THE REASONABLE EFFECTIVENESS OF MODEL THEORY IN MATHEMATICS

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In this article we first provide some background on why (Sections 1 and 2) the applications of model theory across mathematics are reasonable. Section 3 describe some of these applications. While we allude to a number of well-known results over the last seventy years, we focus on three areas that have developed in the last five. We survey the parallel developments of certain combinatorial notions in learning theory (Section 3.1) and in functional analysis (Section 3.2) with fundamental notions of stability theory. Section 4 applies the study of trivial weakly minimal sets, structures very near the base of the stability hierarchy, to count the number of finite models of classes of models closed under substructure.

1. Introduction

In his famous article, *The Unreasonable Effectiveness of Mathematics in the Natural Sciences* [Wig60], Eugene Wigner asserts, 'The first point is that the enormous usefulness of mathematics in the natural sciences is something bordering on the mysterious and that there is no rational explanation for it.' In contrast, we will argue that applications of model theory across mathematics are not mysterious but are easily understood in terms of the basic methodology and motivations of model theory[1]. In his *Introduction to Logic and the Methodology of the Deductive Sciences*, Tarski aimed

> to present to the educated layman . . . that powerful trend . . . modern logic . . . [which] seeks to create a common basis for the whole human knowledge. ([Tar65], xi)

In his 1950 address to the International Conference of Mathematicians, Robinson [Rob52] made this more goal more specific, '. . . we shall be concerned with the effective application of symbolic logic to mathematics proper, more particularly to abstract algebra. Thus, we may hope to find the answer to a genuine mathematical problem by applying a decision procedure to a certain formalized statement.'

After more than a half century of development, we argue that specific formalizations of areas of mathematics are fruitful for those areas and the technology of classification theory provides a common effective basis, not for all mathematics, but to obtain results in many different contexts extending well beyond the Robinson's innovations in abstract algebra. There are three key reasons for this effectiveness.

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[1]Unsurprisingly, I am not the first to appropriate Wigner's metaphor, although most writers maintain Wigner's *IN*effective. See [Hac14] who refers to Corfield and Manders.

The first is representing an area of mathematics as the study of a collection of similar structures for a fixed vocabulary. So one attempts local (area dependent) rather than global foundations for mathematics Second, rather than examining all subsets of those structures, restricting to those defined in a formal logic and thus providing a principled way to isolate of *tame* mathematics. Thirdly, the classification of theories introduced by Shelah [She78], brings to the fore certain combinatorial features that play significant roles in widely distinct areas of mathematics.

At [Bal18, page 2], I wrote,

> In short, the paradigm around 1950 concerned the study of *logics*; the principal results were completeness, compactness, interpolation and joint consistency theorems. Various semantic properties of theories were given syntactic characterizations but there was no notion of partitioning all theories by a family of properties. After the paradigm shift there is a systematic search for a *finite* set of syntactic conditions which divide first order theories into disjoint classes such that models of different theories in the same class have similar mathematical properties.

The finer analysis in the last ten years of the unstable section of the classification has converted the finite italicized in the quote to infinite. This analysis was largely motivated by general model theoretic considerations [She15, MS15]. But as we'll see in Section 3, the combinatorial conditions discovered appear in traditional mathematical settings.

Model theory analyzes the structure of definable sets in any model of theory along two axes: the (quantifier)-complexity of the definition and the combinatorial complexity of the class of definable sets.

Restriction to definable sets is historically very natural. Euclid and Eudoxus developed the method of exhaustion to provide a framework for studying the relations among possibly incommensurable specific pairs of magnitudes such as the diagonal and side of a square. But each example relates to objects which are definable in the modern sense. It is Dedekind who posits the limit exists for *arbitrary* cuts. Speaking polemically, studying only the 'definable' objects in a structure means, 'studying the ones which actually arise'.

As noted, a natural way to 'tame a structure' is to look at definable subsets rather than all sets. This happens automatically in algebraic geometry where the study of solution sets of equations is exactly the study of all definable sets. Tarski [Tar31, Rob52] saw this result in full generality as quantifier elimination or more generally model completeness[2] for an algebraically closed or real closed field, while Chevalley described the key inductive step: constructible sets are closed under projection. This *method of quantifier elimination* provides a general format unifying the Hilbert Nullensatz for a wide range of algebraic applications.

Combinatorial is not quite the right word for the second axis. The central idea is (non)-existence of certain configurations among the definable sets. One such configuration is simply an infinite decreasing sequence of definable sets. On the combinatorial side, replacing the (ascending) descending chain condition (no such sequence exists) on subgroups (ideals) by the (ascending) descending chain condition on *definable* subgroups (ideals) provides a common framework across group

[2]A theory T is model complete if every formula is equivalent to one with only prenex existential quantifiers.

theory, differential algebra, ring theory, etc. Thus, the Wedderburn theorem that certain rings satisfying the descending chain conditions on ideals are represented as matrix rings can be proved for stable theories and so satisfy the dcc on principal (1-generated) ideals [BR77]. The general picture is further clarified by noting that similar variants on the chain condition (e.g., requiring infinite index at each step) for different areas can be unified by recognizing the theory is stable, superstable or ω-stable.

A particularly important example is Berline's [Ber82] proof that Morley rank on algebraically closed fields coincides with the algebraic ranks defined by Krull (on ideals) and by Weil (on the associated algebraic varieties) and all definable sets by (Robinson and Tarski). Surprisingly, the underlying topologies providing the ranks are quite distinct. Morley works with a Stone topology which is totally disconnected and Hausdorff, while the Zariski topology is never Hausdorf.

What I refer to as 'traditional philosophy of mathematics' is dubbed 'philosophy of Mathematics' (Harris, page 30 of [Har15] or [Bal18, page 5]) or 'Foundations of Mathematics' (Simpson in clarifying his view on the Foundations of Mathematics Listserve)). This distinction is transcended in Maddy's recent article, *What do we want a foundation to do?* [Mad18]. She writes

> So my suggestion is that we replace the claim that set theory is a (or the) foundation for mathematics with a handful of more precise observations: set theory provides *Risk Assessment* for mathematical theories, a *Generous Arena* where the branches of mathematics can be pursued in a unified setting with a *Shared Standard of Proof*, and a *Meta-mathematical Corral* so that formal techniques can be applied to all of mathematics at once.

I write from a similar perspective. I am not emphasizing the search for a reliable basis for all mathematics but investigating the organization of mathematics and how particular organizations can productively impact mathematical practice. The clarification of such concepts as function, cardinality, and continuity in the late 19th century had immediate positive impact on mathematics. This effect is usually viewed from the lens of reliability. But Coffa places the relationship between 'reliability and clarity' in historical perspective:

> [We consider] the sense and purpose of foundationalist or reductionist projects such as the reduction of mathematics to arithmetic or arithmetic to logic. It is widely thought that the principle inspiring such reconstructive efforts were basically a search for certainty. This is a serious error. It is true, of course, that most of those engaging in these projects believed in the possibility of achieving something in the neighborhood of Cartesian certainty for principles of logic or arithmetic on which a priori knowledge was to based. But it would be a gross misunderstanding to see in this belief the basic aim of the enterprise. A no less important purpose was the clarification of what was being said. ...
>
> The search for rigor might be, and often was, a search for certainty, for an unshakable 'Grund'. But it was also a search for a clear account of the basic notions of a discipline. ([Cof91], 26)

While the (at least theoretical) reduction of mathematics to set theory provided Maddy's Shared Standard of Proof, it did not (except in basic analysis) provide a fifth criterion that Maddy advances: *essential guidance*. We argue below that the flexibility of model theoretic axiomatizations and the exposure and clarification of common themes provides such essential guidance.

In the first part of this article we outline the paradigm of contemporary model theory and explain why this paradigm might be expected to be useful for proving results in traditional mathematics. In the second we sketch a number of such applications.

2. The Model Theoretic Approach

The first two of the four theses of [Bal18][3] assert:

(1) Contemporary model theory makes formalization of *specific mathematical areas* a powerful tool to investigate both mathematical problems and issues in the philosophy of mathematics (e.g. methodology, axiomatization, purity, categoricity and completeness).
(2) Contemporary model theory enables systematic comparison of local formalizations for distinct mathematical areas in order to organize and do mathematics, and to analyze mathematical practice.

