Three notions of geometry

John T. Baldwin
University of Illinois at Chicago

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Three Themes

1. Axiomatic geometry: Euclid, Hilbert, Tarski, Wu, Szmielew ...
2. Hrushovski constructions and matroids
3. geometries in sense one that have combinatorial geometries in sense 2

Euclid-Hilbert Axioms

Synthetic vrs analytic geometry

Synthetic Geometry

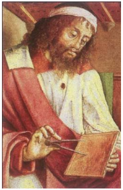
Develop geometry systematically from

1. a short list of primitive notions
2. postulates about these notions
3. introduce more sophisticated notions by definition

Analytic Geometry

Analytic geometry is the algebra of R and R^2 and R^3. It's hypotheses are thus whatever one assumes about the reals-complete archimedean ordered field. It is not really a matter of *proving* theorems, but of *calculating* results. (Craig Smorynski)

Euclid-Hilbert formalization 1900:

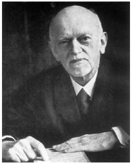
Euclid  Hilbert 

The Euclid-Hilbert (the Hilbert of the Grundlagen) framework has the notions of axioms, definitions, proofs and, with Hilbert, models.
But the arguments and statements take place in natural language.

Euclid uses diagrams essentially; Hilbert uses them only heuristically.

For Euclid-Hilbert logic is a means of proof.

Hilbert-Gödel-Tarski-Vaught formalization 1918-1956:

Hilbert  Gödel  Tarski  Vaught

In the Hilbert (the founder of proof theory)-Gödel-Tarski framework, logic is a mathematical subject.

There are explicit rules for defining a formal language and proof. Semantics is defined set-theoretically.

First order logic is complete. The theory of the real numbers is complete and easily axiomatized. The first order Peano axioms are not complete.

Goals

We describe our vocabulary and postulates in a way immediately formalizable as a first order theory T_{Euclid}.

We will show:

1. T_{Euclid} directly accounts for proportionality and area of polygons.
2. We have to extend T_{Euclid} using methods of contemporary model theory to have formulas for arc length and area.

There is no appeal to the axioms of Archimedes or Dedekind.

Vocabulary

The fundamental notions are:

1. two-sorted universe: points (P) and lines (L).
2. Binary relation $I(A, \ell)$:
   Read: a point is incident on a line;
3. Ternary relation $B(A, B, C)$:
   Read: B is between A and C (and A, B, C are collinear).
4. quaternary relation, $C(A, B, C, D)$:
   Read: two segments are congruent, in symbols $\overline{AB} \approx \overline{CD}$.
5. 6-ary relation $C'(A, B, C, A', B', C')$: Read: the two angles $\angle ABC$ and $\angle A'B'C'$ are congruent, in symbols $\angle ABC \approx \angle A'B'C'$.

τ is the vocabulary containing these symbols.

Note that I freely used defined terms: collinear, angle and segment, in giving the reading.

First order *fully* geometric Postulates

1. Incidence postulates
2. the betweenness postulates (after Hilbert) (yield dense linear ordering of any line).
3. One congruence postulate: SSS
4. parallel postulate
   That, if a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles.

(14 first order geometric axioms) Return

Incidence postulates

Euclid's first 3 postulates in modern language

1. **Postulate 1** Given any two points there is a (unique) line segment connecting them.

$$(\forall p_1, p_2)(\exists \ell)[I(p_1, \ell) \wedge I(p_2, \ell)]$$

$$(\forall p_1, p_2)(\forall \ell_1, \ell_3)[(I(p_1, \ell_1) \wedge I(p_2, \ell_1) \wedge I(p_1, \ell_2) \wedge I(p_2, \ell_2)]) \rightarrow \ell_1 = \ell_2$$

2. **Postulate 2** Any line segment can be extended indefinitely (in either direction).
3. **Postulate 3** Given a point and any segment there is a circle with that point as center whose radius is the same length as the segment.

Section 4: From Geometry to Numbers

From geometry to numbers

We want to define the addition and multiplication of numbers. We make three separate steps.

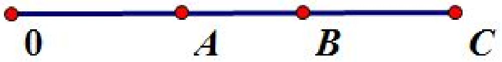
1. identify the collection of all congruent line segments as having a common 'length'. Choose a representative segment OA for this class.
2. define the operation on such representatives.
3. Identify the length of the segment with the end point A. Now the multiplication is on points. And we define the addition and multiplication a little differently.

Today we do step 2; the variant of step 3 is a slight extension.

Defining addition

Adding line segments

The sum of the line segments OA and OB is the segment OC obtained by extending OB to a straight line and then choose C on OB extended (on the other side of B from A) so that $OB \cong AC$.



This gives the structure of $(N, <, +)$
Addition is associative with identity but no inverse.

