Axiomatizing changing conceptions of the geometric continuuum

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Abstract

We begin with a general account of the goals of axiomatization, introducing several variants (e.g. modest) on Detlefsen's notion of 'complete descriptive axiomatization'. We examine the distinctions between the Greek and modern view of number, magnitude and proportion and consider how this impacts the intent of Hilbert's axiomatization of geometry. We list propositions from Euclid, Archimedes, and Descartes that a modern axiomatization must account for. We argue, as indeed did Hilbert, that propositions concerning polygons, area, and similar triangle are derivable (in their modern interpretation in terms of number) from Hilbert's first order axioms. Then we apparently break new mathematical ground by constructing a natural first order theory extending Hilbert's, which justifies formulas for the circumference and area of a circle. We note that Tarski's extension to the first order complete theory $\mathcal{E}^2$ of geometries over real closed fields grounds the geometry of Descartes as well as Euclid but course still not justifying π. We combine these two results to axiomatize a complete first order theory of geometry in which the formula $C = \pi d$ computes the circumference of a circle but which has non-Archimedean models. We argue that Hilbert's continuity properties show much more than the data set of Greek mathematics and thus are an immodest complete descriptive axiomatization.

By the *geometric continuum* we mean the line situated in the context of the plane. Consider the following two propositions[1].

(*) Euclid VI.1: Triangles and parallelograms which are under the same height are to one another as their bases.

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[1]For diagrams illustrating the the Euclidean propositions about area, see Theorem 4.6.4 and Remark 4.6.7.

Hilbert[2] gives the area of a triangle by the following formula.

(**) Hilbert: Consider a triangle ABC having a right angle at A. The measure of the area of this triangle is expressed by the formula

$$F(ABC) = \frac{1}{2}AB \cdot AC.$$

When formulating a new axiom set in the late 19th century Hilbert faced several challenges:

1. Identify and fill 'gaps' or remove 'extraneous hypotheses' in Euclid's reasoning.
2. Reformulate propositions such as VI.1 to reflect the 19th century understanding of real numbers as measuring both length and area.
3. Ground the geometry of Descartes.

Hilbert also accomplished an additional task: grounding calculus. We will argue that in meeting this additional goal, Hilbert added axioms that were unnecessary for purely geometric considerations. We frame this discussion in terms of the notion of *descriptive axiomatization* from [21]. But the axiomatization of a theory of geometry that had been developing for over two millenia leads to several further considerations. First, while the sentences (*) and (**) clearly, in some sense, express the same proposition, the sentences are certainly different. How does one correlate such statements? Secondly, previous descriptions of complete descriptive axiomatization omit the possibility that the axioms might be too strong and obscure the 'cause' for a proposition to hold. We introduce the term 'modest' descriptive axiomatization to denote one which avoids this defect. We give several explicit lists of propositions from Euclid and refer to [41] for an explicit linking of subsets of Hilbert's axioms as justifications for these lists. In particular, we analyze the impact of the distinction between ratios in the language of Euclid and segment multiplication in [45] or multiplication[3] of "numbers". Then, we examine in more detail, certain specific propositions that in the modern interpretation might appear to depend on Dedekind's postulate. The *main mathematical innovation* is that in Sections 5 and 7 we provide first order axiomatizations to justify the formulas for the circumference and area of the circle, even in non-archimedean fields. We conclude that the first order axioms provide a modest complete descriptive axiomatization[4]; while the second order axioms aim at results that are beyond traditional geometry.

[2] Hilbert doesn't state this result as a theorem; and I have excerpted the statement below from an application on page 66 of [45]. Hilbert defines proportionality in terms of segment multiplication on page 50. 'Negative' segments are introduced in Section 17 on page 53.

[3] That is, a multiplication on points rather than segments. See Heyting [44]; the most thorough treatment is in [3].

[4] It is modest with respect to modern conceptions that straight line segments and arcs should be commensurable but not with earlier conceptions discussed in the text.

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1 Introduction

Hilbert groups his axioms for geometry into 5 classes. The first four are first order. Group V, Continuity, contains Archimedes axiom which can be stated in the logic[5] $L_{\omega_1,\omega}$ and a second order completeness axiom equivalent (over the other axioms) to Dedekind completeness[6] of each line in the plane. Hilbert [7] closes the discussion of continuity with 'However, in what is to follow, no use will be made of the "axiom of completeness" '.

Why then did he include the axiom? Earlier in the same paragraph[8], he writes that 'it allows the introduction of limiting points' and enables one 'to establish a one-one correspondence between the points of a segment and the system of real numbers'. We will argue that by invoking 'continuity' Hilbert is justifying a mathematics beyond Euclid's or even Descartes' geometry.

In Section 2, we consider several accounts of the purpose of axiomatization. We adjust Detlefsen's definition to guarantee some 'minimality' of the axioms by fixing on a framework for discussing the various axiom systems: a *modest descriptively complete axiomatization*. One of our principal tools is the notion of 'data set', a collection of sentences to be accounted for by an axiomatization: 'the data set for area X' is time dependent; new sentences are added; old ones are reinterpreted. In Section 3.1, we will consider several concepts of the continuum so as to focus on the notions most relevant here: those which involve order and the embedding of the line in the plane. We review in Section 3.2 the Greek conceptions of proportion and ratio and the fundamental role they play in Euclid's geometry. With this background, Section 3.3 lists data sets that we will axiomatize. We emphasize those propositions of Euclidean, Cartesian, Hilbertian geometry which might be thought to require the Dedekind axiom. Section 4.1 contrasts the arithmetization of geometry program of the 19th century with the grounding

[5] In the logic, $L_{\omega_1,\omega}$, quantification is still over individuals but now countable conjunctions are permitted so it is easy to formulate Archimedes axiom : $\forall x, y(\bigvee_{m\in\omega} mx > y$. By switch the roles of x and y we see each is reached by a finite multiple of the other. {archdef}

[6] Dedekind defines the notion of a cut a linearly ordered set I (a partition of $\mathbb{Q}$ into two intervals (L, U)). Dedekind postulates that each cut has unique realization, a point above all members of L and below all members U -it may be in either L or U (page 20 of [18]. If either the L contains a least upper bound or the upper interval U contains a greatest lower bound, the cut is called 'rational' and no new element is introduced. Each of the other (irrational) cuts introduces a new number. It is easy to see that the *unique* realization requirement implies the Archimedes axiom. By Dedekind completeness of a line, I mean the Dedekind postulate holds for the linear ordering of that line. {Dedpost}

[7] Page 26 of [46].

[8] For a thorough historical description, see the section The Vollständigkeit, on pages 426-435 of [47]. We focus on the issues most relevant to this paper.

of algebra in geometry enunciated by Hilbert. We lay out in Section 4.2 various sets of axioms for geometry and correlate them with the data sets of Section 3.3 in Theorem 4.2.3. Section 4.3 sketches Hilbert's proof that they suffice to define a field. In Section 4.4 we also explore the role of the circle-circle intersection axiom and note that several theorems, which at first glance (or first historical proof) used Dedekind's postulate, are consequences of Hilbert's first order axioms. Section 4.5 has two purposes. On the one hand we distinguish the geometric conception of multiplication as similarity from repeated addition. On the other, we report Hilbert's first order proof that similar triangles have proportional sides. Again, in Section 4.6 we report Hilbert's argument for the computation of area of polygons (avoiding Euclid's implicit use of the Archimedes axiom). Such a *formula* as $A = \pi r^2$ is not justified on the basis of Hilbert's first order axioms (even with Archimedes); but In Section 5, we expand the first order theory of Euclidean geometry EG, by adding a constant π which allows us to computer the area and circumference of a circle. We pause in Section 6 to consider in more detail the differences between the Euclidean and Cartesian data set. We note Cartesian geometry both involves new interpretations of old constructions and an extension of the axiom set. We argue that the Tarski's first order complete geometry is a modest descriptively complete axiom set for this data set. We then combine the last two sections to in Section 7 by providing a complete theory that supports π. Finally, we conclude in Section 8 with 1) an argument that Hilbert's continuity axioms are overkill and 2) with some speculations about the use of 'definable analysis' to justify parts of analysis on first order grounds. The appendix analyzes the distinctions between the completeness axioms of Dedekind and Hilbert.

2 The Goals of Axiomatization

{goalax}

In this section, we place our analysis in the context of recent philosophical work on the purposes of axiomatization. We investigate the connection between axiom sets and data sets of sentences for an area of mathematics. We introduce the notion of a modest descriptively complete axiomatization for a particular data set.

Hilbert begins the Grundlagen [46] with:

> The following investigation is a new attempt to choose for geometry a *simple* and *complete* set of *independent axioms* and to deduce from them the most important geometrical theorems in such a manner as to bring out as clearly as possible the significance of the groups of axioms and the scope of the conclusions to be derived from the individual axioms.

Hallett (page 434 of [47]) presaged much of the intent of this article:

> Thus completeness appears to mean [for Hilbert] 'deductive completeness with respect to the geometrical facts'. ...In the case of Euclidean ge-

> ometry there are various ways in which 'the facts before us' can be presented. If interpreted as 'the facts presented in school geometry' (or the initial stages of Euclid's geometry), then arguably the system of the original Festschrift [i.e. 1899 French version] is adequate. If, however, the facts are those given by geometrical intuition, then matters are less clear.

We begin by considering several ways of construing the word 'fact' as it appears in this discussion. Hintikka doesn't use that word; facts[9] in his 'descriptive use of logic' are a class of models:

> If we use logical notions (such as quantifiers, connectives, etc) for the purpose of capturing a class of structures studied in a particular mathematical theory, we are pursuing the descriptive use of logic. To be more precise, we exploit logic in the sense that we formulate an axiomatization of a mathematical theory in order to describe that class of structures and no other structures, as precisely as we can. Thus, a descriptive use of logic consists, for example, in formulating an axiom system in order to capture the class of structures which number-theory deals with, e.g. the series of natural numbers.
>
> If we want to systematize and formalize mathematicians reasoning about the mathematical structures they are interested in, we are interested in the deductive use of logic. . . . This machinery appeals to the deductive consequence relation provided by the logic, which in turn is defined by a list of inference rules.

Note that Hintikka's 'descriptive use' is a semantic requirement - describe 'as precisely as we can' a *class of structures*. In general this is a hard problem; we don't really know *all* the models. Thus, it is natural that several authors (e.g. [35] [60]) have focused on the situation where, as in Hintikka's example, the class is categorical; we can have a strong conception of a particular structure: *the* natural numbers. This focus has historical roots in Peano, Dedekind and the early axiomatizations of geometry (e.g., Hilbert and Veblen). Blanchette [12] provides an apt moniker for this view: model-centric.

Detlefson [21] provides a syntactic counterpart to Hinktikka's notion of descriptive use, which he calls *descriptive axiomatization*. He motivates the notion with this remark of Huntington (Huntington's emphasis) [50]:

> [A] miscellaneous collection of facts . . . does not constitute a *science*. In order to reduce it to a science the first step is to do what *Euclid* did in geometry, namely, to *select a small number of the given facts as axioms*

[9]We quote Hintikka's discussion ([49] as quoted in [35]) of the descriptive use of logic in full and truncate his account of the deductive aspect.

> *and then to show that all other facts can be deduced from these axioms by the methods of formal logic*.

Detlefsen describes a local descriptive axiomatization as an attempt to deductively organize a data set (a collection of commonly accepted sentences pertaining to a given subject area of mathematics[10]). The axioms are *descriptively complete* if all elements of the data set are deducible from them. This raises two questions. What is a sentence? Who commonly accepts?

From the standpoint of modern logic, a natural answer to the first would be to specify a logic and a vocabulary and consider all sentences in that language. Detlefsen argues (pages 5-7 of [21]) that this is the wrong answer. He thinks Gödel errs in seeing the problem as completeness in the now standard sense of a first order theory[11]. Rather, Detlefsen presents us with an empirical question. We (at some point in time) look at the received mathematical knowledge in some area and want to construct a set of axioms from which it can all be deduced. In general such a data set is more graspable than *all models*. Of course, the data set is inherently flexible; conjectures are proven from time to time. In a way this version reflects Hintikka's as the data set is indeed a description of the class of its models. But it is very far from being model-centric as there is no requirement of categoricity.

Geometry is an example of what Detlefsen calls a local as opposed to a foundational descriptive axiomatization. Beyond the obvious difference in scope, Detlefsen points out several other distinctions. In particular ([21] page 5), the axioms of local axiomatizations are generally among the given facts while those of a foundational axiomatization are found by (paraphrasing Detlefsen) tracing each truth in a data set back to the deepest level where it can be properly traced. Comparing geometry at various times opens a deep question we want to avoid. In what sense do (*) and (**) opening this paper express the same thought, concept etc. Rather that address the issue of what is expressed, we will simply show how to interpret (*) (and other propositions of Euclid) as propositions in Hilbert's system. See Section 3.2 for this issue and Section 3.3 for extensions to the data set over the centuries.

An aspect of choosing axioms seems to be missing from the account so far. Hilbert [48] provides this insight into how axioms are chosen:

> If we consider a particular theory more closely, we always see that a few distinguished propositions of the field of knowledge underlie the construction of the framework of concepts, and these propositions then suffice by themselves for the construction, in accordance with logical principles, of the entire framework. ...

[10]There is an interesting subtlety here. Suppose our body of mathematics is group theory. One might think the data set was the sentences in the vocabulary of group theory true in all groups. (The axioms are evident). But these sentences are not in fact the data set of 'group theory'; that subject is concerned about the properties and relations between groups. So taking the commutative law as a sentence that might illegitimately be added as an axiom for groups is studying the wrong subject.

[11]We argue against this in Remarks 6.17 and 7.16.

> These underlying propositions may from an initial point of view be regarded as the axioms of the respective field of knowledge ...

We want to identify a 'few' distinguished propositions[12] from the data set that suffice for the deduction of the data set. By a *modest* axiomatization[13], we mean one that implies all the data and not too much more[14]. Of course, 'not too much more' is a rough term. One cannot expect a list of known mathematical propositions to be deductively complete. By more, we mean introducing essentially new concepts and concerns or by adding additional hypotheses proving a result that contradict the explicit understandings of the authors of the data set (See the end of Section 4.1. As we'll see below, Hilbert's first order axioms are a modest axiomatization of the data: the theorems in Euclid about polygons (not circles) in the plane. We give an example in Remark 4.4.4 of an immodest first order axiomatization.