Tarski's term, *meta-mathematics* summarises the underlying motif of model theory. By meta-mathematics I mean both developing a general notion of a formal theory as an object of mathematical theory and the study of particular areas of mathematics by formalizing the area in an appropriate theory.

Definition 2.1. *A* full formalization *involves the following components.*

(1) Vocabulary: specification of primitive notions.
(2) Logic:
  (a) Specify a class[4] *of well formed formulas.*
  (b) Specify truth of a formula from this class in a structure.
  (c) Specify the notion of a formal deduction for these sentences
(3) Axioms: specify the basic properties of the situation in question by sentences of the logic.

In other treatments of formalization we have downplayed the deduction system (2c) because for much of model theory compactness (consistency of a set of sentences X follows from consistency of finite subsets of X) is often more important than the existence of a deduction. For 'getting tight results', the recursive deduction system is important but not sufficient. Even primitive recursive upper bounds are far too crude for mathematical' applications. However, the more sophisticated model theoretic techniques obtain mathematically interesting upper bounds.

I have chosen the word 'vocabulary' rather than such rough synonyms as language, similarity type, signature or, even rougher, logic. Examining a particular mathematical topic, the investigator selects certain concepts as fundamental. The vocabulary is a set τ of relation symbols, function symbols, and constant symbols is chosen to represent these basic concepts. A τ-structure with universe A assigns (e.g., to each n-ary relation symbol R a subset R^A of A^n). Thus, many situations in mathematics have led to the now nearly ubiquitous notion of a group. This notion

[3]This introduction heralds many of the notions of the book; we refer to it for further details.
[4]In the instances treated here, this will be a set.

can be formalized in such diverse vocabularies as a single binary function, a single ternary relation, or augmenting, say the binary function with a unary function (inverse) and a constant symbol (identity). Or, returning to the early 19th century, one might focus on somewhat more specific topic such as substitutions or permutation groups. One of model theory's contribution is making rigorous the notion of *interpretation* which allows one to make clear when these different approaches are, or are not, equivalent.

Crucially, fixing a vocabulary, even with suggestive names, has done little work. It is necessary to provide axioms that reflect the topic being studied. Calling a binary relation an order and then positing that it satisfies the axioms of an equivalence relation is madness. But, there has been no strict formal error, just an abuse of the mathematician's right to name concepts arbitrarily. However, a fruitful formalization will respect the previous terminology. Crucially, one must choose an appropriate logic. Dedekind and Peano provided second order axioms which shed great light on the internal structure of the arithmetic of the natural numbers. While these axioms are particularly valued for determining a unique (up to isomorphism[5]) structure, and give a uniform basis for various results in number theory proved by induction, they have not been central in the great 20th century advances in number theory. Rather, these advances are based on considering the natural numbers as substructures of much more tame objects such as geometries over algebraically closed fields. We provide some context in [Bal18, Chapter 5.6] introducing such further sources as [Bou99, HHM07, Mar07, HP00, Sca12].

We focus here on *first order* logic ($L_{\omega,\omega}$) which allows finite Boolean combinations of formulas and quantification over finite strings of individuals. We will make occasional comparisons with *infinitary logic* ($L_{\kappa,\lambda}$) which allows Boolean combinations of $< \kappa$ formulas and quantification over $< \lambda$ individuals. But second order logic will get short shrift. On the one hand, first order set theory is a useful avatar of second order logic [Vää12]; on the other there is almost no model theory of second order logic.

The crucial aspect of modern model theory is the focus not on logics but on the models of a particular theory (usually in first order logic). The crucial ingredient in what I call the *paradigm shift* is Shelah's introduction of a classification of complete first order theories into finitely many kinds. This classification (See `http://www.forkinganddividing.com/#_02_54`.) is roughly syntactic (certainly set theoretically absolute). Morley discovered the significance of ω-stability: a countable first order theory T is ω-stable[6] if for every countable model M of T there are only countably many non-isomorphic 1-element extensions of M. Shelah generalized this notion and gave a long list of equivalent requirements for a theory to be stable (i.e. stable in some infinite cardinal κ). The following three facts indicate the diverse aspects of the notion.

Fact 2.2. *If T is stable*

(1) then T is stable in every cardinal κ with $\kappa^{\aleph_0} = \kappa$.

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[5]Note that isomorphism is not well-defined unless one specifies the vocabulary. See Pierce's paradox in [Bal18].

[6]Non-trivially Morley showed ω-stable implies κ-stable; κ-stable is defined by replacing countable with κ in both occurrences in definition.

(2) *(fundamental theorem of stability theory) then there is no formula* $\phi(\mathbf{x}, \mathbf{y})$ *that has the* order property*: for every n*

$$T \models (\exists \mathbf{x}_1, \ldots \mathbf{x}_n \exists \mathbf{y}_1, \ldots \mathbf{y}_n) \bigwedge_{i<j} \phi(\mathbf{x}_i, \mathbf{y}_j) \wedge \bigwedge_{i \geqslant j} \neg\phi(\mathbf{x}_i, \mathbf{y}_j)$$

and conversely.

(3) *there is a notion of independence on models of T which, locally, generalizes the notion of independence of a vector space.*

The first of these conditions demonstrates that there are fundamental mathematical properties which depend non-trivially on cardinality. In contrast most mathematical result are either very specific to structures of size less than the continuum, e.g., a complete separable ordered field is isomorphic to the real numbers, or completely independent of cardinality, e.g., any Desarguesian plane can be coordinatized by a division ring.

The second condition is syntactic and clearly it is a property that can be checked on the countable models of T. There are consequences for reliability. These notions are clearly described in second order arithmetic and do not depend on higher set theory. The third condition contributes to many of the important applications of stability theory in traditional mathematics.

Initially, I thought of classification theory as the essence of the paradigm shift. But deeper thought led to my current emphasis on *formalization* and the classification of theories. Already in the 1950's (or even the 30's for the special case of real closed fields) such pioneers as Robinson and Tarski realized that showing that those subsets definable in a model of a theory T could be defined by formulas with low quantifier complexity was a powerful tool for studying the theory. This is an epistemological insight. If one formalizes an area of mathematics in a way that all definable sets are 'simple', then one has a much better understanding of the subject. Thus, while the formulas of first order Peano arithmetic have unbounded quantifier-complexity, every definable subset of the complex (or real) field is definable without quantifiers (in a vocabulary with order). The relation between this kind of simplicity and decidability is not obligatory. But many decision problems (e.g. the real field) were solved precisely by reducing to quantifier free formulas where a brute-force analysis was possible.

In the 1980's work of Steinhorn, Pillay, and Van Den Dries [PS86, Dri84] melded this epistemological approach with a combinatorial simplicity of the type discovered by Morley and Shelah. Thus, a theory is strong minimally (best-behaved stable theory) if every definable subset is finite or co-finite. A theory whose models are linearly ordered is *o-minimal* if every definable subset of a model M is a finite union of points and intervals with endpoints in M. This definition captures the essential character of the collection of definable subsets of the real field. This essence is emphasized by the proof [Wil96] that the real exponential field is also o-minimal and model complete. This work was followed by showing other expansions of reals (e.g., by the Γ function) remain o-minimal. Wilkie explains the sense in which o-minimality captures Grothendieck's notion of 'tame topology' in [Wil07]; See also Marker [Mar00]. The subject has been well-integrated with contemporary real algebraic geometry [BCR98] and has had a significant impact in number theory. Half of the 2013 Karp prize[7] was awarded to Kobi Peterzil, Jonathan Pila, Sergei

[7]For award details see `http://vsl2014.at/2014/07/awards-at-the-logic-colloquium/`.

Starchenko, and Alex Wilkie for 'their efforts in turning the theory of o-minimality into a sharp tool for attacking conjectures in number theory, which culminated in the solution of important special cases of the André-Oort Conjecture by Pila.' These results are summarised by Chambert-Loir in a poetic metaphor of unicorns and grasslands while reviewing the collection *O-Minimality and Diophantine Geometry*, [CL17, JW15].