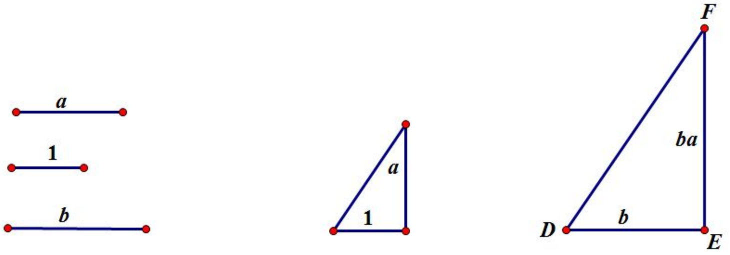
Defining Multiplication

Consider two segment classes *a* and *b*. To define their product, define a right triangle[1] with legs of length *a* and *b*. Denote the angle between the hypoteneuse and the side of length *a* by α.

Now construct another right triangle with base of length *b* with the angle between the hypoteneuse and the side of length *b* congruent to α. The length of the vertical leg of the triangle is *ab*.

[1] The right triangle is just for simplicity; we really just need to make the two triangles similar.

Defining segment Multiplication diagram



Note that we must appeal to the parallel postulate to guarantee the existence of the point F.

Parallel Postulate

Von Staudt, Pasch
Draw similarity picture; Hrushovski group diagram.

Obtaining the field properties

Addition and multiplication are associative and commutative.
There are additive and multiplicative units and inverses.
Multiplication distributes over addition.

(The negative numbers are omitted to avoid complication.)
This particular definition is due to Robin Hartshorne.

Moulton: 1902
' Therefore Desargues's theorem is not a consequence of Hilbert's axioms I 1-2, II, III, IV 1-5, V.'

Lacks SAS

Toward adding π

Describing π

Add to the vocabulary τ a new constant symbol π. Let i_n (c_n) be the perimeter of a regular n-gon inscribed (circumscribed) in a circle of radius 1.
Add for each n,

$$i_n < 2\pi < c_n$$

to give a collection of sentences $\Sigma(\pi)$.

A first order theory for a vocabulary including a binary relation $<$ is *o-minimal* if every 1-ary formula is equivalent to a Boolean combination of equalities and inequalities.

Anachronistically, the o-minimality (every definable subset is a finite union of intervals) of the reals is a main conclusion of Tarski.

The theory with π

Metatheorem

The following set T_π of axioms is *first order complete* for the vocabulary τ along with the constant symbols $0, 1, \pi$.

1. the postulates of a Euclidean plane.
2. A family of sentences declaring every odd-degree polynomial has a root.
3. $\Sigma(\pi)$

If the field is Archimedean there is only one choice for the interpretation of π. If not, there may be many but they are all first order equivalent.

Circumference

Definition

The theory $T_{\pi,C}$ is the extension by definitions of the $\tau \cup \{0, 1, \pi\}$-theory T_π obtained by the explicit definition $C(r) = 2\pi r$.

As an extension by explicit definition, $T_{\pi,C}$ is a complete first order theory.

The Circumference formula

Since

1. by similarity, $i_n(r) = ri_n$ and $c_n(r) = rc_n$,
2. by our definition of multiplication, $a < b$ implies $ra < rb$.
3. and by the approximations of π by Archimedes

Metatheorem

In $T_{\pi,C}$, $C(r) = 2\pi r$ is a circumference function. That, for any r, $C(r)$ is bounded below and above by the perimeter of inscribed and circumscribed regular polygons of a circle with radius r.

Incidence geometries

Hilbert's two sorted framework has been the basis for a large area of mathematics concerned with (finite) geometries, designs, etc.

Definition

A plane is a (two-sorted) system of points and lines satisfying:

- (A) every pair of points determines a unique line;
- (B) every line contains at least two points;
- (C) there exist at least three non-collinear points.

Note that if two points determine a line, two lines can intersect in at most one point. One obtains a projective plane by requiring in addition that every pair of lines intersect in exactly one point.
This is much more general than Hilbert's incidence axioms. While ordered fields are of course infinite, much if not most of the research on incidence systems is concerned with finite structures.

Tarski

Four important characteristics

1. Single sorted
2. ordered field
3. on points not segments
4. The first order theory is complete; the field is real-closed.

Artin/Hrushovski

Rather than working with points or segments, think of the point A on the line as coding the transformation by the segment of length OA and consider the algebra of those transformations.
The bi-interpretability is explicit in Artin 1957.

Hrushovski picks up this notion in a more abstract setting, replacing the translation functions by germs of functions.

Two Diagrams

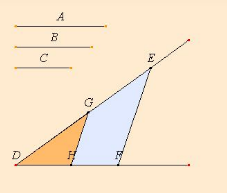


Figure: The fourth proportional

Two Diagrams

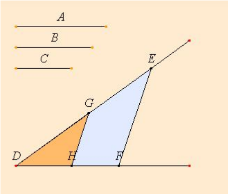


Figure: The fourth proportional

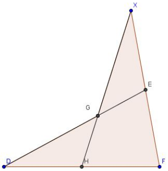


Figure: The group configuration

3 notions of complex geometry

1. loci of linear equations in $\mathcal{C}^2$
2. loci of polynomials in $\mathcal{C}^2$
3. the combinatorial geometry imposed by algebraic closure, since $\mathcal{C}$ is strongly minimal

Two questions

1. What is a synthetic formulation for 1)?
2. Why does 3) have a major impact on 2)?

Wu, Szmielev: axiomatize via parallelism, orthogonality etc. rather than congruence.