The mathematical goal of this paper is to provide a modest descriptively complete axiomatization of plane geometry including the propositions concerning the circumference and area of a circle. If the data set is required to be deductively closed, there would be an easy sufficient condition for a modest axiomatization: the axioms must come from the data set. There is a difficulty with this requirement. First, the data sets stem from eras before 'deductive closure' was clearly defined; so there is an issue of how to apply this requirement to an 'open system' in the sense of [63].

There are two ways in which data sets are destined to change. New theorems will proved from the existing hypotheses; but, more subtly, new interpretations of the basic concepts may develop over time so that sentence attain essentially new meanings. Such is the case with Euclid's VI.1.

We return to our question, "what is a sentence?". The first four groups of Hilbert's axioms are sentences of first order logic: quantification is over individuals and only finite conjunctions are allowed. As noted in Footnote 5, Archimedes axiom can be formulated in $L_{\omega_1,\omega}$. But the Dedekind postulate in any of its variants is a sentence of a kind of second order logic[15]. All three logics are deductive systems so that the set of provable sentences is recursively enumerable. Second order logic (in the standard semantics) fails the completeness theorem but by the Gödel and Keisler [52] completeness theorems every valid sentence of $L_{\omega,\omega}$ or $L_{\omega_1,\omega}$ is provable. In the following discussion we focus on the second order axiom. We take up the Archmidean axiom in detail in Section 8.1 and the role of $L_{\omega_1,\omega}$ in Remark 7.17 on π .

[12]Often, few is interpreted as finite. Whatever Hilbert meant, we should now be satisfied with a small finite number of axioms and axiom schemes. At the beginning of the Grundlagen, Hilbert adds 'simple, independent, and complete'. Such a list including schemes is simple.

[13]We considered replacing 'modest' by 'precise or'safe' or 'adequate'. We chose 'modest' rather than one of the other words to stress that we want a sufficient set and one that is as necessary as possible. As the examples show, 'necessary' is too strong. Later work finds consequences of the original data set undreamed by the earlier mathematicians. Thus just as, 'descriptively complete', 'modest' is a description, not a formal definition.

[14]This concept describes normal work for a mathematician. "I have a proof; what are the actual hypotheses so I can convert it to a theorem."

[15]See the caveats on 'second order' in Section 9.

Adopting this syntactic view, there is a striking contrast between the data set in earlier generations of such subjects as number theory and geometry and axiom systems advanced at the turn of the twentieth century; except for the Archimedean axiom, the data sets are expressed in first order logic. But through the analysis of the concepts involved, Dedekind arrived at second order axioms that formed the capstone of each axiomatization: induction and Dedekind completeness. These axioms answered real problems (especially in analysis). But a primary goal was to describe a particular structure, to attain categoricity.

In the quotation above, Hilbert takes the axioms to come from the data set. But this raises a subtle issue about what comprises the data set. For examples such as geometry and number theory, it was taken for granted that there was a unique model. In one sense this reflects a model-centric view. But even Hilbert (Blanchette's representative of the deductivist view) adds his completeness axiom to guarantee categoricity and to connect with the real numbers. So one can certainly argue that the early 20th century axiomatizers took categoricity as part of the data[16]. But is it essential? Can one obtain the first order data set without making second order assumptions?

Hallett (page 429 of [47]) formulates this issue in words that fit strikingly well in the 'descriptive axiomatization' framework, "Hilbert's system with the Vollständigkeit is complete with respect to 'Cartesian' geometry'." But by no means is Cartesian geometry a part of Euclid's data set.

3 Descriptions of the Geometric Continuum

In the first subsection, we distinguish the 'geometric continuum' from the set theoretic continuum. In Section 3.2 we sketch the background shift from the study of various types of magnitudes by the Greeks, to the modern notion of a collection of real numbers which are available to measure any sort of magnitude. In the third subsection we set out various data sets for 'plane geometry' and discuss the distinctions among them. In the remainder of the paper we will analyze axiomatizations for each data set.

3.1 Conceptions of the continuum

{preformal}

In this section, we motivate our restriction, the geometric continuum is a linearly ordered structure and provide some background on the demand that the line is situated in the plane. Sylvester[17] describes the three divisions of mathematics:

> There are three ruling ideas, three so to say, spheres of thought, which

[16]In fact Huntington invokes Dedekind's postulate in his axiomatization of the complex field in the article quoted above [50].

[17]As quoted in [57].

> pervade the whole body of mathematical science, to some one or other of which, or to two or all of them combined, every mathematical truth admits of being referred; these are the three cardinal notions, of Number, Space and Order.

This is a slightly unfamiliar trio. We are all accustomed to the opposition between arithmetic and geometry. While Newton famously founded the calculus on geometry (see e.g. [22]) the 'arithmetization of analysis' in the late 19th century reversed the priority. From the natural numbers the rational numbers are built by taking quotients and the reals by some notion of completion. And this remains the normal approach today. We want here to consider reversing the direction again: building a firm grounding for geometry and then finding first the field and then some completion and considering incidentially the role of the natural numbers. In this process, Sylvester's third cardinal notion, order, will play a crucial role. In the first section, the notion that one point lies between two others will be fundamental and an order relation will naturally follow; the properties of space will generate an ordered field and the elements of that field will be numbers (but definitely not the set of natural numbers).

We here argue briefly that there is a problem: there are different conceptions of the continuum (the line); hence different axiomatizations may be necessary to reflect these different conceptions. These different conceptions are witnessed by such collections as [24, 67] and further publications concerned with the constructive continuum and various non-Archimdean notions of the continuum.

Feferman [30] lists six[18] different conceptions of the continuum: (i) the Euclidean continuum, (ii) Cantor's continuum, (iii) Dedekind's continuum, (iv) the Hilbertian continuum, (v) the set of all paths in the full binary tree, and (vi) the set of all subsets of the natural numbers. For our purposes, we will identify ii), v), and vi) as essentially cardinality based as they have lost the order type imposed by the geometry; so, they are not in our purview. We want to contrast two essentially geometrically based notions of the continuum: those of Euclid and Hilbert. And we identify Dedekind's and Hilbert's conceptions for reasons described in Section 9.

We began by stipulating that by 'geometric continuum', we meant the line situated in the plane. One of the fundamental results of 20th century geometry is that *any* (projective[19] for convenience) plane can be coordinatized by a 'ternary field'. A ternary plane is a structure with one ternary function $f(x, y, z)$ such that f has the properties that $f(x, y, z) = xy + z$ would have if the right hand side were interpreted in a field. In accord with our concerns with Euclidean geometry here, we assume the axioms of congruence and the parallel postulate; this implies the ternary field is actually a field. But these geometric hypotheses are necessary. In [6], I constructed an $\aleph_1$-categorical projective plane where the ternary field is a wild as possible (in the precise sense of the Lenz-Barlotti classification [83]).

[18] Smorynski [68] notes that Bradwardine already reported five in the 14th century.

[19] That is, any system of points and lines such that two points determine a line, any two lines intersect in a point, and there are 4 non-collinear points.

3.2 Ratio, magnitude, and number

{magnum}

In this section we give a short review of Greek attitudes toward magnitude and ratio as described for example in [59, 29, 71]. We by no means follow the 'geometric algebra' interpretation decried in [38]. We attempt to contrast the Greek meanings of propositions with Hilbert's understanding. When we rephrase a sentence into algebraic notation we try to make clear this is a modern formulation, not the intent of Euclid.

Euclid develops arithmetic in chapters VII-IX. What we think of as the 'number' one, was the unit: a number (Definition VII.2) is a multitude of units. These are counting numbers. So from our standpoint (considering the unit as the number 1) Euclid's numbers (in the arithmetic) can be thought of as the 'natural numbers'. The numbers[20] are a discretely ordered collection of objects.

Following Mueller[21] we work from the interpretation of magnitudes in the Elements as "abstractions from geometric objects which leave out of account all properties of those objects except quantity" : length of line segments, area of plane figures, volume of solid figures etc. Mueller emphasizes the distinction between the properties of proportions of magnitudes developed in Chapter V and those of number in Chapter VII. The most easily stated is implicit in the proof V.5; for every m, every magnitude can be divided in m equal parts[22]. This is of course, false for the numbers.

There is a second use of 'number' in Euclid. It is possible to count unit magnitudes, to speak of, e.g. four copies of a unit magnitude. So (in modern language) Euclid speaks of multiples of magnitudes by positive integers. See Remark 4.3.4 where we give a modern mathematical interpretation of this usage.

Magnitudes of the same type are also linearly ordered and between any two there is a third.[23] Multiplication of line segments yielded rectangles. Ratios are not objects; equality of ratios is a 4-ary relation between two pairs of homogenous magnitudes[24].

Remark 3.2.1. *Here are 4 important definitions or propositions from Chapter V of Euclid.*

1. *Definition V.4 of Euclid [29] asserts: Magnitudes are said to* have a ratio *to one another, which are capable, when multiplied, of exceeding one another.*

2. *Definition V.5 defines 'sameness of two ratios' GG, (in modern terminology): The ratio of two magnitudes x and y are proportional to the ratio of two others*

[20]More precisely, natural numbers greater than 1.

[21]page 121 of [59].

[22]page 122 of [59].

[23]The Greeks accepted only potential infinity. So, while from a modern perspective, the natural numbers are ordered in order type ω, and any collection of homogeneous magnitudes (e.g. areas) are in a dense linear order (which is necessarily infinite); this completed infinity is not the understanding of the Greeks.

[24]Homogeneous pairs means magnitudes of the same type. Ratios of numbers are described in Chapter VII while ratios of magnitudes are discussed in Chapter V.

z, w if for each m, n, $mx > ny$ implies $mz > nw$ (and also if $>$ is replaced by $=$ or $<$).

3. *Definition V.6 says, Let magnitudes which have the same ratio be called proportional.*

4. *Proposition V.9 asserts that 'same ratio' is, in modern terminology, a transitive relation. Apparently Euclid took symmetry and reflexivity for granted and treats proportional as an equivalence relation.*

We now contrast Euclid's notion of proportionality with the segment multiplication of Descartes and Hilbert. We begin with a particular sequence of theorems that illuminate the distinction. Bolzano discusses the 'dissimilar objects' found in Euclid[25] and finds Euclid's approach fundamentally flawed. {BC}

Remark 3.2.2 (Bolzano's Challenge)**.**

> *Firstly triangles, that are already accompanied by circles which intersect in certain points, then angles, adjacent and vertically opposite angles, then the equality of triangles, and only much later their similarity, which however, is derived by an atrocious detour [ungeheuern Umweg], from the consideration of parallel lines, and even of the area of triangles, etc.! (1810, Preface)*

Bolzano's 'atrocious detour' has two aspects: a) the evil of using two dimensional concepts to understand the line[26] and b) the following 'detour' to similarity[27]. In VI.1, using the technology of proportions from chapter V, Euclid determines the area of a triangle or parallelogram; in VI.2, he uses these results to show that similar triangles have proportional sides. The role of the theory of proportions is to show that the area of two parallelograms whose base and top are on the same parallel lines (and so the parallelograms have the same height) but whose bases are of possibly incommensurable lengths have proportionate areas. This step in the arguments uses the properties of proportion in Book V. It is then straightforward to deduce V1.2. And from VI.2, he constructs in VI.12 the fourth proportional to three lines.

Descartes defines the multiplication of line segments to give another segment[28], but he is still relying on Euclid's theory of proportion to justify the multiplication. From segment multiplication, we can regain the notion of proportionality.

[25]This is taken from [32].

[26]In contrast, we take such concepts as fundamental in understanding the *geometric* continuum.

[27]Section 4 reports how Hilbert avoids this detour.

[28]He refers to the construction of the fourth proportional ('ce qui est meme que la multiplication' [20]). See also Section 21 page 296 of [14].

{propdef}

Definition 3.2.3 (Proportionality)**.** *We write the ratio of CD to CA is proportional to that of CE to CB,*

$$CD : CA :: CE : CB$$

which is defined as

$$CD \times CB = CE \times CA.$$

where $\times$ *is taken in the sense of segment multiplication defined as in Descartes by the fourth proportional or as in Definition 4.3.2.*

Now (*) (beginning of this paper) (VI.1) is interpreted as a variant of (**):

$$F(ABC) = \frac{1}{2}\alpha \cdot AB \cdot AC.$$

Here $F(ABC)$ is an area function satisfying the properties discussed in Definition 4.6.6. But the cost is that Euclid does not specify what we now call the proportionality constant while Hilbert must. As we'll see in Definition 4.6.6, Hilbert assigns a proportionality constant (in this case the constant α is one).

In his proof of VI.1 (our *) Euclid applies Definition 5.5 above to deduce the proportionality of the area of the triangle to its base. But this assumes that any two lengths (or any two areas) have a ratio in the sense of Definition V.4. This is an implicit assertion of Archimedes axiom for both area and length[29]. As developed in Section 4, Hilbert's treatment of area and similarity has no such dependence. It is widely understood[30] that Dedekind's analysis is radically different from that of Eudoxus. A principle reason for this, which we emphasize at the end of Section 7, is that Eudoxus applies his method to specific situations; Dedekind demands that every cut be filled. Secondly, Dedekind develops addition and multiplication on the cuts. Thus, Dedekinds's postulate should not be regarded as part of the Euclidean data set. {naming01}

Remark 3.2.4 (Naming $0, 1$)**.** Hilbert shows the multiplication on segments satisfies the semi-field axioms[31]. A last step is to fixing $0, 1$, so that addition and multiplication can be defined on the points of the line through $0, 1$[32]. Hilbert has defined segment

[29] Euclid's development of the theory of proportion and area requires the Archimedean axioms. Our assertion is one way of many descriptions of the exact form and location of the dependence among such authors as [29, 59, 71, 31, 68]. Since our use of Euclid is as a source of sentences, not proofs, this reliance is not essential to our argument.

[30] Stekeler-Weithofer [72] writes, "It is just a big mistake to claim that Eudoxus's proportions were equivalent to Dedekind cuts." Feferman [30] avers, "The main thing to be emphasized about the conception of the continuum as it appears in Euclidean geometry is that the general concept of set is not part of the basic picture, and that Dedekind style continuity considerations of the sort discussed below are at odds with that picture." Stein [71] gives a long argument for at least the compatibility of Dedekind's postulate with Greek thought "reasons … plausible, even if not conclusive- for believing the Greek geometers would have accepted Dedekinds's postulate, just as they did that of Archimedes, once it had been stated."

