Infinitary Model Theory: Covers of Abelian Varieties

John T. Baldwin

August 16, 2009

Short Exact Sequences

$$0 \to N \to V \stackrel{\exp}{\to} \mathbb{A} \to 1. \tag{1}$$

Short Exact Sequences

$$0 \to N \to V \overset{\exp}{\to} \mathbb{A} \to 1. \tag{1}$$

3 cases

1. ${}^{\perp}N$. (Baldwin-Eklof-Trlifaj: APAL 07)
2. Axiomatize in $L_{\omega_1,\omega}$ to guarantee standard kernel: $N = \mathcal{Z}^n$.
   1. $\mathbb{A}$ is $\aleph_1$-free. Complicated examples. (Baldwin-Shelah: JSL 08)
   2. $\mathbb{A}$ is a commutative algebraic group. (Zilber et al). This talk.

Acknowledgements

This talk reports work of Shelah, Zilber, Gavrilovich, and Bays. Few proofs are new but we try to establish a context.

Goal:

What is the role of 'logic' and 'language' in formalizing mathematical notions?

Acknowledgements

This talk reports work of Shelah, Zilber, Gavrilovich, and Bays. Few proofs are new but we try to establish a context.

Goal:

What is the role of 'logic' and 'language' in formalizing mathematical notions?

Logic

First order or Infinitary

Language (Vocabulary)

What are the basic relations of the structure being analyzed.

The 70's

Stability theory developed

1. abstractly with the stability classification
2. concretely by finding the stability class of important mathematical theories and using the techniques of the abstract theory.
3. Concretely by trying to characterize the structures of a given sort that possess specific model theoretic properties.

Cherlin-Zilber Conjecture

A simple group with finite Morley rank is an algebraic group over an algebraically closed field.

Cherlin-Zilber Conjecture

A simple group with finite Morley rank is an algebraic group over an algebraically closed field.

1. Groups may have extra structure.
2. finite Morley rank is non-trivially equivalent to: ranked in the sense of Borovik-Nesin.
3. Finite Morley rank modifies ‘a structure’ not ‘a theory’.

Hierarchy for $L_{\omega_1,\omega}$

1. complete
2. ω-stable
3. excellent

Superstable means ???

Completeness???

Vaught's test

Let T be a set of first order sentences with no finite models, in a countable first order language.

If T is κ-categorical for some $\kappa \geq \aleph_0$, then T is complete.

Needs upward and downward Lowenheim-Skolem theorem for theories

We search for a substitute in $L_{\omega_1,\omega}$.

Small

Let Δ be a fragment of $L_{\omega_1,\omega}$ that contains ϕ.

Definition

A τ-structure M is Δ-small for L^* if M realizes only countably many Δ-types (over the empty set).

Small

Let Δ be a fragment of $L_{\omega_1,\omega}$ that contains ϕ.

Definition

A τ-structure M is Δ-small for L^* if M realizes only countably many Δ-types (over the empty set).

Definition

An $L_{\omega_1,\omega}$-sentence ϕ is Δ-'not so big', if each model of ϕ is small (realizes only countably many complete Δ-types over the empty set).

Small

Let Δ be a fragment of $L_{\omega_1,\omega}$ that contains ϕ.

Definition

A τ-structure M is Δ-small for L^* if M realizes only countably many Δ-types (over the empty set).

Definition

An $L_{\omega_1,\omega}$-sentence ϕ is Δ-'not so big', if each model of ϕ is small (realizes only countably many complete Δ-types over the empty set).

Definition

An $L_{\omega_1,\omega}$-sentence ϕ is Δ-small if there is a set X countable of complete Δ-types over the empty set and each model realizes some subset of X.

'small' means $\Delta = L_{\omega_1,\omega}$

Small implies complet(able)

If M is small then M satisfies a complete sentence.

Small implies complet(able)

If M is small then M satisfies a complete sentence.

