Disjoint n-amalgamation IMP Conference Teheran

John T. Baldwin
University of Illinois at Chicago

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Varieties of amalgamation

background theme: the role of infinitary logic

Goals

1. study *n*-amalgamation toward
   1. existence/amalgamation of atomic models in uncountable cardinals.
   2. 0-1-laws
2. History, aec, and Neo-stability connections

Methods

1. Clarifying generalized Fraïssè constructions
2. amalg, dap, n-amalg, frugal
3. choice of submodel relation
4. excellence

Example and Background

Example

(Laskowski-Shelah)
τ_r has infinitely many r-ary relations R_n and infinitely many $r+1$-ary functions f_n.
We consider the class $\boldsymbol{K}_0^r$ of finite τ_r-structures (including the empty structure) that satisfy the following three conditions.

- The relations $\{R_n : n \in \omega\}$ partition the $(r+1)$-tuples;
- For every $(r+1)$-tuple $\boldsymbol{a} = (a_0, \ldots, a_r)$, if $R_n(\boldsymbol{a})$ holds, then $f_m(\boldsymbol{a}) = a_0$ for every $m \geq n$;
- There is no independent subset of size $r+2$.

Independent means with respect to subalgebra generation.

Bounding Cardinality

Fact

For every $k \in \omega$, if cl is a locally finite closure relation on a set X of size $\aleph_k$, then there is an independent subset of size $k+1$.

Proof. By induction on k. When $k = 0$, take any singleton not included in $\mathrm{cl}(\emptyset)$. Assuming the Fact for k, given any locally finite closure relation cl on a set X of size $\aleph_{k+1}$, fix a cl-closed subset $Y \subseteq X$ of size $\aleph_k$ and choose any $a \in X \setminus Y$. Define a locally finite closure relation cl_a on Y by $\mathrm{cl}_a(Z) = \mathrm{cl}(Z \cup \{a\}) \cap Y$. It is easily checked that if $B \subseteq Y$ is cl_a-independent, then $B \cup \{a\}$ is cl-independent.

Conclusion and Question

Let $\boldsymbol{K}^r$ be the collection of all structures A such that every finite substructure of A is in $\boldsymbol{K}_0^r$.
Let $\mathrm{cl}_A(X)$ be the closure relation on a model A defined by closure under functions.

Conclusion

There is no model of the class $\boldsymbol{K}^r$ of cardinality greater than $\aleph_r$.

Question

Is there a model of the class $\boldsymbol{K}^r$ of cardinality $\aleph_r$?

Axiomatize

ϕ_r

- $\bigwedge_{m \neq n}(R_n(\mathbf{x}) \rightarrow \neg R_m(\mathbf{x}))$
- $\bigvee_n R_n(x_0 \ldots x_r)$
- $(\forall \mathbf{x}) R_n(\boldsymbol{a}) \rightarrow \bigwedge_{m \geq n} f_m(\mathbf{x}) = x_0;$
- There is no independent subset of size $r+2$.
  A slightly more complicated sentence in $L_{\omega_1,\omega}$

Theorem

ϕ_r has no model of cardinality greater than $\aleph_r$

A sharper version is about first order model theory.

The translation

Theorem

[Chang/Lopez-Escobar] Let ψ be a sentence in $L_{\omega_1,\omega}$ in a countable vocabulary τ. Then there is a countable vocabulary τ' extending τ, a first order τ'-theory T, and a countable collection of τ'-types Γ such that reduct is a 1-1 map from the models of T which omit Γ onto the models of ψ.

The proof is straightforward. E.g., for any formula ψ of the form $\bigwedge_{i<\omega} \phi_i$, add to the language a new predicate symbol $R_\psi(\mathbf{x})$. Add to T the axioms

$$(\forall \mathbf{x})[R_\psi(\mathbf{x}) \rightarrow \phi_i(\mathbf{x})]$$

for $i < \omega$ and omit the type $p = \{\neg R_\psi(\mathbf{x})\} \cup \{\phi_i : i < \omega\}$.