[31] In a semi-field there is no requirement of an additive inverse.

[32] And thus all axioms for a field are obtained. Hilbert had done this in lecture notes in 1894 [47].

multiplication on the ray from 0 through 1. But to get negative real numbers he must reflect through 0.

In modern usage we consider natural number and real numbers as different species of the genus 'number'. In mathematical terms this is reflected by speaking of numbers when quantifying over the 'ground field'.

The next step is to identity the point points on the line and the domain of an ordered field by mapping A to OA. This naturally leads to thinking of a segment as a set of points, which is foreign to both Euclid and Descartes. We will discuss the historical significance of this shift just before Theorem 6.16. But even before that we will use the more flexible language of points, especially in Section 5. See Dicta 4.3.5.

We will go further and study (as Dedekind's or Birkhoff's postulates demand) the identification of a straight line segment of the same length as the circumference of a circle. But this contradicts the 4th century view of Eutocius[33], 'Even if it seemed not yet possible to produce a straight line equal to the circumference of the circle, nevertheless, the fact that there exists some straight line by nature equal to it is deemed by no one to be a matter of investigation.'

3.3 Some geometric Data sets

{data}

We begin by distinguishing a number of topics in geometry; these represent distinct data sets in Detlefsen's sense. We label the data sets by names of mathematicians as a convenient references.

1. Euclid I polygonal geometry[34]
2. Euclid II: circle geometry[35]
3. Archimedes: π arc length circumference and area of a circle as proportions[36]
4. Descartes: nth roots[37]
5. Weierstrass: analysis 19th century $A = \pi r^2$, $C = \pi d$.

Table 1 lists five data sets and the proposed descriptively complete axiomatization. For example, the third data set is the geometry of polygons and circles with measurement of arc length. The relevant axiom sets are defined in Notation 4.2.2 for

[33] Archimedis Opera Omnia cum commentariis Eutociis, vol. 3, p. 266. Quoted in: Davide Crippa (Sphere, UMR 7219, Universit Paris Diderot) Reflexive knowledge in mathematics: the case of impossibility

[34] The Elements: Chapter I (except I.1 and (I.22), chapter II, III except for III.1 and III.17, chapters IV-VI.

[35] The Elements: Chapter IV.

[36] The Elements: Euclid XII.2 (area of circle); Archimedes: [2].

[37] [20]

the Euclidean propositions, Theorem 6.16 for Descartes and Theorem 5.0.4 for the area and circle formulas. Theorem 4.2.3 spells out an appropriate axiom system for each topic. I used Weierstrass as the symbol of the 19th century clarification of the foundations of analysis. This activity goes well beyond the traditional area of geometry. As we noted in the introduction, Hilbert was aiming to justify this other phenomena.

Table 1:

Data set/topic	Name	Axiom set
polygons	Euclid I	HP5
circles	Euclid II	EG
arc length π	EG_π	EG_π or $\mathcal{E}_\pi$
nth roots, segment multiplication	Descartes	$\mathcal{E}^2$
limits	Weierstrass	Hilbert

Now we outline some specific theorems, will be addressed below, that might be thought to depend on the continuity axioms. {goal}

Remark 3.3.1. *1. Euclid I*

(a) Similar triangles have proportional sides (Theorem 4.5.2) {sidesplitter}

(b) Area of polygons

(c) Pythagorean theorem {pythag}

(d) laws of sines and cosine {sincos}

2. Euclid II

(a) Euclid 3: circle intersection (Theorem 4.4.5) {cci}

(b) Construction of an equilateral triangle. Euclid I.1

3. Archimedes:

(a) Formulas for circumference of circle: Theorem 5.0.11,

(b) Formulas for area of circle:Theorem 5.0.15

4. Descartes

(a) segment multiplication and coordinate geometry

(b) Extraction of nth roots for all n. Theorem 6.16 {extract}

(c) real algebraic geometric and the association of polynomials with curves {poly}

5. Weierstrass

(a) $\sqrt{2} \cdot \sqrt{3} = \sqrt{6}$. (Theorem 4.4.1) {irrational}

(b) All angles have measure. (Theorem 7.12)

(c) analysis of continuous and even transcendental functions

(d) provable categoricity

We deal in detail below with Euclid I; the crucial point for a), b), c) is that the arguments in Euclid all go through the theory of area which depends on Eudoxus and so has an implicit dependence on the Archimedean axiom; Hilbert eliminates this dependence.

The role of Euclid II appears already in Proposition I of Euclid[38] where Euclid makes the standard construction of an equilateral triangle on a given base. Why do the two circles intersect? While some[39] regard the absence of this axiom as a gap in Euclid, Manders (page 66 of [56]) asserts: 'Already the simplest observation on what the texts do infer from diagrams and do not suffices to show the intersection of two circles is completely safe[40].' For our purposes, here we are content to accept that adopting the circle-circle intersection axiom resolves those continuity issues around circles and lines[41]. We separate this case because Hilbert's first order axioms do not resolve this issue[42]; he chooses to resolve it (implicitly) by an appeal to Dedekind[43].

We discuss Descartes in Section 6. In Sections 5 and 7 we come to the new mathematics of this paper. We provide a first order theory to justify the formulas for circumference and area of a circle. For this, Hilbert implicity uses both continuity axioms to guarantee the existence of π.

Showing a particular set of axioms is descriptively complete is inherently empirical. One must check whether each of a certain set of results is derivable from a given set of axioms. Hartshorne uses this yardstick at various places (e.g Theorem 10.4) in [41] and we summarise the project in Theorem 4.2.3.

[38] `http://aleph0.clarku.edu/~djoyce/java/elements/bookI/propI1.html`

[39] [79], page 4

[40] Manders develops the use of diagrams as a coherent mathematical practice; others have developed the idea of formalizing a deductive system which incorporates diagrams. Here is a rough idea of this program. Properties that are *not* changed by minor variations in the diagram such as subsegment, inclusion of one figure in another, two lines intersect, betweenness are termed *inexact*. Properties that *can be* changed by minor variations in the diagram, such as whether a curve is a straight line, congruence, a point is on a line, are termed *exact*. We can rely on reading inexact properties from the diagram. We must write exact properties in the text. The difficulty in turning this insight into a formal deductive system is that, depending on the particular diagram drawn, after a construction, the diagram may have different inexact properties. The solution is case analysis but bounding the number of cases has proven difficult.

[41] Circle-circle intersection implies line-circle intersection. Hilbert already in [46] shows (page 204-206 of [47]) that circle-circle intersection holds in what we call a Euclidean plane. See Section 4.4.

[42] Hilbert is aware of this and of the alternative discussed here.

[43] Moore suggests in [58] that Hilbert may have added the completeness axiom to the second edition specifically because Sommer in his review of the first edition pointed out it did not prove the line-circle intersection principle.

4 Axiomatizing the geometry of polygons and circles

{axgeom}

In the first section we contrast the goal here of an independent basis for geometry with the 19th century arithmetization project. The second section lays out the first order axioms that will be employed. Section 4.3 sketches Hilbert's definition of a field in a geometry. Section 4.4 distinguishes the role of the circle-circle intersection axiom and notices that a number of problems that can be approached by limits have uniform solutions in any ordered field; completeness of the field is irrelevant. We then return to Bolzano's challenge and derive first, Section 4.5, the properties of similar triangles and then, Section 4.6, the area of polygons.

4.1 From Arithmetic to geometry or from geometry to algebra?

{agorga}

On the first page of *Continuity and the Irrational Numbers*, Dedekind writes:

> Even now such resort to geometric intuition in a first presentation of the differential calculus, I regard as exceedingly useful from the didactic standpoint ... But that this form of introduction into the differential calculus can make no claim to being scientific, no one will deny.

I have no intention of denying that claim. I quote this passage to indicate that Dedekind's motivation was to provide a basis for calculus not geometry. But I will argue that the second order Dedekind completeness axiom is not needed for the geometry of Euclid and indeed for the grounding of the algebraic numbers, although it is for Dedekind's approach. Further I will discuss in Section 8 the possibility that a kind of 'definable' continuity provides a substitute for many (certainly not all) of Dedekind's concerns.

Dedekind provides a theory of the continuum (the continuous) line building up in stages from the structure which is fundamental to him: the natural numbers under successor. This development draws on second order logic in several places. The well-ordering of the natural numbers is required to define addition and multiplication by recursion. Dedekind completeness is a second appeal to a second order principle.

Perhaps in response to Bolzano's insistence, Dedekind constructs the line without recourse to two dimensional objects and from arithmetic. Thus, he succeeds in the 'arithmetization of analysis'.

We proceed in the opposite direction for several reasons. Most important is that we are seeking to ground geometry, not analysis. Further, we would assert that the concept of line arises only in the perception of at least two dimensional space. Dedekind's continuum knows nothing of being straight or breadthless. Hilbert's proof of the existence of the field is the essence of the *geometric continuum*. By virtue of its lying in a plane, the line acquires algebraic properties.

The distinction between the arithmetic and geometric intuitions of multiplication is fundamental. The first is as iterated addition; the second is as scaling or proportionality. The late 19th century developments provide a formal reduction of the second to the first but the reduction is only formal; the intuition is lost. In this paper we view both intuitions as fundamental and develop the second (Section 4.3): with the understanding that development of the first through the Dedekind-Peano treatment of arithmetic is in the background. See Remark 4.3.4 for the connection between the two.

4.2 The geometry of Euclid/Hilbert

{euclid}

We identify two levels of formalization in mathematics. By the Euclid-Hilbert style we mean the axiomatic approach of Euclid along with the Hilbert insight that postulates are implicit definitions of classes of models[44] By the Hilbert-Gödel-Tarski-Vaught style, we mean that that syntax and semantics have been identified as mathematical objects; Gödel's completeness theorem is a standard tool, so methods of modern model theory can be applied[45]. We will give our arguments in English; but we will be careful to specify the vocabulary and the postulates in a way that the translation to a first order theory is transparent. This will allow us to apply the insight that non-elementary properties of models can strengthen the effect of first order assertions (as in Theorem 7.12.f). {ccp}

Postulate 4.2.1. Circle Intersection Postulate *If from points A and B, circles with radius AC and BD are drawn such that one circle contains points both in the interior and in the exterior of the other, then they intersect in two points, on opposite sides of AB.* {HP5}

Notation 4.2.2. We follow [41] in the following nomenclature.

A *Hilbert plane* is any model of Hilbert's incidence, betweenness[46], and congruence axioms. We abbreviate these axioms by HP. We will write HP5 for these axioms plus the parallel postulate.

By the *axioms for Euclidean geometry* we mean HP5 and in addition the circle-circle intersection postulate 4.2.1. We will abbreviate this as EG[47]

By definition, *Euclidean plane* is a model of EG: Euclidean geometry.

[44]The priority for this insight is assigned to such slightly earlier authors as Pasch, Peano, Fano, in works such as [33] as commented on in [13] and chapter 24 of [39].

[45]See Section 8.2 and [9] for further explication of this method and Section 7 for an application.

[46]These include Pasch's axiom (B4 of [41]) as we axiomatize *plane* geometry. Hartshorne's version of Pasch is that any line intersecting one side of triangle must intersect one of the other two.

[47]In the vocabulary here, there is a natural translation of Euclid's axioms into first order statements. The constructions have be viewed as 'for all there exist sentences. The axiom of Archimedes as discussed below is of course not first order. We write Euclid's axioms for those in [29] vrs (first order) axioms for Euclidean geometry, EG. Note that this theory is equivalent to (i.e. has the same models planes over fields where every positive element as a square root) as the system laid out in [4]. That system builds the use of diagrams into the proof rules.

We write $\mathcal{E}^2$ for a geometrical axiomatization of the plane over a real closed field (RCF) (Theorem 5.0.4).

With these definitions we align various subsystems of Hilbert's geometry with certain collections of propositions in Euclidean geometry. {justify}

Theorem 4.2.3. *1. The sentences of Euclid I, polygonal geometry: (Chapter I (except I.1 and (I.22), chapter II, III except for III.1 and III.17, chapters IV-VI.) are provable in HP5.*

2. *The sentences of Euclid II, circle geometry: (Chapter IV, I.1 and I.22, III.1 and III.17 are provable in EG.*

3. *The sentences of Archimedes[48], arc length and π: (Euclid XII.2, area of circle) are provable in Hilbert I-IV plus Archimedes or in the first order EG_π and $\mathcal{E}_\pi$.*

4. *The sentences of Descartes:(nth roots) are provable in RCF ($\mathcal{E}^2$).*

5. *The 19th century analysis of Weierstrass is provable in Hilbert's full system.*

Proof. For 1) and 2) see Sections 20-23 of [41]. For 3) see (see Sections 5 and 7) and for 4) Section 6. For 5) choose an analysis text such as [70]. $\square_{4.2.3}$

In [5] and [8] we give an equivalent set of geometrical postulates, which return to Euclid's construction postulates and stress the role of Euclid's axioms (common notions) [49] in interpreting the geometric postulates. For aesthetic reasons we use SSS rather than SAS as the basic congruence postulate in those notes. Below we explicitly state the postulates only if it seems essential for the development.

We will frequently switch from syntactic to semantic discussions so we stipulate precisely the vocabulary in which we take the axioms above to be formalized. {geovoc}

Notation 4.2.4. *The fundamental relations of plane geometry make up the following vocabulary τ.*

1. *two-sorted universe: points (P) and lines (L).*
2. *Binary relation $I(A, \ell)$:*

   Read: a point is incident on a line;
3. *Ternary relation $B(A, B, C)$:*

   Read: B is between A and C (and A, B, C are collinear).
4. *quaternary relation, $C(A, B, C, D)$: Read: two segments are congruent, in symbols $\overline{AB} \cong \overline{CD}$.*

[48] See the discussion after Theorem 5.0.11.

[49] See http://aleph0.clarku.edu/~djoyce/java/elements/bookI/bookI.html#cns

5. *6-ary relation $C'(A, B, C, A', B', C')$: Read: the two angles $\angle ABC$ and $\angle A'B'C'$ are congruent, in symbols $\angle ABC \cong \angle A'B'C'$.*

Note that I freely used defined terms such as collinear, segment, and angle in giving the reading of these relation symbols.