If ϕ is small then there is a complete sentence ψ_ϕ such that:

$\phi \wedge \psi_\phi$ have a countable model.

So ψ_ϕ implies ϕ.

But ψ_ϕ is not in general unique (real examples).

The $L_{\omega_1,\omega}$-Vaught test

Shelah If ϕ has an uncountable model *M* that is Δ-small for every countable Δ and ϕ is κ-categorical then ϕ is implied by a complete sentence ψ with a model of cardinality κ.

Keisler If ϕ has $< 2^{\aleph_1}$ models of cardinality $\aleph_!$, then for every countable Δ, ϕ is Δ-not so big.
I.e. each model is Δ-small for every countable Δ.

The $L_{\omega_1,\omega}$-Vaught test

Shelah If ϕ has an uncountable model M that is Δ-small for every countable Δ and ϕ is κ-categorical then ϕ is implied by a complete sentence ψ with a model of cardinality κ.

Keisler If ϕ has $< 2^{\aleph_1}$ models of cardinality $\aleph_!$, then for every countable Δ, ϕ is Δ-not so big.
I.e. each model is Δ-small for every countable Δ.

So we effectively have Vaught's test.
But only in $\aleph_1$!
And only for completability!

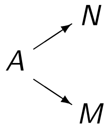
Reducing $L_{\omega_1,\omega}$ to 'first order'

The models of a complete sentence in $L_{\omega_1,\omega}$ can be represented as:
K is the class of atomic models (realize only principal types) of a first order theory (in an expanded language).

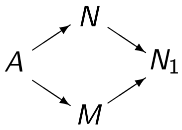
We study $S_{at}(A)$ where $A \subset M \in \mathbf{K}$ and
$p \in S_{at}(A)$ means Aa is atomic if a realizes p.

AMALGAMATION PROPERTY

The class **K** satisfies the *amalgamation property* if for any situation with $A, M, N \in \mathbf{K}$:

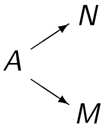


there exists an N_1 such that

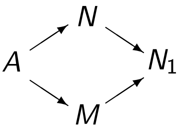


SET AMALGAMATION PROPERTY

The class **K** satisfies the set amalgamation property if for any situation with $M, N \in \mathbf{K}$ and $A \subset M, A \subset N$:



there exists an N_1 such that



Is there a difference?

For a complete first order theory, Morley taught us:

There is no difference.

Is there a difference?

For a complete first order theory, Morley taught us:

There is no difference.

Tweak the language and we obtain set amalgamation.

(Tweak: put predicates for every definable set in the language)

There is a difference!

Zilber's examples of quasiminimal excellent classes have amalgamation over models but the interesting examples do not have set amalgamation.

ψ is categorical in all infinite cardinalities but no model is $\aleph_1$-homogeneous because there is a countably infinite maximal indiscernible set.

Quasiminimality

A class $(\mathbf{K}, \mathrm{cl})$ is *quasiminimal* if cl is a combinatorial geometry which satisfies on each $M \in \mathbf{K}$:

1. there is a unique type of a basis,
2. a technical homogeneity condition: $\aleph_0$-homogeneity over $\emptyset$ and over models.
3. Closure of countable sets is countable

Theorem

A quasiminimal class is $\aleph_1$-categorical.

$L_{\omega_1,\omega}$: The General Case

Quasiminimality is the 'rank one' case

Any geometry has a notion of independent n-system.

In the more general setting

Splitting gives an analogous notion of independent n-system.
And thus a more general notion of excellence.

ω-stability

Definition

ϕ is ω-stable if for every countable model of ϕ, there are only countably many types over M that are realized in models of ϕ (i.e. $|S_{at}(M)| = \aleph_0$).

Recall, these are first order types that are realized in atomic sets.

Essence of Excellence

Let **K** be the class of models of a sentence of $L_{\omega_1,\omega}$.

K is excellent

K is ω-stable and any of the following equivalent conditions hold.