Δ-complete

Let Δ be a fragment of $L_{\omega_1,\omega}$ that contains ϕ.
ϕ is Δ-complete if for every $\psi \in \Delta$
$\phi \models \psi$ or $\phi \models \neg\psi$.
(If Δ is omitted we mean complete for $L_{\omega_1,\omega}$.)

small=complete

Let Δ be a fragment of $L_{\omega_1,\omega}$ that contains ϕ.

Definition

A τ-structure *M* is Δ-small if *M* realizes only countably many Δ-types (over the empty set).

'small' means $\Delta = L_{\omega_1,\omega}$

Generalized Scott's theorem

A structure satisfies a complete sentence of $L_{\omega_1,\omega}$ if and only if it is small.

Reducing complete to atomic

The models of a complete sentence in $L_{\omega_1,\omega}$ can be represented as:
$\boldsymbol{K}$ is the class of atomic models (realize only principal types) of a complete first order theory (in an expanded language).

The general framework

Vocabulary matters

Generalized Fraıssè constructions are very sensitive to:

1. relations versus functions
2. cardinality of the vocabulary

Notation

Let τ be a countable vocabulary (possibly with functions).

Notation

1. $S_\omega(A)$ is the collection of finite substructures of A.
2. $S^*_\omega(A)$ is the collection of finitely generated substructures of A.
3. $S^*_\omega(\boldsymbol{K}) = \boldsymbol{K}^*_0$ is the collection of finitely generated structures in $\boldsymbol{K}$.

Definition

$\boldsymbol{K}$ is locally finite if every finitely generated algebra in $\boldsymbol{K}$ is finite.
I.E. $\boldsymbol{K}_0 = \boldsymbol{K}^*_0$.

Universal classes of models

Function symbols are allowed on this slide without loss of generality.

Definition

A class $(\boldsymbol{K}, \leq)$ of τ-structures and the relation $\leq$ as ordinary substructure that is closed under

1. substructure ($S(\boldsymbol{K}) = \boldsymbol{K}$)
2. unions of increasing chains ($\lim(\boldsymbol{K}) = \boldsymbol{K}$)

is called a *universal class*.

Theorem

A class $(\boldsymbol{K}, \leq)$ of τ-structures is a universal class iff

1. $S(\boldsymbol{K}) = \boldsymbol{K}$
2. $S(A) \subset \boldsymbol{K}$ implies $A \in \boldsymbol{K}$.

More Notation

Definition

Let $(\boldsymbol{K}_0, \leq)$ denote a class of finite τ-structures and let $(\hat{\boldsymbol{K}}, \leq)$ denote the associated (closure under substructure and unions) locally finite universal class.

Logic vrs algebra

Let τ be a countable vocabulary.

Theorem

A universal class $(\boldsymbol{K}, \leq)$ of τ-structures is axiomatized by a universal sentence in $L_{|\boldsymbol{K}_0|^+,\omega}$.
Proof: Allow only finitely generated substructures that are in $\boldsymbol{K}_0^*$.

Note a universal class does not have to be locally finite.
$fg(x) = gf(x) = x$, no cycles.
But it is for a relational language.

Finite Vocabulary Matters

Relations only

finite language Any class of finite relational structures closed under substructures is the class of finite models of a set of universal first order sentences

countable language. Any class of finite relational structures closed under substructures is the class of finite models of a set of universal $L_{\omega_1,\omega}$ sentences

In either case the proper class of finite or infinite models is a universal class.

Generic structures

Definition

Let $(\boldsymbol{K}_0, \leq)$ denote a class of finite τ-structures and let $(\hat{\boldsymbol{K}}, \leq)$ denote the associated (closure under substructure and unions) locally finite universal class.

1. A model $M \in \hat{\boldsymbol{K}}$ is finitely $\boldsymbol{K}_0$*-homogeneous* or *rich* if for all finite $A, B \in \boldsymbol{K}_0$, every embedding $f : A \to M$ extends to an embedding $g : B \to M$. We denote the class of rich models in $\hat{\boldsymbol{K}}$ as $\boldsymbol{K}^R$.
2. The model $M \in \hat{\boldsymbol{K}}$ is *generic* if M is rich and M is an increasing union of a chain of finite substructures, each of which are in $\boldsymbol{K}_0$.

