N^\alpha$, there is a countable $N'' \prec_{\mathbf{K}} N^\alpha$ in M^α with $N' \prec_{\mathbf{K}} N''$. In this case, if $N_0 \prec_{\mathbf{K}} N'$ and $a \in N' \setminus N_0$, then $(N_0, a, N') \sim_0 (N_0, a, N'')$ via the identity map on N''. It follows then, by Lemma 3.3, that for each $y \in P$ coding a Galois type realized by N^α over N_0, $y \in M^\alpha$. Since M^α and $M^{\alpha'}$ have just countably many reals in common for any distinct pair α, α' in $2^{\aleph_1}$,the set of $x \in P$ for which both N^α and $N^{\alpha'}$ realize the Galois type of x over N_0 is countable. $\square$

5.7 Remark. The proof above gives a slightly stronger conclusion. One can get, for instance, that the set of $x \in P$ for which there exist $N_1 \in M^\alpha$ and $N_2 \in M^{\alpha'}$ such that N^α realizes the Galois type of x over N_1 and $N^{\alpha'}$ realizes the Galois type of x over N_2 is countable.

5.8 Remark. The assumption in Theorem 5.6 that the set of reals coding countable structures in $\mathbf{K}$ be analytic can be relaxed to the requirement this set of codes be universally Baire (see [12]), if one is willing to assume the existence of a Woodin cardinal with a measurable cardinal above it (see [9, 10]). However, the corresponding versions of Burgess's Theorem are weaker (see [14]), which means that the range of applications should be narrower.

6 Absoluteness of $\aleph_1$-categoricity

In first order logic, the Baldwin-Lachlan equivalence between '$\aleph_1$-categorical' and 'ω-stable with no two-cardinal models' makes the notion of $\aleph_1$-categoricity Π^1_1 and hence absolute. Shelah provided an example of an AEC, definable in $L(Q)$, which is $\aleph_1$-categorical under MA and has $2^{\aleph_1}$ models in $\aleph_1$ under $2^{\aleph_0} < 2^{\aleph_1}$. It is an open question whether there is such a non-absolute example in $L_{\omega_1,\omega}$. Theorem 6.3.2 of [1] shows that if ϕ is an $\aleph_1$-categorical sentence of $L_{\omega_1,\omega}$ (with an uncountable model) there is a complete $\aleph_1$-categorical sentence of $L_{\omega_1,\omega}$ (with an uncountable model) which implies ϕ. The remainder of the analysis in [1] and in Shelah's work on which it is based restricts to complete sentences. There is a simple argument at the end (25.19 of [1]) that the categoricity characterization extends to incomplete sentences; but the characterization is ostensibly very dependent on assuming $2^{\aleph_n} < 2^{\aleph_{n+1}}$ for $n < \omega$. The importance of the completeness hypothesis manifests itself in considering amalgamation. Theorem 0.1 implies in particular the consistency of: for a complete $L_{\omega_1,\omega}$-sentence, $\aleph_1$-categoricity implies amalgamation in $\aleph_0$. But an easy example in [4] shows (in ZFC) that there is an $\aleph_1$-categorical $L_{\omega_1,\omega}$-sentence which fails amalgamation in $\aleph_0$. [2] shows that $\aleph_1$-categoricity is absolute for a complete sentence of $L_{\omega_1,\omega}$ which satisfies amalgamation and jep in $\aleph_0$ and is ω-stable. In this statement the jep was redundant since such a sentence is $\aleph_0$-categorical. Moreover, for such complete sentences, since ω-stability implies amalgamation in $\aleph_0$ (Corollary 19.14.3 of [1]), the result yields absoluteness of $\aleph_1$-categoricity for ω-stable sentences of $L_{\omega_1,\omega}$. The notion of ω-stability in that analysis of complete sentences (atomic classes) is a syntactic one. Here we generalize this analysis to analytically presented AEC and (almost) Galois

ω-stability; thus we do not need the completeness hypothesis. However, the study of Galois ω-stability in AEC is not sufficiently advanced as to deduce $\aleph_0$-amalgamation from Galois ω-stability.

Shelah's $L(Q)$-example fails amalgamation in $\aleph_0$ and is not ω-stable. We focus here on showing amalgamation is enough to make $\aleph_1$-categoricity absolute for analytically presented AEC. The argument for Theorem 2.1 shows that the existence of an uncountable model is Σ^1_1 in a real parameter and therefore absolute. Amalgamation for countable models in an analytically presented AEC is Π^1_2 and therefore also absolute. This claim should not extend to AEC which are not analytically presented: if membership in $\mathbf{K}$ were Π^1_1 or more complicated, amalgamation would not automatically satisfy Shoenfield absoluteness.