For any finite independent system of countable models with union C:

1. $S_{at}(C)$ is countable.
2. There is a unique primary model over C.
3. The isolated types are dense in $S_{at}(C)$.

Quasiminimal Excellence

means
Quasiminimal and excellent.

QM EXCELLENCE IMPLIES CATEGORICITY EVERYWHERE

QM Excellence implies by a direct limit argument:

Lemma

An isomorphism between independent X and Y extends to an isomorphism of $\mathrm{cl}(X)$ and $\mathrm{cl}(Y)$.

This gives categoricity in all uncountable powers if the closure of finite sets is countable.

What excellence buys

Theorem: Shelah (ZFC)

If an atomic class $\mathbf{K}$ is excellent and has an uncountable model then

1. $\mathbf{K}$ has models of arbitrarily large cardinality;
2. Categoricity in one uncountable power implies categoricity in all uncountable powers.

$\mathcal{Z}$-Covers of Algebraic Groups

Definition A $\mathcal{Z}$-cover of a commutative algebraic group $\mathbb{A}(\mathcal{C})$ is a short exact sequence

$$0 \rightarrow Z^N \rightarrow V \stackrel{\exp}{\rightarrow} \mathbb{A}(\mathcal{C}) \rightarrow 1. \tag{2}$$

where V is a $\mathbb{Q}$ vector space and $\mathbb{A}$ is an algebraic group, defined over k_0 with the full structure imposed by $(\mathcal{C}, +, \cdot)$ and so interdefinable with the field.

Axiomatizing $\mathcal{Z}$-Covers: first order

Let $\mathbb{A}$ be a commutative algebraic group over an algebraically closed field F.
Let T_A be the first order theory asserting:

1. $(V, +, f_q)_{q \in \mathbb{Q}}$ is a $\mathbb{Q}$-vector space.
2. The complete first order theory of $\mathbb{A}(F)$ in a language with a symbol for each k_0-definable variety (where k_0 is the field of definition of $\mathbb{A}$).
3. exp is a group homomorphism from $(V, +)$ to $(\mathbb{A}(F), \cdot)$.

Axiomatizing Covers: $L_{\omega_1,\omega}$

Add to T_A
$\Lambda = \mathcal{Z}^N$ asserting the kernel of exp is standard.

$$(\exists \mathbf{x} \in (\exp^{-1}(1))^N)(\forall y)[\exp(y) = 1 \rightarrow \bigvee_{\mathbf{m} \in \mathcal{Z}^N} \Sigma_{i<N} m_i x_i = y]$$

Categoricity Problem

Is $T_A + \Lambda = \mathcal{Z}^N$ categorical in uncountable powers?

paraphrasing Zilber:

> *Categoricity would mean the short exact sequence is a reasonable 'algebraic' substitute for the classical complex universal cover.*

Two Kinds of Issues

1. What is the proper language to formalize such an 'algebraic substitute'?
2. What are the algebraic consequences of various model theoretic restrictions on the families of examples formalized in a specific way ?

Specifying a family

1. $\mathcal{Z}$-covers
2. E-covers (Replace $\mathbb{Q}$ by the endomorphism group of $\mathbb{A}$.)
3. Gavrilovich's language L_A.

Stabilizing Sequences

Let K be a field and $W \subset K^r$ defined over K.

1. A sequence associated with W, $\mathbf{W}$, is a family of varieties (defined over K) $W^{1/m}$ such that $(W^{1/mk})^k = W^{1/m}$, each $W^{1/m}$ is a minimal K-variety.
2. A sequence stabilizes with respect to $p(\mathbf{x})$, an r-type over the empty set if there exists an ℓ such that for every m, there is a unique K-definable variety V with $V^m = W^{1/\ell}$ and such that $p(\mathbf{x})$ and $\langle \exp(x_1/m\ell), \ldots \exp(x_r/m\ell) \rangle \in V$ is consistent.
3. Sequences stabilize over K if all such sequences stablize.