All rich models are (∞, ω)-omega equivalent.

Generalized Fraïssé

The vocabulary may be infinite and include function symbols.

Lemma

If a class $(\boldsymbol{K}_0, \leq)$ satisfies amalgamation, JEP, and has countably many elements, then there is unique countable generic model, which is rich.

Note that we do *not* need 'closed under substructure'. $\leq$ does not have to be substructure. The new results here are about classes with $\leq$ as substructure and closed under substructure.

The first order theory of the generic is $\aleph_0$-categorical iff there are only finitely many models in each finite cardinality.

The importance of Vocabulary Choice

Definition

A class $\boldsymbol{K}_0$ of finite structures in a countable vocabulary is *separable* if

1. $(\boldsymbol{K}_0, \leq)$ satisfies amalgamation;
2. For each $A \in \boldsymbol{K}_0$, there is a formula $\phi_A(\mathbf{x})$ such that in any $M \in \hat{\boldsymbol{K}}$, $M \models \phi_A(\mathbf{b})$ if and only if $\mathbf{b}$ enumerates a substructure of M that is isomorphic to A.

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Lemma

Suppose $\boldsymbol{K}_0$ is a class of finite τ-structures that is closed under substructure, satisfies JEP, and is separable. Then the generic M is an atomic model of $Th(M)$. Moreover, $\boldsymbol{K}^R = \mathbf{At}$, i.e., every rich model N is an atomic model of $Th(M)$.

k-configurations

Definition

For $k \geq 1$, a *k-configuration* is a sequence $\overline{M} = \langle M_i : i < k \rangle$ of models (not isomorphism types) from $\boldsymbol{K}$. We say $\overline{M}$ *has power* λ if $\| \bigcup_{i<k} M_i \| = \lambda$. An *extension* of $\overline{M}$ is any $N \in \boldsymbol{K}$ such that every M_i is a substructure of N.

Informal: (λ, k)*-disjoint amalgamation*

Any sequence of k models, at least one with λ elements, has a common extension, which properly extends each.

(λ, k)-disjoint amalgamation

Definition

Fix a cardinal $\lambda = \aleph_\alpha$ for $\alpha \geq -1$. We define the notion of a class $(\boldsymbol{K}, \leq)$ having (λ, k)-*disjoint amalgamation* in two steps:

1. $(\boldsymbol{K}, \leq)$ has $(\lambda, 0)$-disjoint amalgamation if there is $N \in \boldsymbol{K}$ of power λ;
2. For $k \geq 1$, $(\boldsymbol{K}, \leq)$ has $(\leq \lambda, k)$-disjoint amalgamation if it has $(\lambda, 0)$-disjoint amalgamation and every k-configuration $\overline{M}$ of cardinality $\leq \lambda$ has an extension $N \in \boldsymbol{K}$ such that every M_i is a proper substructure of N.

For $\lambda \geq \aleph_0$, we define $(< \lambda, k)$-disjoint amalgamation by: has $(\leq \mu, k)$-disjoint amalgamation for each $\mu < \lambda$.

Extending (λ, k)-disjoint amalgamation

Extending (λ, k)-disjoint amalgamation

Fix locally finite $(\boldsymbol{K}, \leq)$ with jep.

Proposition

For all cardinals $\lambda \geq \aleph_0$ and for all $k \in \omega$, if $\boldsymbol{K}$ has $(< \lambda, k+1)$-disjoint amalgamation, then it also has $(\leq \lambda, k)$-disjoint amalgamation.

Push-through construction

Fact

Suppose $\mathbf{C}$ is a configuration that is closed in a configuration $\overline{M}$ and N is an extension of $\mathbf{C}$. Then, there exists a copy N' that is isomorphic to N over $\mathbf{C}$, with $N' \cap \overline{M} = \mathbf{C}$.

Proof Sketch

$\overline{M}$ is a $(\leq \lambda, k)$-configuration.