4.3 From geometry to segment arithmetic to numbers

{num}

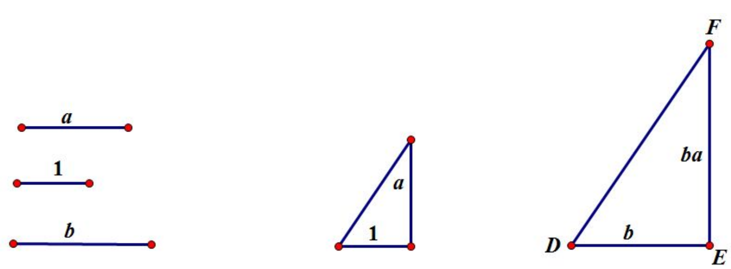
We introduce in this section *segment arithmetic* and sketch Hilbert's definition of the (semi)-field of segments with partial subtraction and multiplication. We assume what we called HP5 in Notation 4.2.2. The details can be found in e.g. [46, 41, 8] {segeq}

Notation 4.3.1. *Note that congruence forms an equivalence relation on line segments. We fix a ray ℓ with one end point 0 on ℓ. For each equivalence class of segments, we consider the unique segment $0A$ on ℓ in that class as the representative of that class. We will often denote the segment $0A$ (ambiguously its congruence class) by a . We say a segment CD (on any line) has length a if $CD \cong 0A$.*

Of course there is no additive inverse if our 'numbers' are the lengths of segments which must be positive. However, this procedure can be extended to a field structure on segments on a line not a ray (so with negatives), either directly as sketched in [5] or by passing through the theory of ordered fields as in Section 19 of [41]. Following Hartshorne [41], here is our official definition of segment multiplication[50]. {segmultdef}

Definition 4.3.2. *[Multiplication] Fix a unit segment class 1. Consider two segment classes a and b. To define their product, define a right triangle with legs of length 1 and a. Denote the angle between the hypoteneuse and the side of length a by α.*

Now construct another right triangle with base of length b with the angle between the hypoteneuse and the side of length b congruent to α. The product ab is defined to be the length of the vertical leg of the triangle.



[50]Hilbert's definition goes directly via similar triangles. The clear association of a particular angle with right muliplication by a recommends Hartshorne's version.

Note that we must appeal to the parallel postulate to guarantee the existence of the point F.

It is clear from the definition that there are multiplicative inverses; use the triangle with base a and height 1. The roughly 3 page proof that multiplication is commutative, associative, distributes over addition, and respects the order uses only the cyclic quadrilateral theorem and connections between central and inscribed angles in a circle.

It is easy[51] to check that the multiplication defined on the positive reals by this procedure is exactly the usual multiplication on the positive reals because they agree on the positive rational numbers. The final extension to make the multiplication on points (rather than segments) is also straight forward. To summarize (details in section 21 of [41]):

Theorem 4.3.3. *The theory of Hilbert fields satisfying the parallel postulate is biinterpretable with theory of ordered pythagorean*[52] *planes. The interpreting formulas are first order with constants naming two points.* {twomult}

Remark 4.3.4. We now have two ways in which we can think of the product $3a$. On the one hand, we can think of laying 3 segments of length a end to end. On the other, we can perform the segment multiplication of a segment of length 3 (i.e. 3 segments of length 1 laid end to end) by the segment of length a. It is an easy exercise to show these are the same. But it makes an important point. The (inductive) definition of multiplication by a natural number is indeed 'multiplication as repeated addition'. But the multiplication by another field element is based on similarity, implies the existence of multiplicative inverses, and so is a very different object.

We pause to discuss the first notion of multiplication in last paragraph which we earlier described as a second meaning of "number" at the beginning of Section 3.2. This is a kind of 'scalar multiplication' by positive integers that can be viewed mathematically as a rarely studied object: a semiring (the natural numbers) acting on a semigroup (positive reals under addition). There is no uniform definition[53] of this *scalar* multiplication within the field; multiplication by $\frac{17}{27}$ is defined in the geometry but not multiplication by $-\frac{17}{27}$.

A mathematical structure more familiar to modern eyes is obtained by extending to the negative numbers and has a well-defined notion of subtraction, both of the scalars which are now the ring $(Z, +, \cdot)$ and on the (now a module) $(\Re, +)$. Now we can multiply by $-\frac{17}{27}$ but the definition is still non-uniform.

Once we have defined the field, we have a uniformly defined multiplication and this intrinsic geometrical multiplication restricts to that imposed by counting where it is defined.

[51] One has to verify that segment multiplication is continuous but this follows from the density of the order since the addition respects order.

[52] A field is Pythagorean if for every a, $1 + a^2$ has a square root.

[53] Instead, there are infinitely many formulas $\phi_q(x, y)$ defining $qx = y$ for each $q > 0$.

{constants1}

Dicta 4.3.5 (Constants 1)**.** To fix the field we have to add constants 0, 1. These constants can name any pair of points in the plane (since the automorphism group acts two transitively (any pair of distinct points can be mapped by an automorphism to any other such pair) as can be proven in EG). But this naming induces an extension of the data set. We have in fact specified the unit. This relects a major change in view from either the Greeks or Descartes. In this situation, the change is small. A sentence $\phi(0, 1)$ holds just if either or both of $\forall x \forall y \phi(x, y)$ and $\exists x \exists y \phi(x, y)$ hold.

4.4 Initial consequences of field arithmetic

{sqrt}

In this section we investigate statements of two sorts: 1) statements of Euclid's geometry that depended on the Archimdean axiom and 2) statements about the properties of real numbers that Dedekind deduces from his postulate but are true in any field associated with a geometry modeling HP5.

We established in Section 4.3 that one could define an ordered field in any plane satisfying HP5. The converse is routine, the ordinary notions of lines and incidence in F^2 creates a geometry over any ordered field, which is easily seen to satisfy HP5. We now exploit this equivalence.

We will prove some algebraic facts using our defined operations, thus basing them on geometry. We begin with Property 3.3.1.5a: square root commutes with multiplication for algebraic numbers. Dedekind (page 22 of [18]) wrote '. . . in this way we arrive at real proofs of theorems (as, e.g. $\sqrt{2} \cdot \sqrt{3} = \sqrt{6}$), which to the best of my knowledge have never been established before.'

Note that this is a problem for Dedekind but not for Descartes. Already Euclid, in constructing the fourth proportional, constructs from segments of length 1, a and b, one of length ab; but he doesn't regard this operation as multiplication. When Descartes interprets this procedure as multiplication of segments, he has no problem. But Dedekind has presented the problem as multiplication in his continuum and so he must prove a theorem to find the product as a real number; that is, he must show the limit operation commutes with product. We report Hilbert's equally rigorous but much more simple proof that any field arising from geometry (e.g. the reals) is closed under multiplication (of any segments). {dedprob}

Theorem 4.4.1. *In an ordered field, for any positive a, if there is an element $b > 0$ with $b^2 = a$, then b is unique (and denoted $\sqrt{a}$). Moreover, for any positive a, c with square roots, $\sqrt{a} \cdot \sqrt{c} = \sqrt{ac}$.*

This fact holds for any field coordinatizing a plane satisfying HP5.

Thus, the algebra of square roots in the real field is established without any appeal to limits. The usual (e.g. [70, 1]) developments of the theory of complete ordered fields invoke the least upper bound principle to obtain the existence of the roots although the multiplication rule is obtained by the same algebraic argument as here. Our approach (like Hilbert's) contrasts with Dedekind[54]; our treatment is based on the geometric concepts and in particular regards 'congruence' as an equally fundamental intuition as 'number'. The justification here for either the existence or operations on roots does not invoke limits.

The shift here is from 'proportional segments' to 'product of numbers'. Euclid had a rigorous proof of the existence of a line segment which is the fourth proportional of $1 : a = b : x$. Dedekind demands a product of numbers; Hilbert provides this by a combination of his interpretation of the field in the geometry and geometrical definition of multiplication.

We now consider Properties 3.3.1, .1c, and .1d. It is well-known that the Pythagorean Theorem is equivalent for Hilbert planes (See Definition 4.2.2 to the parallel postulate. Euclid's proof of Pythagoras I.47 uses an area function as we will justify in Section 4.6. His second proof uses the theory of similar triangles that we will develop Section 4.5. Thus, in both cases Euclid depends on the theory of proportionality (and thus implicitly on Archimedes axiom) to prove the Pythagorean theorem; Hilbert avoids this appeal. Similarly, since the right angle trigonometry in Euclid concerns the ratios of sides of triangles the field multiplication interprets the geometrical operations.

Theorem 4.4.2. *The Pythagorean theorem as well as the law of cosines, Euclid II.11 and the law of sines, Euclid II.13 hold in any Hilbert plane with the parallel postulate (HP5).*

We note Hilbert's first order justification because Euclid's arguments for them implicitly relied on the Archimedean axiom. {exfields}

Example 4.4.3. Hartshorne [41] introduces two instructive examples.

1. A pythagorean field is one closed under addition, subtraction, multiplication, division and for every a, $\sqrt{(1 + a^2)}$. However, the Cartesian plane over a Pythagorean field may fail to be closed under square root and the Poincaire model over a such a field may fail to have equilateral triangles and thus the circle-circle intersection postulate also fails . (Exercise 39.30, 30.31 of [41])
2. On page 146, Hartshorne[55] observes that the smallest ordered field closed under addition, subtraction, multiplication, division and square roots of positive numbers satisfies the circle-circle intersection postulate. We denote this field by F_s for surd field. {impr}

Remark 4.4.4. *Note that if HP5 + CCI were proposed as an axiom set for polygonal geometry it would be a complete descriptive but not modest axiomatization.*

[54] Dedekind objects to the introduction of irrational numbers by measuring an extensive magnitude in terms of another of the same kind (page 9 of [18]).

[55] Hartshorne and Greenberg [40] calls this the constructible field, but given the many meanings of constructible, we use Moise's term surd field. {surd}

Recall that we distinguished a Hilbert plane from a Euclidean plane in Notation 4.2.2. As in [41], we have: {ccstrength}

Theorem 4.4.5. *A Hilbert plane satisfies the circle-circle intersection postulate, 4.2.1 if and only if every positive element of the coordinatizing plane has a square root.*

Similarly, in every Euclidean plane such that every positive element of the coordinatizing plane has a square root, Heron's formula computes the area of a triangle from the lengths of its sides.

Heron's formula demonstrates the hazards of the kind of organization of data sets attempted here. First, Heron apparently lived in the first century AD (and maybe the proof is due to Archimedes) so it doesn't fit in any of my time frames. Secondly, the geometric proof of Heron doesn't involve the square roots of the modern formula [42]. But since in EG we have the field and we have square roots, the modern form of Heron's formula can be proved from EG. Thus as in from (*) to (**), the different means of expressing the geometrical property requires different proofs.

In each case we have considered in this section, Greeks give geometric constructions for what in modern days becomes a calculation involving the field operations and square roots. We still need to complete the argument that HP5 is descriptively complete for polygonal Euclidean geometry. In particular, is the notion of proportional included in our analysis? The test question is the similar triangle theorem. We turn to this issue now.

4.5 Multiplication is not repeated addition

{similar}

In the natural numbers, addition can be defined as iterated succession and multiplication as iterated addition. But the resulting structure is essentially undecidable. However, this structure does not illuminate the essential aspect of multiplication as similarity; many elements have no multiplicative inverse.

Definition 4.5.1. *Two triangles $\triangle ABC$ and $\triangle A'B'C'$ are similar if under some correspondence of angles, corresponding angles are congruent; e.g. $\angle A' \cong \angle A$, $\angle B' \cong \angle B$, $\angle C' \cong \angle C$.*

Various texts define 'similar' as we did, or as corresponding sides are proportional or require both (Euclid). We now meet Bolzano's challenge by showing that in Euclidean Geometry (without the continuity axioms) the choice doesn't matter. Recall that we defined proportional in terms of segment multiplication in Definition 3.2.3. {simtri}

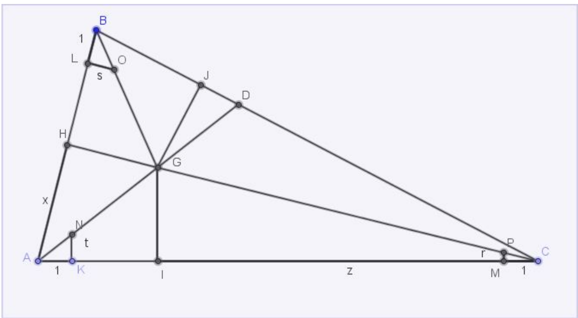
Theorem 4.5.2. *Two triangles are similar if and only if corresponding sides are proportional.*

Here is Hartshorne's proof of the fundamental result.

Proof of Theorem 4.5.2: If ABC and $A'B'C'$ are similar triangles then **using the segment multiplication we have defined**

$$\frac{AB}{A'B'} = \frac{AC}{A'C'} = \frac{BC}{B'C'}.$$

Consider the triangle ABC below with incenter G.



Proof. The point G is the incenter so $HG \cong GI \cong GJ$. Call this segment length a.

Now construct $AK \cong BL \cong MC$ all with standard unit length. Let the lengths of BL be s, NK be t and PM be r.

Let the lengths of $AI \cong AH$ be x, $BH \cong BJ$ be y, and $CI \cong AJ$ be z.

By the definition of multiplication $t \cdot x = s \cdot z = a$. Therefore the length of AC is $\frac{a}{t} + \frac{a}{r} = \frac{a(r+t)}{rt}$.

Duplicate on the second triangle $A'B'C'$ to get the length of $A'C'$ is $\frac{a'}{t} + \frac{a'}{r} = \frac{a'(r+t)}{rt}$. *The crucial point is that because the angles are congruent r, s, t are the same for both triangles.*

But then $\frac{A'C'}{AC} = \frac{a'}{a}$. Now note the same is true for the other two pairs of sides so the sides of the triangle are proportional.

The same ideas allow one to reverse the argument and show triangles with proportional sides are similar. $\square_{4.5.2}$

Remark 4.5.3. As Hilbert showed, in any model M of HP5: similar triangles have proportional sides. There is no assumption that the field is Archimedean or satisfies any sort of completeness axiom. There is no appeal to approximation or limits.