The effectiveness of model theory described in the next section results from a combination of a methodology applicable in many areas of mathematics and a deep understanding of the particular topic.

3. The effectiveness of model theory

In this section we will describe several examples of the interactions of model theory with other areas of mathematics. We pass over the famous examples mentioned above and describe some recent interactions of model theory with other areas of mathematics.

3.1. Parallel Developments I: statistics and learning theory

The Sauer-Shelah lemma was independently discovered by three investigators (Sauer (combinatorics of set systems), Shelah/Perles[8] (model theory/geometry), Vapnik-Chervonenkis (statistics)) around 1972. We adapt the terminology from set system[9].

If $\mathcal{F} = \{S_1, S_2, \dots\}$ is a family of sets, and T is another set, then T is said to be *shattered* by $\mathcal{F}$ if every subset of T (including the empty set and T itself can be obtained as an intersection $T \cap S_i$ between T and a set in the family. The VC dimension of $\mathcal{F}$ is the largest cardinality of a set shattered by $\mathcal{F}$. If there is such an n, we call $\mathcal{F} \subset X$ a Vapnik-Chervonenkis class (or VC class).

In terms of these definitions, the Sauer-Shelah lemma states that if $\mathcal{F}$ is a family of subsets of a set F with $|F| = n$ such that $|\mathcal{F}| > \sum_{i=0}^{k-1} \binom{n}{i}$, then $\mathcal{F}$ shatters a set of size k. Equivalently, if the VC dimension of $\mathcal{F}$ is k, then $\mathcal{F}$ can consist of at most $\sum_{i=0}^{k} \binom{n}{i} = O(n^k)$ sets.

In terms of stability theory, the Sauer-Shelah Lemma asserts that if a formula ϕ *does not have the independence property* (NIP), the number of ϕ-types of a set of size n is a polynomial in n with order the VC dimension of ϕ. This connection was pointed out in [Las92].

A class has finite VC dimension if and only if it is *Probably approximately correct (PAC)* -learnable, in the sense of the following definition [CF18b]. Given an infinite set X with a probability measure μ on X and a collection of measurable subsets of X, denoted by $\mathcal{F}$, one attempts to learn a fixed but unknown $F \in \mathcal{F}$ by sampling from X. For some large n, a sample A of n elements of X is chosen randomly, and the learner is told which points belong to F. The goal is to use the sample to make a prediction $G(A)$ that estimates F with small error. For some $\epsilon > 0$ fixed ahead of time, we say that the sample estimates the set $\mathcal{F}$ ϵ-well if $\mu(G(A) \triangle F) < \epsilon$. The class $\mathcal{F}$ is PAC-learnable if for any δ there is a large enough n such that the measure of the samples of size n (computed using the product measure μ^n) which estimate the sample ϵ-well is greater than $1 - \delta$. Roughly, for large enough sample size, we can get arbitrarily high likelihood that we predict the target set arbitrarily

[8]Shelah [She72] cites 'a little more complex result, of Perles and Shelah'.

[9]The text of this standard definition is taken from the wikipedia article .

well. That is, for a large enough sample size, predictions are *probably approximately correct*.

The connection to model theory follows: when X is taken to be the universe of M, a model of a first order theory T and $\phi(x, y)$ is a formula in the vocabulary of T, we let $\mathcal{F} = \{\phi(M, a)|a \in M\}$. Then [Las92] the VC-dimension of $\mathcal{F}$ is finite if and only if $\phi(x, y)$ is NIP. Much of the interaction has been from learning theory to model theory. In particular, the learning theory notion of a compression scheme [LW86] was adapted to the stability theory context [JL10]. The abstract of [EK19] emphasizes this impact: 'Combining two results from machine learning theory we prove that a formula is NIP if and only if it satisfies uniform definability of types over finite sets (UDTFS). This settles a conjecture of Laskowski.' There has been some feedback to learning theory [LS13].

PAC learning is only one of many models of machine learning. But more recently a surprising new connection arose between 'online learning' and stable theories. In the online learning setting, the learner is presented with a stream of elements and is asked to guess if they belong to the target set. A class is online learnable if there is some N such that the learner has a strategy to make at most N mistakes in learning any set in the class. The notion of thicket dimension (Definition 3.3) takes into account the order in which information is introduced. We set the stage as in [CF18a] with a specific on line learning model. Fix a set X and denote by $\mathcal{P}(X)$ the collection of all subsets of X. A concept class $\mathcal{C}$ on X is a subset of $\mathcal{P}(X)$. In the *equivalence query (EQ) learning model*, a learner attempts to identify a target set $\mathcal{A} \in \mathcal{C}$ by means of a series of data requests called equivalence queries. The learner has full knowledge of $\mathcal{C}$, as well as a hypothesis class $\mathcal{H}$ with $\mathcal{C} \subseteq \mathcal{H} \subseteq \mathcal{P}(X)$. An equivalence query consists of the learner submitting a hypothesis $\mathcal{B} \in \mathcal{H}$ to a teacher, who either returns yes if $\mathcal{A} = \mathcal{B}$, or a counterexample $x \in \mathcal{A} \triangle \mathcal{B}$. In the former case, the learner has learned $\mathcal{A}$, and in the latter case, the learner uses the new information to update and submit a new hypothesis. [CF18a] improve the upper bounds for the number of queries ($LC^{EQ}(C, H)$) required for EQ learning (and the related EQ+MQ) of a class $\mathcal{C}$ with hypotheses $\mathcal{H}$ in terms of the Littlestone dimension of C, denoted $\mathrm{Ldim}(\mathcal{C})$, and the consistency dimension of $\mathcal{C}$ with respect to $\mathcal{H}$, denoted $C(\mathcal{C}, \mathcal{H})$. The consistency dimension is new to learning theory and is related to the model theoretic notion nfcp (the finite cover property fails). Here is a sample result.

Theorem 3.1. *Suppose* $\mathrm{Ldim}(\mathcal{C}) = d < \infty$ *and* $1 < C(\mathcal{C}, \mathcal{H}) = c < \infty$. *Then* $LC^{EQ}(C, H) \leqslant c^d$.

The new insight in [CF18a] is the discovery that Littlestone dimension is an alias for Shelah 2-rank and also for 'thicket' dimension. Littlestone dimension is a rank on set systems while thicket dimension measures systems of sequences. If the set (sequence) system consists of are given by first order formulas then the two ranks are equal. [ALMM19] proves that PAC of 'private learning' (a variant on PAC-learning appropriate when the input data, such as medical records, need to be kept secret) implies finite Littlestone definition, i.e., stability.

A major development in this area is the solution the following analogy

$$\frac{X}{\textit{stability}} = \frac{\textit{VC–dimension}}{\textit{NIP}}.$$

For this, we replace thinking of a path through 2^n as an indicator function for a subset of X, by thinking of a function in 2^X as picking in order a sequence that we would like to list a subset of $A \subseteq X$. Formally,

Definition 3.2. *A* binary element tree *of height n with labels from X is a function $T : 2^{<n} \to X$. A leaf is a binary sequence of length n, $\tau : [n] \to \{0, 1\}$. A* node *is a binary sequence $\sigma \in 2^{<n}$ along with its label, $a_\sigma := T(\sigma)$.*

A leaf τ is properly labeled *by a set A if for all $m < n$, $a_\tau \upharpoonright m \in A$ iff $\tau(m) = 1$.*

Thus Y is 'thicket dimension'.

Definition 3.3. *The* thicket dimension *of a set system $(X, \mathcal{F})$ is the largest $k < \omega$ such that there is a binary element tree of height k with labels from X such that every leaf can be properly labeled by elements of X if such a maximum exists or ∞. The thicket shatter function $\rho_{\mathcal{F}}(n)$ is the maximum number of leaves properly labeled by elements of $\mathcal{F}$ in a binary element tree of height n.*

Thus Y is 'thicket dimension'.