Let us consider for a moment the case where $\mathbf{K}$ is almost Galois ω-stable and satisfies amalgamation and the joint embedding property in $\aleph_0$. In this case, there is a model in $\mathbf{K}$ of size $\aleph_1$ which realizes every Galois type over every one of its countable substructures (i.e., it is $\aleph_1$-Galois saturated). Furthermore, all such saturated models are isomorphic. The question of $\aleph_1$-categoricity for $\mathbf{K}$ then just depends on whether $\mathbf{K}$ has a model of size $\aleph_1$ omitting some Galois type over some countable substructure.

The second part of the following statement is $\utilde{\Sigma}^1_2$ and thus absolute. The relation $\sim_0$ was defined near the beginning of Section 4, and the projection of a tree was defined at the beginning of Section 3.

Theorem 6.1. *Suppose that $\mathbf{K}$ is an analytically presented AEC. Then the following statements are equivalent.*

1. *There exist a countable $M \in \mathbf{K}$ and an $N \in \mathbf{K}$ of cardinality $\aleph_1$ such that*
   - *$M \prec_{\mathbf{K}} N$;*
   - *the set of Galois types over M realized in N is countable;*
   - *some Galois type over M is not realized in N.*
2. *There is a countable model P of* ZFC° *such that*
   - *ω_1^P is well-founded;*
   - *P contains trees on ω projecting to the set of codes for countable elements of $\mathbf{K}$, and to the relations on reals corresponding to $\prec_{\mathbf{K}}$ and $\sim_0$;*
   - *P satisfies statement (1).*

Proof. The implication from (1) to (2) just involves taking the transitive collapse of an elementary submodel. For the reverse direction, fix M and N witnessing (1) in P. In P, there exists a countable set S containing a member of each Galois type over M realized in N, and a member t of a Galois type over M not realized in N. Fixing elements of $\omega^\omega \cap P$ coding t and the elements of S, the statement that t is not Galois-equivalent to any member of S is Π^1_1 in these codes. Since P believes this statement, and since ω_1^P is well-founded, it is true in V also that t is not Galois-equivalent to any member of S.

Then, if P' is an iterate of P by an iteration of length ω_1, and N' is the corresponding image of N, P' thinks that every element of $N' \setminus M$ satisfies a Galois type corresponding to a member of S, which, being countable in P, was fixed by this iteration. It follows that no member of N' satisfies the Galois type corresponding to t. $\square$

A similar argument shows that for an analytically presented AEC $\mathbf{K}$, almost Galois ω-stability is Π^1_2 in a real number coding $\mathbf{K}$, and therefore absolute.

Theorem 6.2. *Suppose that* $\mathbf{K}$ *is an analytically presented AEC. Then the following statements are equivalent.*

1. $\mathbf{K}$ *is not almost Galois* ω*-stable.*
2. *There is a countable model* P *of* ZFC$^\circ$ *such that*
   - ω_1^P *is well-founded;*
   - P *contains trees on* ω *projecting to the set of codes for countable elements of* $\mathbf{K}$*, and to the relations on reals corresponding to* $\prec_{\mathbf{K}}$ *and* $\sim_0$*;*
   - P *thinks that* $\mathbf{K}$ *is not almost Galois* ω*-stable.*

Proof. Again, the forward direction just involves taking the transitive collapse of an elementary submodel. For the reverse direction, fix P, a countable $M \in \mathbf{K} \cap P$ and a tree $T \subseteq \omega^{<\omega}$ in P without terminal nodes, such that the paths through T code distinct Galois types over M. Since ω_1^P is well-founded, P correctly witnesses the fact that no two paths through T give E_M-equivalent reals. Furthermore, every path through T in a $(\mathcal{P}(\omega_1)/\mathrm{NS}_{\omega_1})^P$-ultrapower of P codes a Galois type over M, as witnessed in the ultrapower. The set of such paths is an uncountable analytic set, and therefore contains a perfect set. $\square$

In the case that the set of codes for countable models in $\mathbf{K}$ is Borel (for instance, the set of countable models of a sentence of $L_{\omega_1,\omega}$), almost Galois ω-stability of $\mathbf{K}$ is more easily seen to be Π^1_2 relative to a code for $\mathbf{K}$, as it asserts that for any countable $M \in \mathbf{K}$ and any subtree of $\omega^{<\omega}$ without terminal nodes, either there is a path through the tree not coding a Galois type over M, or there exist distinct E_M-equivalent paths through the tree.