4.6 Area of polygonal figures {area}

In Section 4.5 we saw Bolzano's challenge 3.2.2 is answered by a proof that similar triangles have proportional sides without resorting to the concept of area. But area is itself a vital geometric notion. We show now that using segment multiplication Hilbert grounds the now familiar methods of calculating the area of polygons. As Hilbert writes[56], "We ... establish Euclid's theory of area *for the plane geometry and that independently of the axiom of Archimedes.*"

In this section, we sketch Hartshorne's [41] exposition of this topic. We stress the connections with Euclid's common notions and are careful to see how the notions defined here are expressible in first order logic; in line with our 4th objection to second order axiomatization, this shows that although these arguments are not carried out as direct deductions from the first order axioms, the results are derivable by a direct deduction.

Here is an informal definition[57] of those configurations whose areas are considered in this section.

Definition 4.6.1. *A* figure *is a subset of the plane that can be represented as a finite union of disjoint triangles.*

Hilbert raised a 'pseudogap' in Euclid[58] by distinguishing area and content. In Hilbert two figures have

1. *equal area* if they can be decomposed into a finite number of triangles that are pairwise congruent
2. *equal content* if we can transform one into the other by adding and subtracting congruent triangles.

Euclid treats the equality of areas as a special case of his common notions. The properties of equal content, described next, are consequences for Euclid of the common notions and need no justification. {areaax}

Theorem 4.6.2 (Properties of Equal Content)**.** *The following properties of area are used in Euclid I.35 through I.38 and beyond. They hold for equal content in HP5.*

[56] Emphasis in the original: (page 57 of [46]).

[57] In order to justify the application of the completeness theorem we have to introduce inductively a scheme giving the definition of an n-decomposable figure as the disjoint union of an $(n-1)$-decomposable figure A with a triangle such that a portion of one side of the triangle lies on a portion of one side of the figure A. In Section 5, we give such formal definitions to find area and circumference of a circle.

[58] Any model with infinitessimals shows the notions are distinct and Euclid I.35 and I.36 fail for what Hilbert calls area. Hilbert shows they are equivalent under the axiom of Archimedes. Since Euclid includes preservation under both addition and subtraction in his common notions, his term 'area' clearly refers to what Hilbert calls 'equal content', I call this a pseudogap.

1. *Congruent figures have the same content.*

2. *The content of two 'disjoint' figures (i.e. meet only in a point or along an edge) is the sum of the two content of the polygons. The analogous statements hold for difference and half.*

3. *If one figure is properly contained in another then the area of the difference (which is also a figure) is positive.*

There are serious issues concerning the formalization in first order logic of the notions in this section. Notions such as polygon involve quantification over integers; this is strictly forbidden within the first order system. We can approach these notions with axiom schemes. We want to argue that we can give a uniform metatheoretic definition of the relevant concepts and prove that the theorems hold in all models of the axioms.

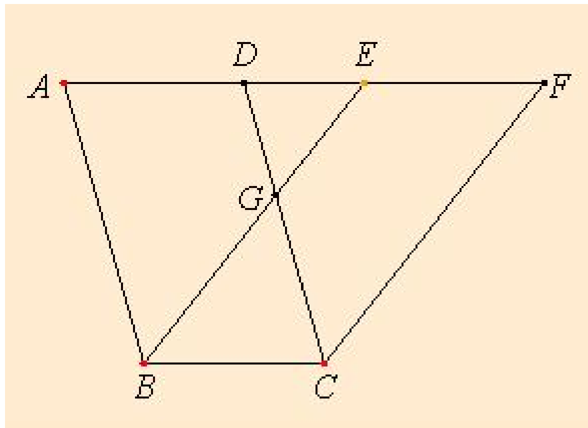
Observe that while these properties concern 'figure', a notion that is not definable by a single formula in first order geometry, we can replace 'figures' by n-gons for each n. For the crucial proof that the area of a triangle or parallelogram is proportional to the base and the height, we need only 'triangles or quadrilaterals'. In general we could formalize formalize these notions with either equi-area predicate symbols[59] or by a schema and a function mapping into the line as in Definition 4.6.6. Here is the basic step.

Definition 4.6.3. *Two figures α and β (e.g. two triangles or two parallelograms) have equal content in one step there exist figures α' and β' such that the disjoint union of α and α' is congruent to the disjoint union of β and β' and $\beta \cong \beta'$.*

Reading equal content for Euclid's 'equal', Euclid's I.35 (for parallelogram) and the derived I.37 (triangles) become the following. With this formulation Hilbert accepts Euclid's proof. {areaprop}

Theorem 4.6.4. *[Euclid/Hilbert] If two parallelograms (triangles) are on the same base and between parallels they have equal content in 1 step.*

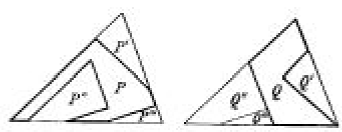
[59]For example, we could have 8-ary relation for quadrilaterals have the same area, 6-ary relation for triangles have the same area and 7-ary for a quadrilateral and a triangle have the same area.



By adding and subtracting figures, Euclid shows $ADBC$ has the same content as $EFBC$.

Now for arbitrary figures:

Definition 4.6.5 (Equal content)**.** *Two figures* P, Q *have* equal content *in* n *steps* [60] *if there are figures* $P'_1 \ldots P'_n$*,* $Q'_1 \ldots Q'_n$ *such that none of the figures overlap, each* P'_i *and* Q'_i *are scissors congruent and* $P \cup P'_1 \ldots \cup P'_n$ *is scissors congruent with* $Q \cup Q'_1 \ldots \cup Q'_n$*.*



Varying Hilbert, Hartshorne (Sections 19-23 of [41]) shows that these properties of content are satisfied in the first order axiom system EG (Notation 4.2.2). The key tool is: {areafn}

Definition 4.6.6. *An* area function *is a map* α *from the set of figures,* $\mathcal{P}$*, into an ordered additive abelian group with* 0 *such that*

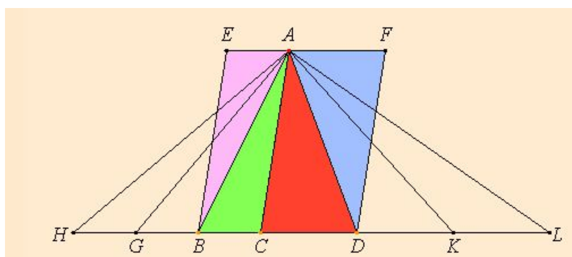
1. *For any nontrivial triangle* T*,* $\alpha(T) > 0$*.*
2. *Congruent triangles have the same content.*
3. *If* P *and* Q *are disjoint figures* $\alpha(P \cup Q) = \alpha(P) + \alpha(Q)$*.*

[60]The diagram is taken from [46].

This formulation hides the quantification over arbitrary n-gons. We clarify the method of translating to first order in Definition 5.0.9.

It is evident that if a plane admits an area function then the conclusions of Lemma 4.6.2 hold. This obviates the need for positing separately De Zolt's axiom that one figure properly included in another has smaller area[61]. In particular this implies Common Notion 4 for 'area'. Using the segment multiplication, Hilbert (compare the exposition in Hartshorne) establishes the existence of an area function for any plane satisfying HP5. The key point is to show that formula $A = \frac{bh}{2}$ does not depend on the choice of the base and height. Thus, Hilbert proves (**) without recourse to the axiom of Archimedes. {incomtri}

Remark 4.6.7. In contrast, recall the diagram for Euclid's, VI.1.



If, for example, BC, GB and HG are congruent segments then the area of ACH is triple that of ABC. But without assuming BC and BD are commensurable, Euclid calls on Definition V.5 of the proportionality chapter to assert that $ABD : ABC :: BD : BD$.

We have now shown that the axioms for Euclidean planes (HP5 + circle-circle intersection) suffice to prove Properties 3.3.1 a to d. Before addressing the question of π, we consider the extensions to Cartesian geometry.

5 Archimedes I: π, circumference and area of circles

{pi1}

The geometry over a Euclidean field (every positive number has a square root) may have no straight line segment of length π, since the model containing only the constructible real numbers does not contain π. We want to find a theory which proves the circumference and area formulas for circles. Our approach is to extend the theory EG so as to guarantee that there is a point in every model which behaves as π does. In this section we will show that in this extended theory there is a mapping assigning a straight line segment to the circumference of each circle. This goal definitely diverges from a

[61] Hartshorne notes that (page 210 of [41]) that he knows no 'purely geometric' (without segment arithmetic and similar triangles) proof for justifying the omission of De Zolt's axiom.

'Greek' data set and indeed is orthogonal to the axiomatization of Cartesian geometry in Section 6. Given that the entire project is modern, we give the arguments entirely in modern style.

Euclid's and the modern approach differ in that the sequences constructed in the study of magnitudes in the *Elements* are of geometric objects, not (even real) numbers. (Implicitly) using Archimedes axiom, Euclid proves (XII.2) that the area of a circle is proportional to the square of the diameter. In a modern account, as we saw already in Section 4.4, where we have interpreted segments Oa as representing the number a, we must identify the proportionality constant and verify that it represents a point in any model of the theory[62].

This shift in interpretation drives the rest of this section. We search first for the solution of a specific problem: is π in the underlying field?

Archimedes is a transitional figure here. By beginning the calculation of expansion of π, he is moving towards the treatment of it as a number. The validation in the theory $\mathcal{E}^2_\pi$ below of the formulas $A = \pi r^2$ and $C = \pi d$ are answering the questions of Hilbert and Dedekind not questions of Euclid or even Archimedes. But the theory EG_π is closer to the Greek origins.

Notation 5.0.1. *Recall that closing a plane under ruler and compass constructions corresponds to closing the coordinatizing ordered field under square roots of positive numbers. Write F_s (surd field) for the minimal field whose geometry is closed under ruler and compasss construction . Note that each member in F_s is definable over the emptyset*[63] *in EG*

Since each model of EG has the field of constructible number embeddable in the field definable any line of the model, we can interpret the Greek theory of proportionality in terms of cuts. Each pair of proportional pairs of magnitudes determines a cut (E.g. page 33-34 of [16].

We have labeled this section, Archimedes, for two reasons. He is the first (Measurement of a Circle in [2]) to prove the circumference of a circle is proportional to the diameter and begin the approximation of the proportionality constant (which wasn't named for another 2000 years). Secondly, his axiom is used not only in his work but implicitly by Euclid in proving the area of a circle is proportional to the square of the diameter.

Recall from Section 4.3, that in interpreting the field in the geometry, we named two arbitrary points as $0, 1$ and the field is definable in this expanded vocabulary. In that case, any pair of points will do. Now we are going to add another constant symbol π and add axioms about that constant to EG guaranteeing that the interpretation of π makes the formulas for area and circumference of a circle hold [64].

[62] For this reason, Archimedes needs only his postulate while Hilbert would also need Dedekind's postulate to prove the circumference formula.

[63] That is for each constructible number a there is a formula $\phi_a(x)$ such that $EG \vdash (\exists 1 x)\phi(x)$.

[64] I have presented parallel arguments for arc length and area. Deriving one from the other might be more

Euclid's 3rd postulate, "describe a circle with given center and radius", implies that a circle is uniquely determined by its radius and center. In contrast Hilbert simply defines the notion of circle and (see Lemma 11.1 of [41]) proves the uniqueness. In either case we have: two segments of a circle are congruent if they cut the same central angle. As the example of geometry over the real algebraic numbers shows, there is no guarantee that there is a straight line segment whose 'length' is π. We remedy this with the following extension of the system.

Dicta 5.0.2. *Constants 2* Note that having named $0, 1$, each element of the maximal real quadratic extension of the rational field [65], F_s, is denoted by a term $t(x, 0, 1)$ built from the field operations and $\sqrt{}$. There may be automorphisms of a model of EG_π, but they must fix F_s pointwise and just move π in its cut in F_s. The proof that EG_π is consistent relies on the compactness theorem. {constants2}

Here we named a further single constant π. But the effect is much different than naming 0 and 1 4.3.5 (Compare Dicta. The new axioms specify the place of π in the ordering of the definable points of the model. So the data set is seriously extended.

We formulate these axioms as properties on the point π rather than on the segment 0π because it is slightly more compact notation and more congenial to a modern reader. But then a few paragraphs later we describe the polygons approximating a circle in term of segments in the geometrical rather than the field language as it is more convenient. {defpiax}

Definition 5.0.3 (Axioms for π)**.** 1. *Add to the vocabulary a new constant symbol π. Let i_n (c_n) be the perimeter of a regular $3{*}2^n$-gon inscribed*[66] *(circumscribed) in a circle of radius* 1. *Let $\Sigma(\pi)$ be the collection of sentences (i.e. type*[67]*)*

$$i_n < 2\pi < c_n$$

for $n < \omega$.

2. *EG_π denotes deductive closure of the following set of axioms in the vocabulary τ in Notation 4.2.4 along with the constant symbols $0, 1, \pi$.*

   (a) *the axioms EG of a Euclidean plane.*

   (b) $\Sigma(\pi)$ {piax}

Theorem 5.0.4. *EG_π is a consistent but incomplete theory. It is not finitely axiomatizable*[68].

efficient but I want to stress the need for careful extension of the definitions of area and length to curves.

[65] See Footnote 55.

[66] I thank Craig Smorynski for pointing out that is not so obvious that that the perimeter of an inscribed n-gon is monotonic in n and reminding me that Archimedes started with a hexagon and doubled the number of sides at each step.

[67] Let $A \subset M \models T$. A type over A is a set of formulas $\phi(\mathbf{x}, \boldsymbol{a})$ where $\mathbf{x}$, $(\boldsymbol{a})$ is a finite sequence of variables (constants from A) that is consistent with T. It is over the empty set if the elements of A are definable without parameters in T (e.g. the constructible numbers). Here we take T as EG.

[68] Ziegler ([84], Remark 6.18) gives that EG is undecidable. It seems likely that his proof can be modified to show the undecidability of EG_π, but I haven't done so.

Proof. A model of EG_π is given by closing $F_s \cup \{\pi\} \subseteq \Re$ under constructibility. To see it is not finitely axiomatizable, for any finite subset Σ_0 of Σ choose a real algebraic number p satisfying Σ_0; close $F_s \cup \{p\} \subseteq \Re$ under constructibility to get a model of EG which is not a model of EG_π. $\square_{5.0.4}$

We have given conditions on the point π names in any model of EG_π. In an non-Archmidean model there will be other realizations of the type of π over the empty set; nevertheless π is a specific point. We now connect the length of the segment 0π with the circumference of a circle. To avoid complications, we restrict our discussion of 'arc length' to circles and straight lines with the following notation. Recall that Euclid uses the word line to refer to any curve and restrictively defines 'straight line'. We have taken straight line as the basic notion. We will use the capitalized word *Line segment* to mean either a straight line segment or an arc (segment of a circle).