Theorem 3.4 (Thicket Sauer-Shelah)**.** [Bha18] *Let $\mathcal{F}$ be a set system of thicket dimension k. Then*

$$\rho_{\mathcal{F}}(n) \leqslant \sum_{i=0}^{k} \binom{n}{i}.$$

Chase and Freitag [CF18c] introduce the notion of *banned sequences* to give a proof that specializes not only to each version of Sauer-Shelah considered here but further improve the Malliaris and Terry improvement [MT18] (using the stability classification to better organize the case analysis) on the bounds in a result of [CKOS16] on a case of the Erdős-Hajnal conjecture. These developments illustrate several ways in which model theory provides essential guidance. The use of stability theory in on line learning not only gives better upper bounds but provides [CF18b, Section 5] a wealth of new examples for the learning theory community.

3.2. Parallel Developments II: functional analysis

In this section we explore some striking analogies between functional analysis and stability theory that turn out not to be at all coincidental. After tracing some of the history we present some suggestions of Khanaki for new methods and problems in stability theory arising from analyzing these analogies.

In [BY14], Ben Yaacov argued that Grothendieck 'first' proved the fundamental theorem of stability theory (Fact 2.2). Like an earlier hybrid, the Gödel-Deligne completeness theorem[10], there is a kernel of truth here; there is a common core to the central argument. But Grothendieck and Shelah have different contexts. That is, as discussed in [BY14, Pil16], there is a topological (functional analytic) core to Shelah's proof that for a first order theory instability (i.e. failure of the order property) is equivalent to the non-definability of types[11]. Pillay [Pil16] strengthens the result to every complete type is generically stable. Grothendieck had earlier isolated this argument as a theorem of *general* topology. Shelah rediscovered the argument in the much more *general* context of complete first order theories, by

[10]Expounded in `https://www.math.princeton.edu/events/godel-deligne-theorem-2016-04-21t163004`).

[11]Another equivalent to stability is that every complete ϕ-type $p \in S_\phi(B)$ is definable; there is a formula $\psi_\phi(y)$ over B such that $\phi(x, a) \in p$ if and only if $\psi_\phi(y)$.

attaching, as described below, a topological space to each subset A of a model of such a theory.

I contrast the two uses of 'general' in the previous paragraph. Grothendieck is finding topological (function analytic) conditions for a certain result. The Stone space (compact, totally disconnected, Hausdorff) topology used by Shelah is a particular example of this situation. But Shelah is proving a general result about first order theories. Thus, he grounds the whole range of applications across mathematics mentioned in this article. He provides a context by which one is enabled to apply the Grothendieck theorem and other results in functional analysis to many diverse areas of mathematics.

This section reports the work of Khanaki [Kha19c, KP18, Kha19a] in transferring theorems of functional analysis to inspire new characterizations of some classes and new classes of first order theories. We isolate these topological phenomena, separating them from the linear space context [Kha19c] so as to focus on the core of the argument. As we are studying the action on a Stone space which is compact, we are able to study the space of functionals with the topology of pointwise convergence rather that engaging various notions of weak topology which arise in functional analysis.

We review some notions and results for the topology of pointwise convergence. If X is any set and A a subset of $\mathbb{R}^X$, then the topology of *pointwise convergence* on A is that inherited from the usual product topology of $\mathbb{R}^X$. A typical neighborhood of a function f is determined by a finite subset $\{x_1, \ldots, x_n\}$ of X and $\epsilon > 0$ as follows:

$$U_f(x_1, \ldots, x_n; \epsilon) = \{g \in \mathbb{R}^X : |f(x_i) - g(x_i)| < \epsilon \text{ for } i \leqslant n\}.$$

$C(X) \subseteq \mathbb{R}^X$ denotes the space of continuous functions from X into $\mathbb{R}$; it is naturally a linear space under pointwise addition and is equipped with sup norm. For $A \subseteq M \models T$ and a formula $\phi(x; y)$ (here x,y represent finite sequences of variables) $S_\phi(A)$ is the collection of types containing formulas $\phi(x, a)$ or $\neg\phi(x, a)$ for $a \in A$. $S_{\phi^{\text{opp}}}(A)$ reverses the roles of x and y; now formulas $\phi(a, y)$ are in the type. With this notation we describe the relevant function space following [KP18, 1,2] and [Kha19b, 2.1].

Notation 3.5. *We fix the usage of A and X in this paragraph. Let T be a first order theory, M a model of T, and M^* a sufficiently saturated elementary extension of M. Specifically, fix $\phi(x, y)$ with $\lg(x) = n$ and A a set of n-tuples contained in M^*. Let X be $S_{\phi^{\text{opp}}}(A)$, the set of ultrafilters generated by formulas $\phi(a, y)$ with $a \in A$. Now define a collection of functions $\phi(a, y)$ from X into 2 by $\phi(a, q) = 1$ iff $\phi(a, y) \in q$. As ϕ is fixed* we can identify this set of functions with A. *Since each function in A takes only the values* 0 *and* 1*, A is uniformly bounded. Moreover, the logic topology guarantees that each $\phi(a, y)$ is continuous. So $A \subseteq C(X)$.*

In general a space of functions from X to $\mathbb{R}$ has the interchangeable double limit property if for sequences of functions $f_n \in \mathbb{R}^X$ and points $x_m \in X$

$$\lim_n \lim_m f_n(x_m) = \lim_m \lim_n f_n(x_m)$$

when the limits on both sides exist. We translate this to our context:

Definition 3.6. *Let $A \subseteq M \models T$, and X be $S_{\phi^{\text{opp}}}(A)$.*

(A, X) has the interchangeable double limit property *if for any infinite sequences $\boldsymbol{a} = \langle a_n : n < \omega \rangle \in A$ and $\mathbf{b} = \langle b_n : n < \omega \rangle \in X$*

$$\lim_n \lim_m \phi(a_n, b_m) = \lim_m \lim_n \phi(a_n, b_m)$$

when the limits on both sides exist.

If $\boldsymbol{a}$ and $\mathbf{b}$ are infinite sequences we denote by $(\boldsymbol{a}\hat{}\mathbf{b})$ the sequence obtained by concatenating at each n, $\langle a_1 b_1, a_2 b_2, \ldots \rangle$.

Observation 3.7. *Note that T is stable exactly if for each $A \subseteq M \models T$, with $X = S_{\phi^{\text{opp}}}(A)$, (A, X) has the interchangeable double limit property.*

Proof. If there exist $\boldsymbol{a}, \mathbf{b}$ with the order property, $\phi(a_i, b_j)$ if and only if $i < j$ then $\lim_n \lim_m \phi(a_n, b_m) = 1$ since for fixed n and a tail of m $\phi(a_n, b_m)$ is true. But the value is 0 when the limit is taken in the opposite direction.

Conversely, suppose T is stable, Fix $\boldsymbol{a}, \mathbf{b}$ so that both limits exist. Fix $n > 2$. By the Ramsey theorem we can find a subsequence of $(\boldsymbol{a}\hat{}\mathbf{b})$ that is ϕ-n-indiscernible (any two properly ordered n-tuples from the sequence have the same ϕ-type). Since T is stable this sequence is 'set' n-indiscernible[12]. Hence, the double limits are equal. And since the sequences have double limits they must be the limits of the subsequences. ∎

Definition 3.8. *Let A be a subset of a topological space X, then*

(i) *The set A is* relatively compact *in X if its closure in X is compact.*
(ii) *The set A is* relatively sequentially compact *(RSC) in X if each sequence of elements of A has a subsequence converging to an element of X.*

The following Theorem applies to A and $X = S_{\phi^{\text{opp}}}(A)$. See [BY14, Pil16].

Fact 3.9 (Grothendieck's criterion)**.** *Let X be a compact topological space. Then the following are equivalent for a norm-bounded subset $A \subseteq C(X)$:*

(i) *A is relatively compact in $C(X)$.*
(ii) *A has the interchangeable double limit property.*

Since the interchangeable double limit property is equivalent to ϕ does not have the order property (Observation 3.7), we have:

Theorem 3.10 (stable)**.** *ϕ does not have the order property if and only if for each M model of T and $A \subset M$, the epononymous[13] $A \subset X = S_{\phi^{\text{opp}}}(A)$ is relatively compact in $C(X)$.*

Recall that a formula $\phi(\mathbf{x}, \mathbf{y})$ has the *strict* order property in a model M if there are $\mathbf{b}_i \in M$, for $i < \omega$, such that

$$M \models (\forall \mathbf{x})\phi(\mathbf{x}, \mathbf{b}_i) \to \phi(\mathbf{x}, \mathbf{b}_j) \text{ iff } i \leqslant j.$$

A crucial theorem of Shelah is that T is stable if and only if every formula is both NIP (fails the independence property) and NSOP (fails the strict order

[12] That is, we can drop the requirement of 'properly ordered' [She78, Definition 2.4]. Note that the proof that (T is stable) implies (order indiscernability implies set indiscernability) in [Bal88, Theorem 1.3.i)] finds an ordering formula from a sequences that is not set indiscernibles in the same number of variables as the offending formula.