On its surface, Galois ω-stability for an analytically presented AEC $\mathbf{K}$ is Π^1_4 in a code for $\mathbf{K}$, as it says that for every M, if $M \in \mathbf{K}$ then there are countably many reals such that every suitable real is E_M-equivalent to one of them. Statements of this type are also forcing-absolute in the presence of suitable large cardinals, though not in ZFC. For all we know, there exists an analytically presented almost Galois ω-stable AEC whose Galois ω-stability (or lack thereof) is not absolute.

The proof of Theorem 5.5, as given in [5], however, shows that if an analytically presented $\mathbf{K}$ satisfies amalgamation and is almost Galois ω-stable but not Galois ω-stable, then $\mathbf{K}$ contains uncountable small models of uncountably many distinct Scott ranks. The existence of two (or any number up to ω

many) uncountable small models in $\mathbf{K}$ of distinct Scott ranks is equivalent, by the iteration construction of this paper, to the existence of a countable model of ZFC° which is well-founded up to the supremum of these ranks and thinks there are such models. Again, this latter statement is Σ^1_2 and therefore absolute. Summarizing, we have the following.

Theorem 6.3. *Let* $\mathbf{K}$ *be an analytically presented almost Galois* ω*-stable AEC satisfying amalgamation in* $\aleph_0$*, and having an uncountable model. Then the* $\aleph_1$*-categoricity of* $\mathbf{K}$ *is equivalent to a* Π^1_2*-sentence, and therefore absolute.*

Proof. In this situation, $\aleph_1$-categoricity is equivalent to the conjunction of three conditions:

1. Joint embedding of any two countable models that have uncountable extensions.
2. All uncountable small models in $\mathbf{K}$ have the same Scott rank.
3. Part (1) of Theorem 6.1 fails.

Clearly $\aleph_1$-categoricity implies the first two conditions. The second condition implies that $\mathbf{K}$ is Galois ω-stable, by Theorem 5.5 and the argument of the last paragraph before Theorem 6.3. Then $\aleph_1$-categoricity also implies the third condition, as Galois ω-stability and amalgamation in $\aleph_0$ imply the existence of a Galois-saturated model in $\aleph_1$ which realizes only countably many Galois types over each countable submodel.

Conversely, if condition 2) holds then as in the first paragraph $\mathbf{K}$ is Galois ω-stable. Condition 1), Galois ω-stability and amalgamation imply the existence of Galois-saturated model in $\aleph_1$ which realizes only countably many Galois types over each countable submodel. Condition 3) asserts that each model in $\aleph_1$ has this property.

Finally, these conditions are absolute. The negation of Condition 1) is Σ^1_2, using the idea behind Theorem 2.1 to verify the uncountable extensions. As remarked in the preceding paragraph, the negation of Condition 2) is equivalent to the existence of two small models in $\mathbf{K}$ of distinct Scott ranks and thus is absolute. Finally, by Theorem 6.1, condition 3) is equivalent to 2 of Theorem 6.1, which is absolute. $\square$

6.4 Remark. We should point out that our absoluteness results in this section and the previous one relied only on the fact that the Galois types are induced by an analytic equivalence relation. In the same way, the results of Section 2 were analyzing Borel equivalence relations. Each approach then can be applied much more generally, though we have no applications for this degree of generality at this time.

7 Questions

The following questions have been left unresolved.

7.1 Question. It is shown in [5] that if an almost Galois ω-stable $PC\Gamma(\aleph_0, \aleph_0)$ class satisfying amalgamation *has only countably many models in* $\aleph_1$, then it is Galois ω-stable. By Theorem 0.1, Theorem 0.5 and the main theorem of [5], the amalgamation hypothesis is not needed if $2^{\aleph_0} < 2^{\aleph_1}$.

Can amalgamation be eliminated from the hypotheses? More strongly, is it consistent with ZFC that there is an analytically presented AEC with amalgamation and joint embedding in $\aleph_0$, that is almost Galois ω-stable but not Galois ω-stable, despite being $\aleph_1$-categorical?

Even assuming amalgamation, can the assumption of only countably many models in $\aleph_1$ be weakened to assuming less than $2^{\aleph_1}$ many?

7.2 Question. Can there be an almost Galois ω-stable analytically presented AEC whose Galois ω-stability (or lack thereof) is not absolute?

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