Dicta 5.0.5 (Definitions or Postulates)**.** In Definition 5.0.7 we extend the ordering on segments by adding arcs of circles to the domain. Two approaches to this step are a) (as here) introduce an explicit but inductive definition or b) add a new predicate to the vocabular and new axioms specifying it behavior. This alternative reflects in a way the trope that Hilbert's axioms are *implicit definitions*. But this choice is not available for the initial axiomatization. It is only because we have already established a certain amount of geometric vocabulary that we can take choice a). Crucially the definition of bent lines (and thus the perimeter of certain polygons) is not a single definition but a schema of formulas ϕ_n defining the property for each n.

Definition 5.0.6. *By a* bent line[69] $b = X_1 \dots X_n$ *we mean a sequence of straight line segments* X_iX_{i+1} *such that each end point of one is the initial point of the next.*

1. *Note each bent line* $b = X_1 \dots X_n$ *has a length* $[b]$ *given by the straight line segment composed of the sum of the segments of* b*.*
2. *An* approximant *to the arc* $X_1 \dots X_n$ *of a circle with center* P*, is a bent line satisfying:*
   (a) $X_1, \dots X_n, Y_1, \dots Y_n$ *are points such that all* PX_i *are congruent and* Y_i *is in the exterior of the circle.*
   (b) *Each of* X_1Y_1*,* Y_iY_{i+1}*,* Y_nX_n *is a straight line segment.*
   (c) X_1Y_1 *is tangent to the circle at* X_1*;* $Y_{n-1}X_n$ *is tangent to the circle at* X_n*.*
   (d) *For* $1 \le i < n$*,* Y_iY_{i+1} *is tangent to the circle at* X_i*.* {extorder}

Definition 5.0.7. *Let* $\mathcal{S}$ *be the set (of equivalence classes of) straight line segments. Let* $\mathcal{C}_r$ *be the set (of equivalence classes under congruence) of arcs on circles of a given radius*[70] r*. Now we extend the linear order on* $\mathcal{S}$ *to a linear order* $<_r$ *on* $\mathcal{S} \cup \mathcal{C}_r$ *as follows. For* $s \in \mathcal{S}$ *and* $c \in \mathcal{C}_r$

[69]This is less general than Archimedes (page 2 of [2]) who allows segments of arbitrary curves 'that are concave in the same direction'.

[70]It at least requires some work to compare the length of arcs on circles of different radius and with chords of different lengths. We work around the issue now; our assignment of angle measure in Lemma 7.12 solves the problem in some models. Is there a more direct/more general solution?

1. *The segment $s <_r c$ if and only if there is a chord XY of a circular segment $AB \in c$ such that $XY \in s$.*
2. *The segment $s >_r c$ if and only if there is an approximant $b = X_1 \ldots X_n$ with length s [b] =to c with $[X_1 \ldots X_n] >_r c$.*

It is easy to see that this order is well-defined since each chord of an arc is shorter than any approximant to the arc and shorter than the arc.

Remark 5.0.8. *In a situation such as Definition 5.0.7 there are two ways to proceed. We can define $<$ on the extended domain or we can add an $<^*$ to the vocabulary and postulate that $< *$ extends $<$ and that (e.g. chords are less than arcs let than approximate. We prefer the definition route but both work.*

Now we want to argue that π, as implicitly defined by the theory EG_π, serves its geometric purpose. For this, we add a new unary function symbol C mapping our fixed line to itself and satisfying the following scheme asserting that for each n, $C(r)$ is between the perimeter of a regular inscribed n-gon and a regular circumscribed n-gon of a circle with radius r. For this we specialize our notion of approximant to the perimeter of the circumscribed (inscribed) polygons. {circfn}

Definition 5.0.9. *Consider the following properties of a unary function $C(r)$ mapping $\mathcal{S}$, the set of equivalence classes (under congruence) of straight line segments, into itself.*

ι_n *For any points $P, X_1, \ldots, X_n$ such that all the segments PX_i are congruent with length r, and all the segments X_iX_{i+1} (including X_nX_1) are congruent, the sum $i_n(r)$ of the lengths of the segments X_iX_{i+1} (including X_nX_1) is less than $C(r)$.*

γ_n *For any points $P, X_1, \ldots X_n, Y_1, \ldots Y_n$ such that all PX_i are congruent with length r, Y_i is in the exterior of the circle, X_i is the midpoint of Y_iY_{i+1}, and all Y_iY_{i+1} (including Y_nY_1) are congruent the sum $c_n(r)$ of the lengths of the segments Y_iY_{i+1} (including Y_nY_1) is greater than $C(r)$.*

Any function $C(r)$ satisfying these conditions is called a circumference function*, we call $C(r)$ the* circumference *of a circle with radius r.* {circumfn}

Definition 5.0.10. *The theory $EG_{\pi,C}$ is the extension of the $\tau \cup \{0, 1, \pi\}$-theory EG_π obtained by the explicit definition $C(r) = 2\pi r$.*

Since by similarity of the polygons $i_n(r) = ri_n$ and $c_n(r) = rc_n$, the ordering specified in Definition 5.0.9 will be satisfied if $C(r)$ is replaced by 'the circumference of a circle of radius r'. Note that while the approximations are given by standard 3×2^n-gons, defined by a schema, the translation to circles of different radius is done by multiplication within the geometry. So the approximations can be calculated for circles of any radius (including infinite or infinitessimal radius if the field is non-archimedean.)

Thus we have shown that for each r there is an $s \in \mathcal{S}$ whose length, $2\pi r$ is less than the perimeters of all inscribed polygons and greater that those of the inscribed polygons. We can verify that by choosing n large enough we can make i_n and c_n as close together as we like (more precisely, for given m differ by $< 1/m$). In phrasing this sentence I follow Heath's description[71] of Archimedes statements, "But he follows the cautious method to which the Greeks always adhered; he never says that given curve or surface is the *limiting form* of the inscribed or circumscribed figure; all that he asserts is that we can approach the curve or surface as nearly was we please".

Our definition of EG_π then makes the following metatheorem immediate. {circform}

Theorem 5.0.11. *In $EG^2_{\pi,C}$, $C(r) = 2\pi r$ is a circumference function (i.e. satisfies all the conditions ι_n and γ_n).*

We have *not* established this claim for each arc in $\mathcal{C}_r$ for even one r. We will accomplish that task in Lemma 7.12.

In an Archimedean field there is a unique interpretation of π and thus a unique choice for a circumference function with respect to the vocabulary without the constant π. By adding the constant π to the vocabulary we get a formula which satisfies the conditions in every model. But in a non-Archimedean model, any point in the monad of $2\pi r$ would equally well fit our condition for being the circumference.

We now sketch the extension of EG_π , analogous to that for circumference, to treat the area of a circle. We extend the impact of the properties of equal content in Theorem 4.6.2 by expanding the notion of *figure*. We denote the expanded class by a capital F.

Definition 5.0.12. *A* Figure *is a figure or a sector of a circle. That is, either a sector of circle or a subset of the plane that can be represented as a finite union of disjoint triangles.* {expproppi}

Lemma 5.0.13 (another approximation of π)**.** *Let I_n and C_n denote the area of the regular 3×2^n-gon inscribed or circumscribing the unit circle.*

$$I_n < \pi < C_n$$

for $n < \omega$

Then EG_π proves each of these sentences is satisfied by π.

Proof. The (I_n, C_n) define the cut for π in the constructible reals and the (i_n, c_n) define the cut for 2π.

[71] [43], introduction Kindle location 393.

Now, as in the circumference case, by formalizing a notion of equal area, including a schema for approximation by finite polygons (which for conciseness we omit), we can show define a formal area function $A(r)$ now defined does indeed compute the area. {careafn}

Definition 5.0.14. *The theory $EG_{\pi,C,A}$ is the extension of the $\tau \cup \{0, 1, \pi\}$-theory EG_π obtained by the explicit definition $A(r) = \pi r^2$.*

In the vocabulary with this function named we have, since the $I_n(C_n)$ converge to one half of the limit of the $i_n(C_n)$ and we describe the same cut: {circarea}

Theorem 5.0.15. *In $EG_{\pi,C,A}$, the area of a circle is $A(r) = \pi r^2$.*

We have extended our descriptively complete axiomatization from the polygonal geometry of Hilbert's first order axioms (HP5) to Euclid's results on circles and beyond. Euclid doesn't deal with arc length at all and we have assigned straight line segments to both the circumference and area of a circle. So this would not qualify as a modest axiomatization of Greek geometry but only of the modern understanding of these formulas. This distinction is not a problem for the notion of descriptive axiomatization. The facts are sentences. The formulas for circumference and are not the same sentences as the Euclid/Archimedes statement in terms of proportions.

We could however better argue that $EG_{\pi,A}$ is modest axiomatization of Euclid, than $EG_{\pi,C,A}$ given the absence of arc length from Euclid (and the presence of VI.1).

6 From Descartes to Tarski

{DT}

It is not our intent to give a detailed account of Descartes' impact on geometry. We want to bring out the changes from the Euclidean to the Cartesian data set. For our purposes, the most important is to explicitly (on page 1 of [20]) define the multiplication of line segments to give a line segment which breaks with Greek tradition[72]. And later on the same page to announce constructions for the extraction of nth roots for all n. The second of these cannot be done in EG, since it is satisfied in the field which has solutions for all quadratic equations but not those of odd degree[73].

Marco Panza [63] formulates in terms of ontology a key observation,

> The first point concerns what I mean by 'Euclid's geometry'. This is the theory expounded in the first six books of the Elements and in the Data. To be more precise, I call it 'Euclids plane geometry', or EPG, for short. It is

[72]His proof is still based on Eudoxus.

[73]See section 12 of [41].

not a formal theory in the modern sense, and, a fortiori, it is not, then, a deductive closure of a set of axioms. Hence, it is not a closed system, in the modern logical sense of this term. Still, it is no[74] more a simple collection of results, nor a mere general insight. It is rather a well-framed system, endowed with a codified language, some basic assumptions, and relatively precise deductive rules. And this system is also closed, in another sense ([51] 311-312), since it has sharp-cut limits fixed by its language, its basic assumptions, and its deductive rules. In what follows, especially in section 1, I shall better account for some of these limits, namely for those relative to its ontology. More specifically, I shall describe this ontology as being composed of objects available within this system, rather than objects which are required or purported to exist by force of the assumptions that this system is based on and of the results proved within it. This makes EPG radically different from modern mathematical theories (both formal and informal). One of my claims is that Descartes geometry partially reflects this feature of EPG.

In our context we interpret 'composed of objects available within this system' model theoretically as the existence of certain starting points and the closure of each model of the system under admitted constructions. We take Panza's 'open' system to refer the diversity of constructions[75], such as ruler and compass, conic, 'mechanical' available in Greek geometry and (at least partially) systematized by Descartes. It is exactly in this way that we have argued that the first order axioms solve Problem 3.3.1 1a.-d. But Descartes allows many more constructions (again cf. [63]) than the ruler and compass licensed in EG. Descartes endorses such 'mechanical' constructions as the duplication of the cubic as geometric. The construction he uses are what we would now call algebraic. He rejects as non-geometric any method for quadrature of the circle. In fact, Descartes believed that his construction encompassed all algebraic curves[76].

In particular Descartes constructions license the extraction of nth roots for all n and the solution of many (all) higher degree polynomials. For our purpose we take the common identification of Cartesian geometry with real algebraic geometry: the study of polynomial equalities and inequalities in the theory of real closed fields. To justify this geometry we adapt Tarski's 'elementary geometry. This move makes a significant conceptual step away from Descartes whose constructions were on segments and who did not regard a line as a set of points while Tarski's axiom are given entirely formally in a one sorted language of relations on points. But in our modern construction of an axiom set the translation is routine (Remark 3.2.4), but anachronistic.

From Tarski [73] we get {Tarskiax}

Theorem 6.16. *Tarski [73] gives a theory equivalent to the following system of axioms* $\mathcal{E}^2$. *It is first order complete for the vocabulary in Notation 4.2.4.*

[74] There appears to be a typo. Probably 'more a" should be deleted. jb

[75] The types of constructions allowed are analyzed in detail in Section 1.2 of [63] and the distinctions with the Cartesian view in Section 3.

[76] The exact verification of this assertion is an historical issue.

1. *Euclidean geometry*
2. *Either of the following two sets of axioms which are equivalent over i)*
    (a) *An infinite set of axioms declaring that every polynomial of odd-degree has a root.*
    (b) *The axiom schema of continuity described just below.*

Admittedly, this step may be criticized for being too strong, as we have criticized the Dedekind postulate. There are several responses to this claim.

Our axioms a) are in the spirit of Descartes – asserting the solutions of certain equations. They provide a complete descriptive axiomatization of the Cartesian data set. Of course this makes sense only if we allow the translation from segments to points. b) If the real algebraic numbers are the natural model for Cartesian geometry then we have exactly the first order sentences in the vocabulary of he geometry.

The connection with Dedekind's approach is seen by Tarski's actual formulation as in [37]; the first order completeness of the theory is imposed by an **Axiom Schema of Continuity** - a definable version of Dedekind cuts:

$$(\exists a)(\forall x)(\forall y)[\alpha(x) \wedge \beta(y) \rightarrow B(axy)] \rightarrow (\exists b)(\forall x)(\forall y)[\alpha(x) \wedge \beta(y) \rightarrow B(xby)],$$

where α, β are first-order formulas, the first of which does not contain any free occurrences of a, b, y and the second any free occurrences of a, b, x. This schema allow the solution of odd degree polynomials. By the completeness of real closed fields, this theory is also complete[77]. {gc}

Remark 6.17 (Gödel completeness)**.** In Detlefsen's terminology we have found a Gödel complete axiomatization of (in our terminology Cartesian) plane geometry. This guarantees that if we keep the vocabulary and continue to accept the same data set no axiomatization can account for more of the data. There are certainly open problems in plane geometry [53]. But however they are solved the proof will be formalizable in $\mathcal{E}^2$. Of course, more perspicuous axiomatizations may be found. Or one may discover the entire subject is better viewed as an example in a more general context.

In the case at hand, however, there are more specific reasons for accepting the geometry over real closed fields as 'the best' descriptive axiomatization. It is the only one which is decidable and 'constructively justifiable'. {Zieg}

Remark 6.18 (Undecidability and Consistency)**.** Ziegler [84] has shown that every nontrivial finitely axiomatized subtheory of RCF[78] is not decidable.