[13] I was a little worried that this word was too fancy. But `https://www.merriam-webster.com/dictionary/eponymous` shows it is exactly what I mean.

property). Khanaki [Kha19a] refines this result in several ways by characterizing various notions in functional analytic style. For this, we introduce a property A_ϕ that yields a new characterization of NSOP.

Definition 3.11. *We say* $\boldsymbol{a} =< a_i : i < \omega >$ *and* $\mathbf{b} =< b_j : j < \omega >$ *witness that* ϕ *satisfies* A_ϕ *in* T *if*

(1) *the* independence property is uniformly blocked for $\phi(x, y)$ on $\boldsymbol{a}$. *That is, there exist* $(N_{\phi,\mathbf{a}}, E_{\phi,\mathbf{a}})$ *with* $N < \omega$ *and* $E \subseteq N$ *such that for any subset* $(a_{i_1}, \ldots a_{i_j}, \ldots a_{i_N})$ *of distinct elements of* $\boldsymbol{a}$*:*

$$\neg\exists y(\bigwedge_{j\in E} \phi(a_{i_j}, y) \wedge \bigwedge_{j\notin E} \neg\phi(a_{i_j}, y)).$$

(2) $\boldsymbol{a}, \mathbf{b}$ *witness* ϕ *has the order property.*

Now Khanaki shows by fairly standard model theoretic arguments:

Theorem 3.12. (1) [Kha19a, Proposition 2.4] *If* A_ϕ *holds witnessed by some* $\boldsymbol{a}, \mathbf{b}$ *then some Boolean combination of instances of* ϕ *has the strict order property.*

(2) [Kha19a, Proposition 2.7] *T has the NSOP if and if only there is no formula and sequence that witness* A_ϕ *is true.*

We will say ϕ *engenders* the SOP if some Boolean combination of instances of ϕ has SOP. Now we[14] deduce from Theorem 3.12 an 'intrinsic' characterization of those formulas ϕ which have the Independence Property but not the Strict Order Property. The characterization asserts that the type of a countable sequence $\boldsymbol{a}$ that indexes an independent family of sets is omitted and a second type of a countable sequence $\boldsymbol{a}\frown\mathbf{b}$ that witnesses the strict order property is realized in all sufficiently saturated models.

Theorem 3.13. ϕ *has NIP but engenders SOP if and only if*

For every $\boldsymbol{a}$ *in the monster (or any* $\aleph_1$*-saturated) model,* M^**, of* T *the independence property is uniformly blocked for* $\phi(x, y)$ *by some* $(N_{\phi,\mathbf{a}}, E_{\phi,\mathbf{a}})$ *with* $N_{\phi,\mathbf{a}} < \omega$ *on* $\boldsymbol{a}$ *and there exists* $\boldsymbol{a}, \mathbf{b}$ *that witness the order property for* ϕ*.*

Note that by compactness that although $N_{\phi,\mathbf{a}}$ varies with $\boldsymbol{a}$ there must be a uniform bound N or there would be a sequence in M^* that is not bounded. This uniformity illustrates two instruments for the effectiveness of model theory: i) the compactness theorem allows one to 'concentrate' an unbounded phenomenon in a single instance and ii) the ability to choose models with special properties that focus a problem. In our case, we posit a saturated model to realize the concentrated phenomenon. In another situation, the prime model might show a certain configuration can be avoided.

In [Bal18, Chapter 2.3] I distinguish between a virtuous property of a theory T and a dividing line. A property is *virtuous* if it has sigificant mathematical consequences for the theory or its models. A property is a *dividing line* if it and its negation are both virtuous. We now find some further virtuous properties suggested by the study of Baire functions in analysis.

[14] This characterization was extracted from [Kha19a].

Definition 3.14. *(1) A real valued function from a complete metric space is said to be Baire-*1 *if it is a pointwise limit of a sequence of continuous functions.*

(2) $f \in \mathbb{R}^X$ is DBSC if it is a difference of two bounded semi-continuous functions. This is a proper subclass of the Baire-1 functions.

It is standard (e.g. [Adl08]) that any formula ϕ which does not have the independence property has an alternation number n_ϕ, the maximal number of elements n_ϕ such that there exists an indiscernible sequence $\boldsymbol{a}$ and a b such that $\phi(a_i, b) \leftrightarrow \neg\phi(a_{i+1}, b)$ for $i < n$. We use a *wider notion of alternation number* by not requiring $\boldsymbol{a}$ to be indiscernible. Khanaki shows in [Kha19a, Lemma 2.6] a topological result which translates into model theory[15] as follows.

Fact 3.15. *If the independence property is uniformly blocked on $\boldsymbol{a}$ then ϕ has alternation number $n_{\phi,\boldsymbol{a}}$ on $\boldsymbol{a}$ and consequently $\phi(a_n, x)$ converges pointwise to a function $f \in \mathbb{R}^X$ which is DBSC.*

Note the distinction in form between the two propostions in the next theorem. The first is an unconditional statement that there is a subsequence whose limit is DBSC; the second is conditioned on the sequence being uniformly blocked.

Theorem 3.16. *(1) (NIP)* [Kha19a, Remark 2.11] *ϕ is NIP if and only if for every sequence $\boldsymbol{a}$, there is a subsequence a_{i_j} such that $\phi(a_i, y)$ converges to an $f \in \mathbb{R}^X$ which is DBSC.*

(2) (NSOP) [Kha19a, Remark 2.8] *A complete first order theory T is NSOP if and only if*

for any formula ϕ and infinite sequence $\mathbf{a}$ if the independence property is uniformly blocked on $\mathbf{a}$ by some $(N_{\phi,\mathbf{a}}, E_{\phi,\mathbf{a}})$ then $\phi(x, a_i)$ converges to an f that is continuous.

Proof. 1) It is well known that NIP is equivalent to every sequence $\phi(\boldsymbol{a}_n, x)$ has a subsequence with bounded alternation number and so the subsequence converges. The statement here just adds that the limit function is DSBC, which follows from Fact 3.15.

2) Suppose T is NSOP. Then, by Theorem 3.12.ii) there is no formula ϕ and sequences $\boldsymbol{a}, \mathbf{b}$ that satisfy both conditions of A_ϕ. Suppose there is an $\boldsymbol{a}$ satisfying condition i) of A_ϕ. Since condition ii) of A_ϕ fails, for any $\mathbf{b}$, the pair $\boldsymbol{a}, \mathbf{b}$ do not witness the order property. Pillay [Pil16, Proposition 2.2] shows that if ϕ does not satisfy the order property in M, then for any sequence $\boldsymbol{a} \in M$, $\lim \phi(a_n, x)$ converges to a continuous function. Thus, f is continuous.

Conversely, suppose T has SOP witnessed by the formula ϕ so there is a sequence $\boldsymbol{a}$ such that $\forall y \phi(a_i, y) \rightarrow \phi(a_j, y)$ if and only if $i < j$. Thus, if $j < i$, $\exists y(\phi(a_i, y) \wedge \phi(a_j, y)$. In particular, there is a $\mathbf{b}$ so that $\boldsymbol{a}\mathbf{b}$ witness the order property for ϕ; so, condition ii) of A_ϕ holds. But then the independence property is blocked on $\boldsymbol{a}$ by $N = 2$ and $E = \{1\}$ and condition i) of A_ϕ is satisfied contrary to hypothesis. ■

Khanaki states [Kha19a, Fact 3.1] the following version of the Eberlein-Šmulian theorem for the topology of pointwise convergence on $C(X)$. We are interested in

[15]The 'consequently' Lemma 3.15 is ii) implies iii) of the topological Lemma 2.6. 'ii) implies iii)' requires the additional assumptions which here amount to the observation that the $\phi(a_n, x)$ are continuous and $S_{\phi^{opp}}(\boldsymbol{a})$ is a metric space. This last condition depends on the countability of $\boldsymbol{a}$. For large A, $S_{\phi^{opp}}(A)$ is not a metric space although it is compact.

the result when X and A are as described in Notation 3.5. As noted A is uniformly bounded. See [Whi67] for a short proof.