Pseudo-generating sequences

Let $\mathbb{V} = (V, A) \models T_A$. $\langle \tau_1, \ldots, \tau_N \rangle \in V$ is a *pseudogenerating tuple* of $\Lambda(V)$ if for each $m \in \mathcal{Z}$:

$$n_1\tau_1 + \ldots, +n_N\tau_N \in m\Lambda \text{ iff } \gcd(n_1, \ldots, n_N) \in m\mathcal{Z}.$$

We write $\mathrm{PG}^N(\tau_1, \ldots, \tau_N)$.

Why Pseudo-generating sequences

lemma

Let $\mathbb{V} = (V, A) \models T_A$ with $V \approx V_0 \oplus (\Lambda(V) \otimes \mathbb{Q})$. If $\langle \tau_1, \ldots, \tau_N \rangle \in V$ is a pseudogenerating tuple of $\Lambda(V)$ and

$$V' = V_0 \oplus \mathbb{Q}\tau_1 \oplus \ldots \oplus \mathbb{Q}\tau_N$$

then $\mathbb{V}' = (V', A)$ is a model of T_A with standard kernel.

Smallness Implies

If $T_A + \Lambda(V) = \mathcal{Z}^N$ is small,

1. Sequences stabilize over K if K is finitely generated over $\mathbb{Q}$;

This argument is specific to $\mathcal{Z}$-covers; it relies on

1. the role of pseudogenerators
2. the invariant factor theorem holds in $\mathcal{Z}$.

Smallness Implies

If $T_A + \Lambda(V) = \mathcal{Z}^N$ is small,

2. every model of $T_A + \Lambda(V) = \mathcal{Z}^N$ is atomic in L^*;
3. $T_A + \Lambda(V) = \mathcal{Z}^N$ admits elimination of quantifiers in L^*;
4. every countable model of $T_A + \Lambda(V) = \mathcal{Z}^N$ + 'infinite dimension' is ω-homogeneous.

JB $T_A + \Lambda(V) = \mathcal{Z}^N$ has a finite number of completions that have uncountable models.

Aside: Characteristic p

[Bays, Zilber] Consider

$$0 \to Z[1/p] \to V \to F_p^* \to 0.$$

where $Z[1/p]$ is the localization at p and F_p^* is an infinite dimensional algebraically closed field of characteristic p.

$T_A + \Lambda(V) = \mathcal{Z}^N$ is not small. There are $2^{\aleph_0}$ completions - distinct minimal models.

The theories must be analyzed separately; each is quasiminimal excellent.

ω-stable Implies

If $T_A + \Lambda(V) = \mathcal{Z}^N$ is ω-stable

1. $T_A + \Lambda(V) = \mathcal{Z}^N$ admits elimination of quantifiers in L^*.
2. Every countable model of $T_A + \Lambda(V) = \mathcal{Z}^N$ + 'infinite dimension' is ω-homogeneous over elementary submodels
3. Sequences stabilize over K if K is a countable acf.

ω-stability: Kummer theory

F is a countable algebraically closed field and $b_1, \ldots b_k$ are multiplicatively independent over F.

For any m, let $F_m = F(b_1^{\frac{1}{m}}, \ldots, b_k^{\frac{1}{m}})$.
For fixed ℓ, let $G_m = \mathrm{Gal}(F_{m \cdot \ell}/F_\ell)$.

If $T_A + \Lambda(V) = \mathcal{Z}^N$ is ω-stable.

$$G_m \approx \mathbb{A}_m(F)^k \approx (\mathcal{Z}/m\mathcal{Z})^{Nk}$$

Group automorphisms are field automorphisms

Algebraic Formulations of Excellence

Let $\mathcal{S} = \{F_s : s \subset n\}$ be an independent n-system of algebraically closed fields contained in a suitable monster $\mathcal{M}$. Denote the subfield of $\mathcal{M}$ generated by $(\bigcup_{s \subset n} F_s)$ as k.