[77] Tarski proves the equivalence of geometries over real closed fields with his axiom set in [73]. He calls the theory elementary geometry, $\mathcal{E}^2$.

[78] RCF abbreviates 'real closed field'; these are the ordered fields such that every positive element has a square root and every odd degree polynomial has at least one root. The theory is complete and recursively axiomatized so decidable. By nontrivial subtheory, I mean one satisfied by one of $\mathbb{C}, \Re$, or a p-adic field Q_p. For the context of Ziegler result and Tarski's quantifier elimination in computer science see [55].

Thus both to more closely approximate the Dedekind continuum and to obtain decidability we restrict to planes over RCF; Tarski [37] gave a geometric axiomatization (and proved biinterpretability) between RCF and the theory of all planes over real closed fields. The crucial fact that makes decidablity possible is that the natural numbers are *not first order definable* in the real field. The geometry can represent multiplication as repeated addition in the sense of a module over a $\mathbb{Z}$ but not with the full ring structure. As Tarski noticed and Friedman [34] proved, RCF is provably consistent in exponential function arithmetic (EFA).

Of course, another crucial contribution of Descartes is coordinate geometry. Tarski provides a converse; his interpretation of the plane into the coordinatizing line [75] underlies our smudging of the study of the 'geometry continuum' with axiomatizations of 'geometry'.

7 Archimedes II: π, arc length and area of circles

{pi}

We now combine the theories from Section 5 and Section 6, to find a modest descriptive axiomatization of the modern conception of Tarski's elementary geometry plus the elements of angle measure. As in Section 6, we will obtain a complete first order theory. This will illuminate the delicate dependence of descriptive axiomatization on the vocabulary in which the facts are expressed.

Dedekind (page 37-38) observes that what we would now call the real closed field with domain the field of real algebraic numbers is 'discontinuous everywhere' but 'all constructions that occur in Euclid's elements can . . . be just as accurately effected as in a perfectly continuous space'. Strictly speaking, for *constructions* this is correct. But the proportionality constant between a circle and its circumference π is absent, so, even more, not both a straight line segment of the same length as the circumference and the diameter are in the model[79]. We want to find a theory which proves the circumference and area formulas for circles and countable models of the geometry over RCF, where 'arc length behaves properly'.

Mueller (page 236 of [59]) makes a important point distinguishing Euclid/Eudoxus from Dedekind's use of cuts.

> One might say that in applications of the method of exhaustion the limit is given and the problem is to determine a certain kind of sequence converging to it, . . . Since, in the *Elements* the limit always has a simple description, the construction of the sequence can be done within the bounds of elementary geometry; and the question of constructing a sequence for any given arbitrary limit never arises.

[79]Thus, the compass postulate derived from [11] is violated. (See Remark 7.9.)

This distinction can be expressed in another way. We speak of the method of Eudoxus: a technique to solve certain problems, which are specified in each application. In contrast, Dedekind's postulate provides a solution for $2^{\aleph_0}$ problems.

Now that we have established that each model of EG has the field of constructible number embeddable in the field definable any line of the model, we can interpret the Greek theory of proportionality in terms of cuts. Each pair of proportional pairs of magnitudes determines a cut (E.g. page 33-34 of [16]. The non-first order postulates of Hilbert play complementary roles. The Archimedean axiom is minimizing; each cut is realized by at most one point so each model has cardinality at most $2^{\aleph_0}$. The Veronese postulate (See Footnote 6.) or Dedekind's postulate is maximizing; each cut is realized, the set of realizations could have arbitrary cardinality.

Descartes eschews the idea that there can be a ratio between a straight line segment and a curve. As [17] writes, "Descartes[80] excludes the exact knowability of the ratio between straight and curvilinear segments":

> ... la proportion, qui est entre les droites et les courbes, nest pas connue, et mesme ie croy ne le pouvant pas estre par les hommes, on ne pourroit rien conclure de l qui fust exact et assur.

Nevertheless we will define in Section 7, a theory $\mathcal{E}^2_\pi$ analogous to EG_π which does not on the Dedekind axiom but can be obtained in a first order way. Using the completeness of $\mathcal{E}^2$, we can continue the process to find a model where every circular arc has a length.

At this point we need some modern model theory to guarantee the completeness of the theory we are defining. A first order theory T for a vocabulary including a binary relation $<$ is *o-minimal* if every 1-ary formula is equivalent in T to a Boolean combination of equalities and inequalities [19]. Anachronistically, the o-minimality of the reals is a main conclusion of Tarski in [74]. {2piax}

Theorem 7.1. *The following set $\mathcal{E}^2_\pi$ of axioms is first order complete for the vocabulary τ in Notation 4.2.4 along with the constant symbols $0, 1, \pi$.*

1. *$\mathcal{E}^2$, that is:*
   (a) *the axioms EG of a Euclidean plane.*
   (b) *A family of sentences declaring every odd-degree polynomial has a root.*
2. $\Sigma(\pi)$

Proof. We have established that there are well-defined field operations on the line through 01. By Tarski, the theory of this real closed field is complete. The field

[80] Descartes, Oeuvres, vol. 6, p. 412. Crippa also quotes Averros as emphatically denying the possibility of such a ratio and notes Vieta held similar views.

is bi-interpretable with the plane [75] so the theory of the geometry T is complete as well. Further by Tarski, the field is o-minimal. The type over the empty set of any point on the line is determined by its position in the linear ordering of the subfield of constructible numbers F_0; each constructible number is definable over the empty set. Each i_n, c_n is an element of the field F_0. This position in the linear order of 2π in the linear order on the line through 01 is given by Σ. Thus $T \cup \Sigma$ is complete. $\square_{5.0.4}$

We now rely on the definitions of bent line, cirumference function etc. from Section 5. Using them we extend the theory $\mathcal{E}^2_\pi$. {2circumfn}

Definition 7.2. *The theory $\mathcal{E}^2_{\pi,C}$ is the extension of the $\tau \cup \{0, 1, \pi\}$-theory $\mathcal{E}^2_\pi$ obtained by the explicit definition $C(r) = 2\pi r$.*

As an extension by explicit definition, $\mathcal{E}^2_{\pi,C}$ is complete and o-minimal. As before, by similarity $i_n(r) = ri_n$ and $c_n(r) = rc_n$, the approximations of π by inscribed and circumscribed polygons the conditions of Notation 5.0.9 will be satisfied if $C(r)$ is replaced by 'the circumference of a circle of radius r. As in Theorem 5.0.11, our definition of $\mathcal{E}^2_\pi$ then gives. {2circform}

Theorem 7.3. *In $\mathcal{E}^2_{\pi,C}$, $C(r) = 2\pi r$ is a circumference function (i.e. satisfies all the ι_n and γ_n). Moreover, $\mathcal{E}^2_{\pi,C}$ is a complete decidable o-minimal theory.*

Proof. Each finite approximation is a theorem as before. Adding a definable function preserves o-minimality and completeness. The theory is recursively axiomatized and complete so decidable. $\square_{7.3}$

As before in an Archimedean field, such as the reals, we have the usual interpretation of the circumference function. In a non-Archimedean field we have chosen one circumference from the monad in which it lies. The extension to area is as in Section 5. In particular as in Lemma 5.0.13, we have: {2exproppi}

Lemma 7.4. *Let, for $n < \omega$, I_n and C_n denote the area of the regular 3×2^n-gon inscribed or circumscribing the unit circle:*

$$I_n < \pi < C_n.$$

Then $\mathcal{E}^2\pi$ proves each of these sentences is satisfied by π.

Now, as in the EG_π case, by formalizing a notion of equal area, we can define a formal area function $A(r)$. {2areafn}

Definition 7.5. *The theory $\mathcal{E}^2_{\pi,C,A}$ is the extension of the $\tau \cup \{0, 1, \pi\}$-theory T_π obtained by the explicit definition $A(r) = \pi r^2$.*

In the vocabulary with this function named, just as before, we have:

{2circarea}

Theorem 7.6. *In $\mathcal{E}^2_{\pi,C,A}$, the area of a circle is $A(r) = \pi r^2$.*

Remark 7.7. Since o-minimality is preserved by naming constants and by explicit definition, the theory of the ordered field defined the geometry $T_{0,1,\pi,A,C}$ is o-minimal. The theory $T_{0,1,\pi,A,C}$ verifies both the area and circumference functions. It is biinterpretable with the theory of the field. And thus it is constructive consistent (i.e provably consistent in EFA and therefore in primitive recursive arithmetic ([34])).

Remark 7.8. We have so far, in the spirit of the quote from Mueller at the beginning of this section, tried to find the proportionality constant only for a specific proportion. In the remainder of the section, we consider several ways of systematizing the solution of families of such problems. First, still in a specific case we look for models where every angle determines and arc that corresponds to the length of a straight line segment. Then we consider several model theoretic schemes to organize such problems. {Birk}

Remark 7.9. Birkhoff [11] introduced the following axiom in his system[81].

> POSTULATE III. The half-lines ℓ, m, through any point O can be put into $(1, 1)$ correspondence with the real numbers $a(\text{mod} 2\pi)$, so that, if $A \neq O$ and $B \neq O$ are points of ℓ and m respectively, the difference $a_m - a_\ell(\text{mod} 2\pi)$ is $\angle AOB$.

Birkhoff takes the real numbers as an unexamined background object. He argues that his axioms define a categorial system isomorphic to $\Re^2$. So it is equivalent to Hilbert's.

The next task is to find a more modest version of Birkhoff's postulate: a first order theory with countable models which assign a measure to each angle between 0 and 2π. Recall that we have a field structure on the line through 01 and the number π on that line. We will make one further explicit definition.

Definition 7.10. *A* measurement of angles *function is a map μ from congruence classes of angles into $[0, 2\pi)$ such that if $\angle ABC$ and $\angle CBD$ are disjoint angles sharing the side BC, $\mu(\angle ABD) = \angle ABC + \angle CBD$*

If we omitted the additivity property this would be trivial: Given an angle $\angle ABC$ less than a straight angle, let C' be the intersection of a perpendicular to AC through B with AC and let $\mu(\angle ABC) = \frac{BC'}{AB}$. (It is easy to extend to the rest of the angles.) To obtain the additivity, we proceed as follows. {arclength}

Theorem 7.11. *For every countable model M of $\mathcal{E}^2_\pi$, there is a countable model M' containing M such that a measure of angles function μ is defined on the (congruence class of) each angle determined by points $P, X, Y \in M'$.*

[81] This is the axiom system used in virtually all U.S. high schools since the 1960's.

Proof. We adapt the proof of Theorem 5.0.11. Fix an angle XPY where X, Y are on the circumference of a unit circle with center P. Replace the inscribed and circumscribed polygons of Definition 5.0.9 by building polygons inscribed and circumscribing the sector (also using the two radii as two sides, but choosing new points to refine the approximation by bisecting each central angle at each stage). As in the proof of Theorem 5.0.4, we obtain the arc length as a type over PXY in $\mathcal{E}^2_\pi$.

Given a model N, let N' be a countable elementary extension of N realizing all the, countably many, angle measure cuts in N. Now proceed inductively, let $M_0 = M$ and $M_{n+1} = M'_n$. Then M_ω is required model where μ is defined on all angles. $\square_{7.11}$

Since each of the cuts we realized in the previous construction was given by a recursive type over a finite set, a recursively saturated model[82] will realize the relevant type to verify the following theorem. {getarcl}

Corollary 7.12. *If M is a countable recursively saturated model of $\mathcal{E}^2_\pi$ a measure of angles function μ is defined on the (congruence class of) each angle determined by points $P, X, Y \in M$.*

Remark 7.13. We have constructed a countable model M such that each arc of a circle in M has length measured by a straight line segment in M. There is no Archimedean requirement; adding the Archimdean axiom there would determine a unique number rather than a monad.

Remark 7.14. Note that in any model satisfying the hypotheses of Corollary 7.12, we can carry out elementary right angle trigonometry (angles less than 180^o). Unit circle trigonometry, where periodicity extends the sin function to all of the line violates o-minimality. (The zeros of the $\sin$ function are an infinite discrete set.)

Remark 7.15. As suggested by the quote from Mueller [59] opening this section, the requirement of recursive saturation provides one principle for distinguishing the application of the method of Eudoxus by the Greeks from the *Deus ex machina* of Dedekind. Putatively, to state a problem is to state it recursively, so the proportions for which we seek a proportionality constant will be satisfied in any model where each recursive type over a finite set is realized. {gc1}

Remark 7.16 (Gödel completeness again)**.** It might be objected that such minor changes as adding to $\mathcal{E}$ the name of the constant π, or adding the definable functions C and A undermines the claim in Remark 6.17 that $\mathcal{E}^2$ was descriptively complete for Cartesian geometry. But the data set has changed. We add these new constants and functions because the modern view of 'number' requires them.

Descriptive completeness is *not* always preserved by naming constants even to a Gödel -complete theory. Compare Dicta 5.0.2.

[82] See for example [10].

{inflog}

Remark 7.17 ($L_{\omega_1,\omega}$)**.** Another tool for formalizing the axiomatization is to work in $L_{\omega_1,\omega}$. This allows the statement of the axiom of Archimedes. But it can give finer information; we can stipulate the (unique!) realization of certain cuts. By passing to $L_{\omega_1,\omega}$, we lose the full use of the compactness theorem. But there is still a completeness theorem and 'Barwise compactness' (chapter 4 and 9 of [52]).

We could even take the Scott sentence[83] of some favored (perhaps, the unique countable recursively saturated) model and obtain $\aleph_0$ categoricity. Unfortunately, we cannot do this with the Hilbert model; it has no countable $L_{\omega_1,\omega}$-elementary submodel.

Tarski ends [73] by comparing the properties of three first order theories of geometry $\mathcal{E}^2$, EG, and the weak second order theory[84] of $\Re^2$. Tarski concludes:

> The author feels that, among these various conceptions, the one embodied in $\mathcal{E}^2$ distinguishes itself by the simplicity and clarity of underlying intuitions and by the harmony and power of its metamathematical implications.

We hope the ability to develop the formulas for the area and circumference of circles in a very mild extension of $\mathcal{E}^2$ bears witness to his judgement.