Theorem 3.17 (Eberlein-Šmulian variant)**.** *A is relatively compact in $C(X)$ if and only if both i) and ii) hold where*

i (RSC[16]) Every sequence of A has a convergent subsequence in $\mathbb{R}^X$ and
ii (SCP[17]) the limit of every convergent sequence from A is continuous.

We say a theory has RSC (SCP) if for every $A \subseteq M \models T$ and $X = S_{\phi^{opp}}(A)$, (A, X) has RSC (SCP).

Note that SCP of a theory is a strengthening of the characterization of NSOP in Theorem 3.16 as SCP drops the hypothesis of the implication defining NSOP.

Since we know stability is equivalent to the relative compactness of A in $C(X)$ the following theorem just states the model theoretic translation of Theorem 3.17.

Theorem 3.18. [Kha19a, Remark 3.2] *The following are equivalent:*

(1) ϕ is stable for T.
(2) For every $A \subseteq M^*$ and every ϕ, the pair (A, X) is both RSC and SCP.

The novelty here is that SCP strictly implies NSOP and NIP is equivalent to RSC. This is a splitting of unstable into two classes (NRSC and NSCP) that overlap differently than IP and SOP do. There is a theory [Kha19a, Remark 3.5,Example 2.15] that is NSOP and IP but does not have SCP.

Khanaki [Kha19c] introduced the notion on NIP in a model and with Pillay [KP18, Kha19a] has demonstrated the interest of such first order properties in a fixed model. Khanaki suggests in [Kha19a] that the Kechris-Louveau hierarchy of Baire-1 functions could be translated by the scheme outlined here to a hierarchy of theories defined analogously to RSC and SCP above. In particular, he suggests investigating the class of theories such that convergent sequences of functions $\phi(a_n, x)$ are DBSC. These suggestions appear to be a very interesting way in which functional analysis could aid in the neo-stability project.

Several questions arise. Are these properties virtuous? Are they dividing lines? Do they separate interesting theories? In particular, do they give applications in other areas of math? Shelah assures us that one should explore the universe without worrying about this last question. But experience with model theoretic classification as exemplified in other sections of this paper give a positive answer. So it is worth looking.

I have discussed here the use of functional analytic methods in refining the stability classification. Let me quickly mention some applications of model theory to functional analysis. In particular there is a lot of work around C^* and Von Neumann algebras. Showing specific classes are function algebras are elementary in continuous logic is a key tool. Model theory of C*-algebras will appear as a Memoir of the American Mathematical Society [FHL+]. Hart's web page `https://ms.mcmaster.ca/~bradd/#Research` contains links to many papers as well as this memoir. [BYBHU08] provides the background in continuous logic. The study of metric abstract elementary classes provides another perspective and links to category theory [BGL+16, HH12, LR16].

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[16]Relatively sequentially compact in $\mathbb{R}^X$

[17]Sequential completeness property

4. Finite Combinatorics

I cannot attempt to survey the interactions of model theory with combinatorics. Recent interactions are with such topics as the Erdős-Hajnal conjecture, Szmerédi's theorem, and the Elekes-Szabó theorem. Examples come from various places in the stability hierarchy, especially the new notion of distal theories. Here I will concentrate on one particular investigation that involves very nicely behaved structures from a model theoretic standpoint.

Graph theorists count graphs that have a specified property. One standard sort of problem is to fix a class of finite graphs $\mathcal{H}$ that is *hereditary* (closed under substructure and isomorphism) and count. The model theorists eyes light up. One of the earliest theorems of model theory, the Łoś-Tarski theorem, asserts a class $\mathcal{H}$ is hereditary exactly if it is defined by a set of universal sentences. And counting the number of models of each cardinality was the motivating problem for classification. The *speed* of $\mathcal{H}$ is the function sending n to $|\mathcal{H}_n|$ where $|\mathcal{H}_n|$ is the members of $\mathcal{H}$ with universe n. Work in the 2000's by Alon, Balogh, Bollobás, Morris, Thomason, Weinreich (in various combinations) almost completely classified the possible speeds for an *hereditary class of graphs* as follows:

Theorem 4.1. *Let $\mathcal{H}$ be an hereditary class of finite graphs.*

- *(poly/exp) For some k, $|\mathcal{H}_n|$ is a sum of terms $p_i(n)i^n$ for $i < k$, where each $p_i(n)$ is a rational polynomial*
- *(factorial) $|\mathcal{H}_n| = n^{(1-\frac{1}{k}-o(n))n}$ for some $k > 1$.*
- *(penultimate) $|\mathcal{H}_n|$ is caught between a function growing slightly slower than n^n and one slightly below 2^{n^2}.*
- *(exponential in n^2) $|\mathcal{H}_n|$ grows as $2^{Cn^2+o(n^2)}$.*

The penultimate range is both the 'next to fastest' growth rate and, importantly, a range. There is an $\mathcal{H}$ whose growth rate is close to the lower limit on one infinite set of natural numbers and close to the upper limit on another [BBBW0a].

A graph is a structure with one symmetric binary relation. Here is the basic question. Can the kind of analysis carried out for graphs be extended to an arbitrary finite relational language[18] ? Noting that $|\mathcal{H}_n|$ is counting the number of quantifier-free n-types of the theory $T_{\mathcal{H}}$ consisting of the universal sentences true in $\mathcal{H}$ links the problem with classical (late 1950's) model theory. Strikingly, the solution by Laskowski and Terry depends on the fine analysis of the stability hierarchy. Their work illustrates one of the themes underlying the effectiveness of model theory: approximating the finite by the infinite [Bal00]. The crucial step is to study the class $\mathcal{H}$ of finite models by studying infinite models of completions of $T_{\mathcal{H}}$. We need a little history to see how more sophisticated model theory enters the picture.

An element a is said to be in the *algebraic closure* of a set B, $a \in \mathrm{acl(B)}$ if there is a formula $\phi(x, \mathbf{y})$ and a sequence $\mathbf{b} \in B$ such that $\phi(a, \mathbf{b})$ and there are only finitely many solutions of $\phi(x, \mathbf{b})$ (written $(\exists^{\leqslant k} x)\phi(x, \mathbf{b})$.) A definable set D is *strongly minimal* if every definable subset of D is finite or cofinite. This implies that there is a unique non-algebraic type of elements in D. On a strongly minimal set algebraic closure behaves as closure does in a vector space. Morley's categoricity theorem for countable vocabularies was reformulated in [BL71] by showing every

[18]Spencer was surprised that his 0-1 law for graphs with edge probability $n^{-\alpha}$[SS88] (α irrational) extended to arbitrary finite relational languages [BS97].

model of an $\aleph_1$-categorical theory is controlled by a strongly minimal set. In generalizing Morley's results to uncountable vocabularies, Shelah introduced the notion of a *weakly minimal* set: an infinite definable set $W(x)$ such that every complete type over a model M that contains the formula $W(x)$ has a unique non-algebraic extension to any $N \succ M$. Strongly (weakly) minimal theories are the best behaved ω-stable (superstable) theories.

A structure is said to have *trivial algebraic closure* if $\mathrm{acl(A)} = \bigcup_{\mathrm{a}\in\mathrm{A}} \mathrm{acl(a)}$ for every subset A. Laskowski [Las13] defines that *a τ-formula $\phi(\mathbf{z})$ is mutually algebraic* if there is an integer K so that $M \models \forall \mathbf{x} \exists^{\leqslant K} \mathbf{y} \phi(\mathbf{x}; \mathbf{y})$ for every proper partition $\mathbf{z} = \mathbf{x}\hat{\ }\mathbf{y}$. If every formula with parameters is equivalent to a Boolean combination of mutually algebraic formulas the *structure is mutually algebraic*. Laskowski proves that each model of a complete theory T is mutually algebraic if and only if T is weakly minimal and algebraic closure is trivial on models of T. An incomplete theory T is mutually algebraic if and only if every completion is.