1. Choose $\mathbf{C}\alpha = \langle C^0_\alpha, \ldots C^{k-1}_\alpha \rangle$ such that each C^i_α is closed in M_i and $\bigcup_\alpha C^i_\alpha = M_i$
   by letting $Y_\alpha = \{a_\alpha\} \cup \bigcup_i C^i_\alpha$
   and $C^i_{\alpha+1} = \mathrm{cl}(Y \cap M_i)$.
2. Choose D_α extending $\mathbf{C}_\alpha$ with $(D_\alpha \cap \bigcup_i M_i) = \emptyset$

Getting models

Theorem

Suppose $1 \leq r < \omega$ and $\boldsymbol{K}_0$ has the $(< \aleph_0, r+1)$-disjoint amalgamation property.
Then for every $0 \leq s \leq r$, $(\hat{\boldsymbol{K}}, \leq)$ has the $(\leq \aleph_s, r-s)$-disjoint amalgamation property.

In particular, $\hat{\boldsymbol{K}}$ has models of power $\aleph_r$.

Moreover, if there are only countably many isomorphism types in $\boldsymbol{K}_0$, then rich models of power $\aleph_r$ exist and the class $\boldsymbol{K}^R$ also has $(\leq \aleph_s, r-s)$-disjoint amalgamation.

Getting Rich Models

Lemma

Fix $\lambda \geq \aleph_0$. If ***K*** has $(< \lambda, 2)$-disjoint amalgamation and has at most λ isomorphism types of finite structures, then

1. every $M \in \boldsymbol{K}$ of power λ can be extended to a rich model $N \in \boldsymbol{K}$, which is also of power λ.
2. and consequently there is a rich model in λ^+.

The Theorem

Theorem

For every $r \geq 1$, the class $\mathbf{At^r}$ satisfies:

1. there is a model of size $\aleph_r$, but no larger models;
2. every model of size $\aleph_r$ is maximal, and so 2-amalgamation is trivially true in $\aleph_r$;
3. disjoint 2-amalgamation holds up to $\aleph_{r-2}$;
4. 2-ap fails in $\aleph_{r-1}$.

More technically, amalgamation for elementary submodels in $\hat{\boldsymbol{K}}^r$ also fails in $\aleph_{r-1}$.

The Amalgamation spectrum

The *finite amalgamation spectrum* of an abstract elementary class $\boldsymbol{K}$ is the set $X_{\boldsymbol{K}}$ of $n < \omega$ (for $\aleph_n \geq LS(\boldsymbol{K})$) and $\boldsymbol{K}$ satisfies amalgamation in $\aleph_n$. There are many examples where the finite amalgamation spectrum of a complete sentence of $L_{\omega_1,\omega}$ is either $\emptyset$ or ω.
This is the first example of a complete sentence an aec where the spectra was not: all, none, or just $\aleph_0$.

Question

Can the amalgamation spectrum of complete sentence of $L_{\omega_1,\omega}$ have a proper alternation?

Hjorth's theorem

Theorem: Hjorth

There is a sequence of countable families Φ_α of complete sentences of $L_{\omega_1,\omega}$, such that some $\phi \in \Phi_\alpha$ has a model in $\aleph_\alpha$ and no bigger.

Theorem: BKL

There is a sequence ϕ_n of complete sentences of $L_{\omega_1,\omega}$, such that ϕ_n has a model in $\aleph_n$ and no bigger.

Question

Can the second theorem be extended to all countable α?

How many models of sentences with bounded spectrum?

Fact

If $(\boldsymbol{K}, \prec_{\boldsymbol{K}})$ has the amalgamation property in κ then models of cardinality κ^+ can be amalgamated over models of cardinality κ.

Corollary

If all models in cardinality κ^+ are maximal and $\boldsymbol{K}$ had ap in κ then there are at most 2^κ models in κ^+.

Question

Is there any complete sentence in $L_{\omega_1,\omega}$ with such behavior?