8 Conclusion

{concl}

We discuss first the role of the Archimedean postulate in the Grundlagen. Then we list objections to second order axioms for 'geometry' and then suggest in Section 8.3 that Hilbert is conceiving of geometry in a somewhat broader sense. In a final section, we speculate on broader uses of 'definable mathematics'.

8.1 The role of the Axiom of Archimedes in the Grundlagen

{Archrole}

The discussions of the Axiom of Archimedes in the Grundlagen fall into several categories. i) Those, in Sections 9 -12 (from [46]) are metamathematical - concerning the consistency and independence of the axioms. ii) In Section 17, the axiom of Archimedes is used justify the coordinatization of n-space by n-tuples of real numbers. iii) In Sections 19 and 21, it is shown that the Archimedean axiom is necessary to show equicomplementable (equal content) is the same as equidecomposable (in 2 or more dimensions). These are all metatheoretical results.

[83] For any countable structure M there is a 'Scott' sentence ϕ_M such that all countable models of ϕ_M are isomorphic to M; see chapter 1 of [52].

[84] Weak second order logic allows quantification over finite sequences of elements.

Coordinitization is certainly a central geometrical notion. But it does not require the axiom of Archimedes to coordinatize n-space by the line in the plane. The axiom is used to assign (a binary representation of a real) to each point on the line. That is, to establish a correspondence between object defined in the geometry and an extrinsic notion of real number. Thus, it is not a proof in the system advanced by Hilbert.

The use of the Archimedean axiom to prove equidecomposable is the same as equicomplementable is certainly a proof in the system. But an unnecessary one. As we argued in Section 4.6, Hilbert could just have easily defined 'same area' as 'equicomplementable' (as is in a natural reading of Euclid).

Thus, we find no theorems in the Grundlagen proved from its axiom system that essentially depend on the Axiom of Archimedes.

8.2 Against the Dedekind Posulate for Geometry

{againstded}

Our motto is: 'Make no unnecessary hypothesis.' Our fundamental claim is that (slight variants on) Hilbert's first order axiom provide a modest descriptively complete axiomatization of most of Greek geometry. We spelt out in Section 3.3 a careful collection of different data sets and showed in sections 4, 5, 6 that appropriate sets of first order axioms are modest descriptively complete axioms for each them. We then showed a slight extension of Tarski's first order axiomatization accounts not only for the Cartesian data set but the basic properties of π. In the Appendix 9, we lay out the reasons that the discussion in this paper concerns Dedekind's rather than Hilbert's formulation of the continuity postulate.

As we pointed out in Section 3 of [9] various authors have proved under $V = L$, any countable or Borel structure can be given a categorical axiomatization. We argued there that this fact undermines the notion of categoricty as an independent desiderata for an axiom system. There, we gave a special role to attempting to axiomatize canonical systems. Here we go further, and suggest that even for a canonical structure there are advantages to a first order axiomatization that trump the loss of categoricity.

We argue then that the Dedekind postulate is inappropriate (in particular immodest) as an attempt to axiomatize the Euclidean or Cartesian or Archimedean data sets for several reasons:

1. The requirement that there be a straight-line segment measuring any circular arc is clearly contrary to the intent of Descartes.
2. It is not part of the data set but rather an external limitative principle.

   The notion that there was 'one' geometry (i.e. categoricity) was implicit in Euclid. But it is not a geometrical statement. Indeed, Hilbert described his completeness axiom (page 23 of [45]), 'not of a purely geometrical nature'. This is most clearly seen in Hilbert's initial metamathematical formulation: that the model of Axiom groups I-IV and Archimedes axiom must be maximal. {overkill}

3. It is not needed to establish the properly geometrical propositions in the data set. We first reviewed in Theorem 4.2.3 the literature showing the first two data sets in Section 3.3 are provable from the axioms we labeled in Notation 4.2.2 as EG (HP5 + CCI) (Notation 4.2.2). Then we extended to Tarski's $\mathcal{E}^2$ and our T_π to give first order axioms accounting for data sets 3 and 4.

4. Proofs from Dedekind's postulate obscure the true geometric reason for certain theorems. Hartshorne writes[85]:

   > '... there are two reasons to avoid using Dedekind's axiom. First, it belongs to the modern development of the real number systems and notions of continuity, which is not in the spirit of Euclid's geometry. Second, it is too strong. By essentially introducing the real numbers into our geometry, it masks many of the more subtle distinctions and obscures questions such as constructibility that we will discuss in Chapter 6. So we include the axiom only to acknowledge that it is there, but with no intention of using it.

5. The use of second order logic undermines a key proof method – informal proof. A crucial advantage of a first order axiomatization[86] is that it licenses the kind of argument[87] described in Hilbert and Ackerman[88]:

   > Derivation of Consequences from Given Premises; Relation to Universally Valid Formulas
   >
   > So far we have used the predicate calculus only for deducing valid formulas. The premises of our deductions, viz Axioms a) through f), were themselves of a purely logical nature. Now we shall illustrate by a few examples the general methods of formal derivation in the predicate calculus .... It is now a question of deriving the consequences from any premises whatsoever, no longer of a purely logical nature.
   >
   > The method explained in this section of formal derivation from premises which are not universally valid logical formulas has its main application in the setting up of the primitive sentences or axioms for any particular field of knowledge and the derivation of the remaining theorems from them as consequences . We will examine, at the end of this section, the question of whether every statement which would intuitively be regarded as a consequence of the axioms can be obtained from them by means of the formal method of derivation.

[85] page 177 of [41]

[86] See Section 9 for more detail on this argument.

[87] We noted that Hilbert proved that a Desarguesian plane in 3 space by this sort of argument in Section 2.4 of [7].

[88] Chapter 3, §11 Translation taken from [12].

We exploited this technique in Section 6 to provide axioms for the calculation of the circumference and area of a circle.

Venturi[89] formulates a distinction, which nicely summarises our argument: 'So we can distinguish two different kinds of axioms: the ones that are *necessary* for the development of a theory and the *sufficient* one used to match intuition and formalization.' In our terminology only the necessary axioms make up a '*modest* descriptive axiomatization'. For the geometry Euclid I (basic polygonal geometry), Hilbert's first order axioms meet this goal. With T_π, a modest complete descriptive axiomatization is provided even including the basic properties of π. The Archimedes and Dedekind postulates have a different goal; they secure the 19th century conception of $\Re^2$ to be the unique model and thus ground elementary analysis.

8.3 Why does axiom group V exist?

{whyV}

We return to the second paragraph of the introduction. Hilbert wrote that V.1 and V.2 allow one 'to establish a one-one correspondence between the points of a segment and the system of real numbers'. Here he is exhibiting a model centric rather than a descriptive (in Detlefsen's syntactic sense) approach to axiomatization. I think these should be recognized as two distinct motivations: to make Euclid rigorous and to ground analytic geometry and calculus. These are complementary but distinct projects. We have noted here that the grounding of real algebraic geometry is fully accomplished by Tarski's axiomatization. And we have provided a first order extension to deal with the basic properties of the circle.

Hilbert has in fact a more modern notion of geometry which includes both 'metric' and 'algebraic' geometry. Modern topologists usually regard themselves as geometers as do algebraic geometers while recognizing the differences in goals and methods among the subjects. Hilbert seems really to be aiming at the foundations of both. But his devotion of the text to axiomatic treatment of Euclidean geometry suggests a more narrow reading of his goals.

8.4 But what about analysis?

{extensions}

We have expounded a procedure [41] to define the field operations in an arbitrary Euclidean plane. We argued that the first order axioms of EG suffice for the geometrical data sets Euclid I and II, not only in their original formulation but by finding proportionality constants for the area formulas of polygon geometry. By adding axioms to require the field is real closed we obtain a complete first order theory that encompasses many of Descartes innovations. The plane over the real algebraic numbers satisfies this theory; thus, there is no guarantee that there is a line segment of length π. Using the

[89] page 96 of [80]

o-minimality of real closed fields, we can guarantee there is such a segment by adding a constant for π and requiring it to realize the proper cut in the rationals. However, guaranteeing the uniqueness of such a realization requires the $L_{\omega_1,\omega}$ Archimedean axiom.

Hilbert and the other axiomatizers of 100 years ago wanted more; they wanted to secure the foundations of calculus. In full generality, this surely depends on second order properties. But there are a number of directions of work on 'definable analysis'.

One of the directions of research in o-minimality has been to prove the expansion of the real numbers by a particular functions (e.g. the Γ-function on the positive reals [69]); our example with arc length might be extended to consider other more interesting transcendental curves.

Peterzil and Starchenko study the foundations of calculus in [64]. They approach the complex analysis through o-minimality of the real part in [65]. The impact of o-minimality on number theory was recognized by the Karp prize of 2014. And a non-logician, suggests using methods of Descartes to teach Calculus [66].

In a sense, our development is the opposite of Ehrlich's in [28]. Rather than trying to unify all numbers great and small, we are interested in the minimal collection of numbers that allow the development of a geometry according with our fundamental intuitions.

9 Appendix: Hilbert and Dedekind on Continuity

{Hilbdeb}

Hilbert's formulation of the completeness axiom reads [46]:

> Axiom of Completeness (Vollständigkeit): To a system of points, straight lines, and planes, it is impossible to add other elements in such a manner that the system thus generalized shall form a new geometry obeying all of the five groups of axioms. In other words, the elements of geometry form a system which is not susceptible of extension, if we regard the five groups of axioms as valid.

We have used in this article the following adaptation of Dedekind's postulate for geometry (DG):

> DG: The linear ordering imposed on any line by the betweenness relation is Dedekind complete[90].

[90] See footnote 6.

While this formulation is convenient for our purposes, it misses an essential aspect of Hilbert's version. DG implies the Archimedean axiom and Hilbert was aiming for an independent set of axioms. Hilbert's axiom does not imply Archimedes. A variant VER (see [15]) on Dedekind's postulate that does not imply the Archimedean axiom was proposed by Veronese in [81][91]. If we substituted VER for DG, our axioms would also satisfy the independence criterion.

Hilbert's completeness axiom in [46] asserting any model of the rest of the theory is maximal, is inherently model-theoretic. The later line-completeness [45] is a technical variant[92]. Giovannini's account [36] includes a number of points already made here; but I note three further ones. First, Hilbert's completeness axiom is not about deductive completeness (despite having such consequences), but about maximality of every *model* (page 145). Secondly (last line of 153) Hilbert expressly rejects Cantor's intersection of closed intervals axiom because in relies on a sequence of intervals and 'sequence is not a geometrical notion'. A third intriguing note is an argument due to Baldus in 1928 that the parallel axiom is an essential ingredient in the categoricity of Hilbert's axioms [93].

Here are two reasons for choosing Dedekind's (or Veronese's) version. The most basic is that one cannot formulate Hilbert's version as sentence Φ in second order logic[94] with the intended interpretation

$$(\Re^2, \mathbf{G}) \models \Phi.$$

The axiom requires quantification over subsets of an extension of the model which putatively satisfies it. Here is a second order statement[95] of the axiom, where ψ denotes the conjunction of Hilbert's first four axiom groups and the axiom of Archimedes.

$$(\forall X)(\forall Y)\forall \mathbf{R})[[X \subseteq Y \wedge (X, \mathbf{R}\restriction X) \models \psi \wedge (Y, \mathbf{R}) \models \psi] \rightarrow X = Y]$$

[91]The axiom VER asserts that for a partition of a linearly ordered field into two intervals L, U (with no maximum in the lower L or minimum in the upper U) and third set in between at most one point, there is a point between L and U just if for every $e > 0$, there are $a \in A, b \in B$ such that $b - a < e$. Veronese derives Dedekinds postulate from his plus Archimedes in [81] and the independence in [82]. In [54] Levi-Civita shows there is a non-Archimean ordered field that is Cauchy complete. I thank Philip Ehrlich for the references and recommend section 12 of the comprehensive [27]. See also the insightful reviews [62] and [61] where it is observed that Vahlen [78] also proved this axiom does not imply Archimedes.

[92]Since any point in the definable closure of any line and any one point not one the line, one can't extend any line without extending the model. Since adding either the Dedekind postulate and or Hilbert completeness gives a categorical theory satisfied by a geometry whose line is order isomorphic to $\Re$ the two axioms are equivalent (over HP5 + Arch).

[93]Hartshorne (sections 40-43 of [41] gives a modern account of Hilbert's argument that replacing the parallel postulate by the axiom of limiting parallels gives a geometry that is determined by the underlying (definable) field. With V.2 this gives categoricity.

[94]Of course, this analysis is anachronistic; the clear distinction between first and second order logic did not exist in 1900. By $\mathbf{G}$, we mean the natural interpretation in $\Re^2$ of the predicates of geometry introduced in Section 4.2.

[95]I am leaving out many details, $\mathbf{R}$ is a sequence of relations giving the vocabulary of geometry and the sentence 'says' they are relations on Y; the coding of the satisfication predicate is suppressed.

This anomaly has been investigated by Väänänen who makes the distinction between the last two displayed formulas (on page 94 of [76]) and expounds in [77] a new notion 'Sort Logic' which provides a logic with a sentence Φ which by allowing a sort for an extension axiomatizes geometry in the sense of 9. The second reason is that Dedekind's formulation, since it is about the geometry, not about its axiomatization, directly gives the kind of information about the existence of transcendental numbers that we discuss in the paper.

In [76], Väänänen discusses the categoricity of natural structures such as real geometry when axiomatized in second order logic (e.g. DG). He has discovered the striking phenomena of 'internal categoricity'. Suppose the second order categoricity of a structure A is formalized by the existence of sentence Ψ_A such that $A \models \Psi_A$ and any two models of Ψ are isomorphic. If this second clause in provable in a standard deductive system for second order logic, then it is valid in the Henkin semantics, not just the full semantics.

Philip Ehrlich has made several important discoveries concerning the connections between the two 'continuity axioms' in Hilbert and develops the role of maximality. First, he observes (page 172) of [25] that Hilbert had already pointed out that his completeness axiom would be inconsistent if the maximality were only with respect to the first order axioms. Secondly, he [25, 26] systematizes and investigates the philosophical significance of Hahn's notion of Archimedean completeness. Here the structure (ordered group or field) is not required to be Archimedean; the maximality condition requires that there is extension which fails to extend an Archimedean equivalence class[96]. This notion provides a tool (not yet explored) for investigating the non-Archimedean models studied in Section 7. The use in that section of various weakenings of saturation to study these models has natural links with Ehrlich's rephrasing the maximality condition in terms of homogeneous universal structure [23].

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