Simplifying (abusing) the original notation we say a quantifier-free n-type p over a finite set $A \subset M$ is *m-large* in M if there are m pairwise disjoint realizations of p. And T has *unbounded arrays* if for arbitrarily large m and N there is an $M \models T$ such that for some finite A there are at least N m-large types over A. The authors show:

Theorem 4.2. [LT19b] *An incomplete theory T is mutually algebraic if and only if every atomic formula has uniformly bounded arrays in every model M of T.*

In [LT19a] these model theoretic notions support new results measuring speeds.

Theorem 4.3. *Let $\mathcal{H}$ be an hereditary class of finite structures in a language with finitely many relation symbols with maximal arity r.*

- *(poly/exp) For some k, for sufficiently large n, $|\mathcal{H}_n|$ is a sum of terms $p_i(n)i^n$ for $i < k$, where each $p_i(n)$ is a rational polynomial.*
- *(factorial) $|\mathcal{H}_n| = n^{(1-\frac{1}{k}-o(n))n}$ for some $k > 1$.*
- *(penultimate) $|\mathcal{H}_n|$ is caught between a function growing slightly slower than n^n and one with growth approximately $2^{n^{r-\epsilon}}$.*
- *(exponential in n^r) $|\mathcal{H}_n|$ grows as $2^{Cn^r+o(n^r)}$.*

Note that for the first two cases, the results are the same as in graphs. But the faster speeds depend on the maximal arity r of relations in the language. As in the graph case, there are examples showing the range of solutions in the penultimate case actually occur. The argument divides into two main cases. On the one hand the authors show theories with unbounded arrays (so not mutually algebraic by Theorem 4.2) fall into classes 3) and 4) and then analyze the distinction. On the other, they break the mutually algebraic theories into three classes; each of them yields speeds in one of classes 1), 2), and 3).

This extension of a result for graphs to arbitrary relational languages uses not only a refinement of the stability classification that gives very precise control over definable sets but invokes the precise model theoretic notion of interpretation to control the mutually algebraic structures by ones which are 'totally bounded'.

5. The value of formalization

This article focuses on understanding why model theory has so many applications across mathematics. Our choice of topics was restricted by space and time, the desire to emphasize the widening range of applications, and the need to avoid areas which where the technical mathematical prerequisites are huge. Two, more or less random examples of the last are [BHP, CN08]. The key is that formalizing a topic in mathematics both forces a clarification of concepts and allows the systematic investigation for analyzing the relations among theories. This key is also used by other areas of logic. Definability plays not only a central role in exploring relations within set theory (V=L, determinacy, etc.) but via the notion of Borel isomorphism in classifying problems arising in many areas. Two surveys of such applications are [Kec10, Ros11]. Computability theory has contributed to the general theory of randomness; the large literature was summarised in [Nie12]. In his retiring presedential address at the ASL meeting in Prague, Ulrich Kohlenbach, described proof-mining as 'local proof theory' [Koh]. In the general setting of abstract metric spaces, he describes results in fixed point and ergodic theory, convex optimization, geodesic geometry, Cauchy problems, game theory etc. General metatheorems are applied to the formal proof of theorem in specific areas that have been formalized in an appropriate way. This is analogous to applying results about ω-stable theories to differentially closed fields as well as compact complex manifolds.

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cellular equivalent TO mutually algebraic and omega categorical

T is k-cellular implies $|\mathcal{H}_n| \sim o(^{(1-1/k)n}$.

If T is not cellular then for all $\epsilon > 0$, $|\mathcal{H}_n| \geqslant n^{(1-\epsilon)n}$

i.e. $|\mathcal{H}_n| \geqslant O(n^n)$.

all Hrushovski construction are included in the 3 rd growth class although not mutually algebraic.

References

[Adl08] H. Adler. Introduction to theories without the independence property. *Archive for Mathematical Logic*, 2008. `http://www.logic.univie.ac.at/~adler/docs/nip.pdf`.

[ALMM19] Noga Alon, Roi Livni, Maryanthe Malliaris, and Shay Moran. Private pac learning implies finite littlestone dimension. In *51st Symposium on the Theory of Computing (STOC)*, pages 91–99. 2019. Translation and comments by T.L. Heath.

[Bal88] John T. Baldwin. *Fundamentals of Stability Theory*. Springer-Verlag, 1988.

[Bal00] John T. Baldwin. Finite and infinite model theory: An historical perspective. *Logic Journal of the IGPL*, 8, 2000.

[Bal18] John T. Baldwin. *Model Theory and the Philosophy of Mathematical Practice: Formalization without Foundationalism*. Cambridge University Press, 2018.

[BL71] John T. Baldwin and A.H. Lachlan. On strongly minimal sets. *Journal of Symbolic Logic*, 36:79–96, 1971.

[BR77] John T. Baldwin and B. Rose. $\aleph_0$-categoricity and stability of rings. *Journal of Algebra*, 45:1–17, 1977.

[BS97] John T. Baldwin and S. Shelah. Randomness and semigenericity. *Transactions of the American Mathematical Society*, 349:1359–1376, 1997.

[BBBW0a] József Balogh, Béla Béla Bollobas, and David Weinreich. The penultimate rate of growth for graph properties. *European Journal of Combinatorics*, 22:277–289, 200a.

[BHP] M. Bays, B. Hart, and A. Pillay. Universal covers of commutative finite morley rank groups. `https://www3.nd.edu/~apillay/papers/universalcovers-BHP.pdf`.

[BY14] I. Ben Yaacov. Model theoretic stability and definability of types, after A. Grothendieck. *Bull. Symb. Log.*, 20:491–496, 2014.

[BYBHU08] I. Ben Yaacov, A. Berenstein, C. W. Henson, and A. Usvyatsov. Model theory for metric structures. In Z. Chatzidakis and et al., editors, *Model theory with applications to algebra and analysis. Vol. 2*, volume 350 of *London Math. Soc. Lecture Note Ser.*, pages 315–427. Cambridge Univ. Press, Cambridge, 2008.

[BCR98] Jacek Bochnak, Michel Coste, and Marie-Francoise Roy. *Real Algebraic Geometry.* A Series of Modern Surveys in Mathematics. Springer-Verlag, 1998.

[Ber82] Chantal Berline. Déviation des types dans les corps algébriquement clos. In Bruno Poizat, editor, *Seminaire Théories Stable: 1980-82*, volume 3, pages 3.01–3–10. Poizat, 1982.

[Bha18] Siddharth Bhaskar. Thicket Density. Math Archiv:1702.03956, 2018.

[BGL+16] W. Boney, G. Grossberg, M. Lieberman, Jirí Rosický, and S. Vasey. μ-abstract elementary classes and other generalizations. *Journal of Pure and Applied Algebra*, 220:3048–66, 2016.

[Bou99] E. Bouscaren, editor. *Model Theory and Algebraic Geometry : An Introduction to E. Hrushovski's Proof of the Geometric Mordell-Lang Conjecture.* Springer-Verlag, Heidelberg, 1999.

[CN08] James Casale, Guy andFreitag and Joel Nagloo. Ax-Lindemann-Weierstrass with derivatives and the genus 0 Fuchsian groups. arXiv:1811.06583, 2008.

[CL17] Antoine Chambert-Loir. Review of 'o-minimality and diophantine geometry '. *The Bulletin of Symbolic Logic*, 23:115–117, 2017. DOI:10.1017/bsl.2017.3.

[CF18a] H. Chase and J. Freitag. Bounds in query learning. Math arXiv:1904.10122, 2018.

[CF18b] H. Chase and J. Freitag. Model theory and machine learning. Math arXiv:1801.06566, 2018.

[CF18c] H. Chase and J.xe43 Freitag. Model theory and combinatorics of banned sequences. Math arXiv:1904.10122, 2018.