If $T_A + \Lambda(V) = \mathcal{Z}^N$ is excellent,

Canonical completions

$$\mathcal{A}(k) = A^n \oplus \prod_{s \subset n} \mathcal{A}(F_s)$$

where A^n is a free Abelian group.

Almost Quasiminimal Excellence

Let **K** be a class of L-structures which admit a function cl_M mapping $X \subseteq M$ to $\mathrm{cl}_M(X) \subseteq M$ with a distinguished sort U.
K is quasiminimal if:

1. cl_M satisfies is a monotone idempotent closure operator with $\mathrm{cl}_M(X) \in \mathbf{K}$
2. For $X, Y \subset U$, $\mathrm{cl}(X) \cap \mathrm{cl}(Y) = \mathrm{cl}(X \cap Y)$.
3. cl_M satisfies exchange on U.
4. $M = \mathrm{cl}_M(U)$.
5. The usual homogeneity conditions are satisfied.

If in addition, the excellence condition holds for special subsets of U, the class is almost quasiminimal excellent.

ω-stable

Every ω-stable cover is $\aleph_1$-categorical.
But, unlike the first order case,
this doesn't automatically imply categoricity in all cardinals
-not even the continuum.

Excellence

The following are equivalent under VWGCH ($2^{\aleph_n} < 2^{\aleph_{n+1}}$)

1. The $\mathcal{Z}$-cover of $\mathbb{A}$ is categorical in all uncountable κ.
2. The $\mathcal{Z}$-cover of $\mathbb{A}$ is categorical in all $\aleph_n$ for $n < \omega$.
3. The $\mathcal{Z}$-cover of $\mathbb{A}$ is almost quasiminimal excellent.
4. The $\mathcal{Z}$-cover of $\mathbb{A}$ is almost quasiminimal excellent and $\mathbb{A}$ satisfies the algebraic conditions for excellence: has canonical completions.

Where did the set theory come from?

VWGCH: $2^{\aleph_n} < 2^{\aleph_{n+1}}$ for $n < \omega$.

Where did the set theory come from?

VWGCH: $2^{\aleph_n} < 2^{\aleph_{n+1}}$ for $n < \omega$.

VWGCH: Shelah 1983

An atomic class **K** that has at least one uncountable model and is categorical in $\aleph_n$ for each $n < \omega$ is excellent.

What is true?

From the algebraic side,
If $\mathbb{A}$ is

$\mathcal{Z}$-covers

1. $(\mathcal{C}, \cdot)$ then quasiminimal excellent (Zilber)
2. $(\tilde{F}_p, \cdot)$ then not small.
   Each completion is quasiminimal excellent. (Bays-Zilber)
3. elliptic curve w/o cm then ω-stable (Gavrilovich/Bays)
4. elliptic curve w cm then not ω-stable.

Paths and Categoricity

Gavrilovich writes:

> *We propose in this work a model theoretic structure which contains an abstract substitute for the notion of path. The substitute must possess the familiar properties of paths appearing a topological context, rich enough to imply a useful theory of paths; in particular, they must determine the notion of path on an abstract algebraic variety uniquely up to isomorphism. Model theory provides a framework to formulate the uniqueness property in a mathematically rigorous fashion. Following Zilber we use the notion of categoricity in uncountable cardinals on non-elementary classes.*

Gavrilovich continues

In this work we introduce a language L_A which is appropriate for describing the basic homotopy properties of algebraic varieties in their complex topology, and prove some partial results towards stability and categoricity of associated structures in that language.

What else is true?

E-covers$/L_A$

1. elliptic curve w cm: ω-stable as an $\mathrm{End}(E)$-module (Gavrilovich)
2. higher dimensional: open

Relies on number theoretic results of Serre, Bashmakov

Two approaches to Infinitary Logic

Analysis above is direct study of $L_{\omega_1,\omega}$.

But Abstract Elementary Classes provide a more general approach.

GALOIS TYPES: Algebraic Form

Suppose **K** has the amalgamation property. Then there is a monster model $\mathcal{M}$.