Other notions

1. Kruckman and pseudo-finiteness.
2. Excellence and Notop; Zilber/Shelah
3. characterizing cardinals and Vaught conjecture
4. Homology and n-ap
5. Further Spectrum problems

n-ap and pseudo-finiteness

Definition

The theory T is pseudofinite if every sentence in T has a finite model.

2 causes of psuedofiniteness in countably categorical theories

1. tame behavior - algebraic e.g. totally categorical theories
2. randomness

this is an observation of examples by Kruckman.

2 outrageous conjectures

1. (Kim-Pillay) Every countably categorical pseudofinite theory is simple.
   False T_{feq} random equivalence relations is a counterexample. (asserted Shelah, detailed proof Kruckman)
2. (Kruckman) Every countably categorical pseudofinite theory is $NSOP_1$.

Kruckman's theorem

Context: Fix a countably categorical theory in a countable relational theory T that arises as a Fraïssè limit.
Define n-amalgamation by the amalgamation of types in T.

Question

What is the connection with the n-ap discussed earlier? Relations vrs functions.

Theorem (Kruckman)

Every relational countably categorical theory with disjoint n-ap for all n is pseudofinite.

Random graph with edge probablity $n^{-\alpha}$

The 0-1 law holds for irrational α: the Spencer-Shelah random graph.
First fully correct proof Baldwin-Shelah Randomness and Genericity `http://homepages.math.uic.edu/~jbaldwin/model11.html` near bottom; see also stable generic structures (with Shi) and
Laskowski: A simpler axiomatization of the Shelah-Spencer almost sure theories
for the closest to the framework here.
`http://www.math.umd.edu/~laskow/Pubs/ss.pdf`

Kruckman pointed out that the $\boldsymbol{K}_\alpha$ class fails 3-dap the n-ap is not necessary for pseudo-finiteness.

Pseudo-finiteness for Functional vocabularies

Pseudo-finiteness

The first order theories T^r of the classes $\mathbf{At}^\mathbf{r}$ are not pseudo-finite; there are surjective non-injective functions.

Question

How does one assign probability measures to obtain 0-1 laws for such vocabularies?
Does (uniform) local finiteness matter?

Compton in the 80's developed an extensive probability for classes of algebra that were closed under direct product and had unique factorization into indecomposables. Compare Kruckman's algebraic case.
`http://www.math.uwaterloo.ca/~snburris/htdocs/density.html`

Excellence

Let $\boldsymbol{K}$ be an ω-stable class of atomic models.
Shelah and Zilber: For Shelah the independence is the sense of ω-stability. This was the origin of the notion of otop. For Zilber, there is an ambient geometry and so the independence is closure in that geometry.

Definition

A set of $\boldsymbol{K}$-structures $\overline{N} = \langle N_u : u \subsetneq k \rangle$ is a -system if it is a directed system of $\boldsymbol{K}$-structures with cardinality λ indexed by the proper subsets of k.

Definition

The class $\boldsymbol{K}$ has (λ, k) excellence ((λ, k)-good) if every independent (λ, k)-system has a unique prime amalgam. For Shelah, excellent is $(\aleph_0, n)$ excellence for every n.

Consequences of Excellence

1. Shelah: An excellent class that is categorical up to $\aleph_\omega$ is categorical in all uncountable powers.
2. Zilber: A quasi-minimal excellent class is categorical in all uncountable powers.
3. Shelah ($2^{\aleph_n} < 2^{\aleph_{n+1}}$) An atomic class that has at most $2^{\aleph_n}$ models in $\aleph_{n+1}$ for all n is excellent.

Homology groups of types in model theory

http://pages.towson.edu/akolesni/research_papers.html

Abstract: Amalgamation functors and boundary properties in simple theories

John Goodrick, Byunghan Kim, Alexei Kolesnikov
This paper continues the study of generalized amalgamation properties. Part of the paper provides a finer analysis of the groupoids that arise from failure of 3-uniqueness in a stable theory. We show that such groupoids must be abelian and link the binding group of the groupoids to a certain automorphism group of the monster model, showing that the group must be abelian as well.