[CKOS16] M. Chudnovsky, R. Kim, Sang-Il Oum, and P. Seymour. Unavoidable induced subgraphs in large graphs with no homogeneous sets. *J. Combinatorial Theory, Ser. B,*, 118, 2016. Math ArXiv 1504.05322.

[Cof91] Alberto Coffa. *The semantic tradition from Kant to Carnap: To the Vienna Station.* Cambridge University Press, 1991.

[Dri84] L. van den Dries. Remarks on Tarski's problem concerning $(r; +, \cdot, \exp)$. In G. Lolli, G. Longo, and G. Marcja, editors, *Logic Colloquium '82*, pages 97–121. Springer-Verlag, 1984. MR 86g:03052.

[EK19] Shlomo Eshel and Itay Kaplan. On uniform definability of types over finite sets for nip formulas. 2019 manuscript arXiv:1904.10336v1, 2019.

[FHL+] I. Farah, B. Hart, M. Lupini, L. Robert, A. Tikuisis, A. Vignati, and W. Winter. *Model theory of* C*-*algebras.* Memoirs of the American Mathematical Society. American Mathematical Society, Providence.

[Hac14] Ian Hacking. *Why is there a philosophy of mathematics at all?* Cambridge University Press, 2014.

[Har15] Michael Harris. *Mathematics without Apologies: Portrait of a Problematic Vocation.* Princeton University Press, 2015.

[HH12] Åsa Hirvonen and T. Hyttinen. Metric abstract elementary classes with perturbations. *Fund. Math.*, 217:123–170, 2012.

[HHM07] D. Haskell, E. Hrushovski, and H.D. MacPherson. *Stable domination and independence in algebraically closed valued fields.* Lecture Notes in Logic. Association for Symbolic Logic, 2007.

[HP00] E. Hrushovski and A. Pillay. Effective bounds for the number of transcendental points on subvarieties of semi-abelian varieties. *American Journal of Mathematics*, 122:451–463, 2000.

[JL10] Johnson, Hunter R. and Laskowski, Michael C. Compression schemes, stable definable families, and o-minimal structures. *Discrete & Computational Geometry*, 43:914–926, 2010.

[JW15] G.O. Jones and A.J. Wilkie. *O-Minimality and Diophantine Geometry*, volume 421 of *London Mathematical Society Lecture Note Series.* Cambridge University Press, 2015.

[Kec10] A. Kechris. *Global aspects of ergodic group actions.* Number 160 in Mathematical Surveys and Monographs. American Mathematical Society, 2010.

[Kha19a] K. Khanaki. Dividing lines in unstable theories and subclasses of Baire 1 functions. preprint: 2019 arXiv:1904.09486v5, 2019.

[Kha19b] K. Khanaki. Remarks on the strict order property. preprint: 2019 `https://arxiv.org/pdf/1902.05229v1.pdf`, 2019.

[Kha19c] K. Khanaki. Stability, NIP, and NSOP; model theoretic properties of formulas via topological properties of function spaces. to appear: MLQ (on Arxiv 2014) arXiv:1410.3339, 2019.

[KP18] K. Khanaki and A. Pillay. Remarks on the NIP in a model. *Math. Log. Quart.*, 64:429–434, 2018.

[Koh] U. Kohlenbach. Proof-theoretic methods in nonlinear analysis. In *Proceedings of the ICM, 2018*. to appear.

[Las92] Michael C. Laskowski. Vapnik-Chervonenkis classes of definable sets. *Journal of the London Mathematical Society*, 45:377–384, 1992.

[Las13] Michael C. Laskowski. Mutually algebraic structures and expansions by predicates. *Journal of Symbolic Logic*, 78:185–194, 2013.

[LT19a] Michael C. Laskowski and Caroline Terry. Jumps in speeds of hereditary properties in finite relational languages. preprint, 2019.

[LT19b] Michael C. Laskowski and Caroline Terry. Uniformly bounded arrays and mutually algebraic structuress. preprint, 2019.

[LR16] M. Lieberman and Jirí Rosický. Classification theory for accessible categories. *Journal of Symbolic Logic*, 81:151–165, 2016.

[LW86] Nick Littlestone and Manfred Warmuth. Relating data compression and learnability. Technical report, University of California, Santa Cruz, 1986.

[LS13] Roi Livni and Pierre Simon. Honest compressions and their application to compression schemes. In *Conference on Learning Theory*, pages 77–92. 2013.

[Mad18] P. Maddy. What do we want a foundation to do? Comparing set-theoretic, category-theoretic, and univalent approaches. In S. Centrone, D. Kant, and D. Sarikaya, editors, *Reflections on Foundations: Univalent Foundations, Set Theory and General Thoughts*. 2018.

[Mar00] D. Marker. Review of: Tame topology and o-minimal structures, by Lou van den Dries,. *Bulletin (New Series) of American Mathematical Society*, 37:351–357, 2000.

[Mar07] D. Marker. The number of countable differentially closed fields. *Notre Dame Journal of Formal Logic*, 48:99–113, 2007.

[MS15] M. Malliaris and S. Shelah. Keisler's order has infinitely many classes. *Israel Journal of Mathematics*, 2015. arXiv:1503.08341, to appear.

[MT18] M. Malliaris and C. Terry. On unavoidable induced subgraphs in large prime graphs. *J. Graph Theory*, 88:255–270, 2018.

[Nie12] Andres Nies. *Computability and Randomness*. Oxford Logic Guides. Oxford, Oxford, 2012.

[Pil16] A. Pillay. Generic stability and Grothendieck. *South American Journal of Logic*, 2:437–442, 2016.

[PS86] A. Pillay and C. Steinhorn. Definable sets in ordered structures I. *Transactions of the American Mathematical Society*, 295, 1986.

[Rob52] A. Robinson. On the application of symbolic logic to algebra. In *Proceedings of the International Congress of Mathematicians, Cambridge, Massachusetts, U.S.A., August 30-September 6,1950*, volume 1, pages 686–694. American Mathematical Society, Providence, Rhode Island, 1952. reprinted in Collected works Vol 1. Yale University press, 1971.

[Ros11] Christian Rosendal. Descriptive Classification Theory and Separable Banach Spaces. *Notices of the American Mathematical Society*, pages 1251–1262, 2011.

[Sca12] T. Scanlon. Counting special points: Logic, Diophantine geometry, and transcendence theory. *Bulletin of the American Mathematical Society (N.S.)*, 49:51–71, 2012.

[She15] S. Shelah. Dependent theories and the generic pair conjecture. *Communications in Contemporary Math*, 17:64 pp, 2015. Sh index 900: had circulated since early 2000's.

[She72] Saharon Shelah. A combinatorial problem; stability and order for models and theories in infinitary languages. *Pacific Journal of Math*, 41:247–261, 1972.

[She78] S. Shelah. *Classification Theory and the Number of Nonisomorphic Models*. North-Holland, 1978.

[SS88] S. Shelah and J. Spencer. Zero-one laws for sparse random graphs. *Journal of A.M.S.*, 1:97–115, 1988.

[Tar31] Alfred Tarski. Sur les ensemble définissable de nombres réels I. *Fundamenta Mathematica*, 17:210–239, 1931.

[Tar65] Alfred Tarski. *Introduction to Logic and to the Methodology of the Deductive Sciences*. Oxford: Galaxy Book, 1965. First edition: Oxford 1941.

[Vää12] Jouko Väänänen. Second order logic or set theory. *Bulletin of Symbolic Logic*, 18:91–121, 2012.

[Whi67] R. Whitley. An elementary proof of the Eberlein-Šmulian theorem. *Mathematische Annalen*, pages 116–119, 1967.

[Wig60] Eugene Wigner. The unreasonable effectiveness of mathematics in the natural sciences. *Communications in Pure and Applied Mathematics*, 13, 1960.

[Wil96] A. Wilkie. Model completeness results for expansions of the real field by restricted Pfaffian functions and exponentiation. *Journal of American Mathematical Society*, pages 1051–1094, 1996.

[Wil07] A. Wilkie. O-minimal structures. *Séminaire Bourbaki*, 985, 2007. `http://eprints.ma.man.ac.uk/1745/01/covered/MIMS_ep2012_3.pdf`.