Definition

Let $M \in \mathbf{K}$, $M \prec_{\mathbf{K}} \mathbb{M}$ and $a \in \mathbb{M}$. The Galois type of a over M is the orbit of a under the automorphisms of $\mathbb{M}$ which fix M.

We say a Galois type p over M is realized in N with $M \prec_{\mathbf{K}} N \prec_{\mathbf{K}} \mathbb{M}$ if $p \cap N \neq \emptyset$.

Tameness

tame is short for $(\aleph_0, \infty)$ tame:

Distinct Galois types differ on a countable submodel.

Grossberg and VanDieren focused on the idea of studying 'tame' abstract elementary classes.

Tameness-Algebraic form

Suppose **K** has the amalgamation property.

K is (χ, μ)-tame if for any model M of cardinality μ and any $a, b \in \mathcal{M}$:

Tameness-Algebraic form

Suppose **K** has the amalgamation property.

K is (χ, μ)-tame if for any model M of cardinality μ and any $a, b \in \mathcal{M}$:

If for every $N \prec_{\mathbf{K}} M$ with $|N| \leq \chi$ there exists $\alpha \in \mathrm{aut}_{\mathrm{N}}(\mathcal{M})$ with $\alpha(a) = b$,

then there exists $\alpha \in \mathrm{aut}_{\mathrm{M}}(\mathcal{M})$ with $\alpha(a) = b$.

Consequences of Tameness

Suppose **K** has arbitrarily large models and amalgamation.

Theorem (Grossberg-Vandieren)

*If $\lambda > \mathrm{LS}(\mathbf{K})$, **K** is λ^+-categorical and $(\lambda, < \infty)$-tame then **K** is categorical in all $\theta \geq \lambda^+$.*

Theorem (Lessmann)

*If **K** with $\mathrm{LS}(\mathbf{K}) = \aleph_0$ is $\aleph_1$-categorical and $(\aleph_0, \infty)$-tame then **K** is categorical in all uncountable cardinals*

AQE and covers

Assume: $T_A + \Lambda = \mathcal{Z}^N$ is ω-stable (VWGCH)

The following are equivalent

1. $T_A + \Lambda = \mathcal{Z}^N$ is $(\aleph_0, \infty)$-tame.
2. $T_A + \Lambda = \mathcal{Z}^N$ is almost quasiminimal excellent.
3. $T_A + \Lambda = \mathcal{Z}^N$ is categorical in all uncountable cardinalities.

Are there $\mathbb{A}$ whose covers are ω-stable but not excellent?
There are ϕ that are $\aleph_1$-categorical but not tame (Baldwin-Kolesnikov).

Mordell-Weil Theorem

For $\mathbb{A}$ a smooth elliptic curve,
If k is a finitely generated extension of $\mathbb{Q}$, $\mathbb{A}(k)$ is a finitely generated abelian group.

Smallness and Mordell-Weil

$\mathbb{A}_\ell(k)$ is the k-rational points of order ℓ.
$\mathbb{A}_{\rm tor}(k)$ is the k-rational points of any finite order.

For *any* commutative algebraic group $\mathbb{A}$:

If $T_A + \Lambda(V) = \mathcal{Z}^N$ is small.

If k is finitely generated over $\mathbb{Q}$, $\mathbb{A}_{\rm tor}(k)$ is finite.

Reprise: $L_{\omega_1,\omega}$-Vaught test

Naive translation is false

An $\aleph_1$-categorical sentence in $L_{\omega_1,\omega}$ need not be complete.
(e.g. finite dimension)

From models to sentences

A complete sentence can be assigned to small models.
But 'small' is a nontrivial attribute.

Reprise: Two Directions

1. Investigate the model theoretic properties in $L_{\omega_1,\omega}$ of various covers in the appropriate algebraic language.
2. Find an appropriate language to draw general conclusions about covers from model theoretic hypotheses.