Work by Hrushovski, Kolesnikov, Kim, Goodrick

Other Spectra

JEP

B-Koerwien-Souldatos

If $\langle \lambda_i : i \leq \alpha < \aleph_1 \rangle$ is a strictly increasing sequence of characterizable cardinals whose models satisfy JEP($< \lambda_0$), there is an $L_{\omega_1,\omega}$-sentence ψ such that

1. The models of ψ satisfy JEP($< \lambda_0$), while JEP fails for all larger cardinals and AP fails in all infinite cardinals.
2. There exist $2^{\lambda_i^+}$ non-isomorphic maximal models of ψ in λ_i^+, for all $i \leq \alpha$, but no maximal models in any other cardinality; and
3. ψ has arbitrarily large models.

Maximal models

B-Souldatos

There are complete sentences of $L_{\omega_1,\omega}$, which

1. maximal models in κ and κ^+.
2. Assume for simplicity that $2^{\aleph_0} > \aleph_\omega$. For each $n \in \omega$, there is a complete $L_{\omega_1,\omega}$-sentence ϕ'_n with maximal models in cardinalities $2^{\aleph_0}, 2^{\aleph_1}, \ldots, 2^{\aleph_n}$.
3. Assume κ is a homogeneously characterizable cardinal and for simplicity let $2^{\aleph_0} \geq \kappa$. Then there is a complete $L_{\omega_1,\omega}$-sentence ϕ_κ with maximal models in cardinalities 2^λ, for all $\lambda \leq \kappa$.

The big gap

Theorem. B-Boney

The Hanf number for Amalgamation is at most the first strongly compact cardinal

The best known lower bound is $\aleph_\omega$.

Homogeneous Characterizability

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The complete sentence ϕ with countable model M homogenously characterizes κ if

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Theorem (Gao)

If countable structure has a set of absolute indiscernibles, there is an $L_{\omega_1,\omega}$ equivalent model in $\aleph_1$.

Mergers

Mergers

1. Let θ be a complete sentence of $L_{\omega_1,\omega}$ and suppose M is the countable model of θ and $V(M)$ is a set of absolute indiscernibles in M such $M - V(M)$ projects onto $V(M)$. We will say θ is a *receptive* sentence.
2. For any sentence ψ of $L_{\omega_1,\omega}$, the *merger* of ψ and θ is the sentence $\chi = \chi_{\theta,\psi}$ obtained by conjoining with θ, $\psi \restriction N$.
3. For any model M_1 of θ and N_1 of ψ we write $(M_1, N_1) \models \chi$ if there is a model with such a reduct.

Getting receptive models

Suppose $\boldsymbol{K}_0$ is a class of finite τ structures with disjoint amalgamation and θ_0 is the Scott sentence of the generic.

Construction

Add to τ unary predicates U, V and binary P.
Require that the predicates U and V partition the universe and restrict the relations of τ to hold only within the predicate V. We set $\boldsymbol{K}_1$ as the set of finite τ_1-structures (V_0, U_0, P_0) where $V_0 \restriction \tau \in \boldsymbol{K}$ and P_0 is the graph of a partial function from V_0 into U_0.

To amalgamate, use disjoint amalgamation in the V-sort; extend the projection by the union of the projections. If the disjoint amalgamation contains new points, project them arbitrarily to U. Let $\mathcal{M}$ be the generic model for $\boldsymbol{K}_1$.

A back and forth argument shows $U(\mathcal{M})$ is a set of absolute indiscernibles.

Ancient history

Morley asks:
if every ω-stable theory has a model in each infinite cardinality with a set of absolute indiscernibles of the same size as the model.

Applying merger

Theorem: (Hjorth, B-Friedman-Koerwien-Laskowski)

There is a receptive sentence that characterizes (has only maximal models) $\aleph_1$.

Corollary: (B-Friedman-Koerwien-Laskowski)

If there is a counterexample to Vaught's conjecture there is one that has only maximal models in $\aleph_1$.

crux: Disjoint amalgamation

Unexplored territory

1. What is the stability classification of the new first order theories constructed to find $L_{\omega_1,\omega}$- examples?
2. Are any psuedo-finite?