Classifying strongly minimal sets and their geometries

The trichotomy

Zilber Conjecture

Every strongly minimal first order theory is

1. disintegrated
2. vector space-like
3. non-locally modular : conjecturally field-like

Disintegrated is often called trivial but see next section.
After Hrushovski's example the third class splits:

not locally modular = 1-ample

1. not 2-ample
2. ample for various $n \geq 2$; all n implies interprets a field

I will discuss geometries that are 1-ample but not 2-ample.

What happened to the condition: A geometry is 'n-based' if $a \in \mathrm{cl}(X)$ implies) there exists $X_0 \subset X$, $a \in \mathrm{cl}(X_0$?

Disintegrated but hardly trivial

Axiomatic analysis

Axiomatic analysis studies behavior of fields of functions with operators but *without* explicit attention in the formalism to continuity but rather to the algebraic properties of the functions. The function symbols of the vocabulary act on the functions being studied; the functions are elements of the domain of the model.

Differential Algebra

The axioms for *differentially closed fields* are a first order sentences in the vocabulary $(+, \times, 0, 1, \partial)$ (where ∂f is interpreted as the derivative The first order formulation is particularly appropriate because many of the fields involved are non-Archimedean.

Differentially closed fields

Blum provided the first axiomatizations and she showed the characteristic 0 theory was ω-stable.

By Shelah's uniqueness theorem for prime models over sets for ω-stable theories, differential closures are unique up to isomorphism. But Kolchin, Rosenlicht and Shelah independently showed they are not minimal (by constructing in different ways a strongly minimal set with trivial geometry).

Differentially closed fields II

Hrushovski and Itai lay out as an application of 'Shelah's philosophy' the following model theoretic fact (based on Buechler's Dichotomy) fundamental to the study of differential fields:

'an *algebraically* closed differential field K is *differentially* closed if every strongly minimal formula over K has a solution in K'.

Even more, by the general theory of superstability, their study reduces to the study of strongly minimal sets and definable simple FMR groups that are associated with strongly minimal sets.

Painlevè equations

Consider the theory of differentially closed fields with constant field the complex numbers $\mathcal{C}$.
In 1900 Painlevè began the study of nonlinear second order ordinary differential equations (ODE) satisfying the Painlevè property (no movable singularities). In general such an equation has the form

$$y'' = f(y, y')$$

with f a rational function (i.e. in $\mathbb{C}(t)$).
He classified such equations into 50 canonical forms and showed that 44 of these were solvable in terms of 'previously known' functions.
Here is a canonical form for the third of the remaining classes; the Greek letters are the constant coefficients; t is the independent variable satisfying $y' = 1$ and the goal is to solve for y.

$$P_{III}(\alpha, \beta, \gamma, \delta) : \qquad \frac{d^2y}{dt^2} = \frac{1}{y}\left(\frac{dy}{dt}\right)^2 - \frac{1}{t}\frac{dy}{dt} + \frac{1}{t}(\alpha y^2 + \beta) + \gamma y^3 + \frac{\delta}{y}$$

Problem 1

Show that a generic equation (i.e. the constant coefficients are algebraically independent) of each of the six forms is irreducible.

For this, one must take on the logicians task: ‘What does *not reducible* mean’?

By reducible Painlevè meant, solvable from ‘known functions’.

The Japanese school clarified ‘solvable’ to mean, roughly speaking: generated from solutions to order one ordinary differential equations (ODE) and algebraic functions through a fixed family of constructions (integration, exponentiation, etc.).

In the formal setting, this is equivalent to showing that:
If an order two differential equation is strongly minimal; then there can be no *classical solutions*.

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If an order two differential equation is strongly minimal; then there can be no *classical solutions*.
This problem was solved (without the formalization) in each of the six cases by the Japanese school (led by Umemura) in the late 1980's.

Problem II

Conjecture

If there are *n* algebraically independent solutions of a generic strongly minimal Painlevè equation then that set along with its first derivatives is also algebraically independent.

The new step is to invoke the Zilber trichotomy which holds for differentially closed fields.
To reduce to a disintegrated strongly minimal set. Pillay and Nagloo show the other alternatives are impossible in this situation and indeed that the strongly minimal set is $\aleph_0$-categorical.
Using the geometric triviality (from the Zilber trichotomy) heavily and tools from the Japanese analysts they show the required algebraic independence.
In contrast, Freitag and Scanlon show the order three algebraic differential equation over $\mathbb{Q}$ satisfied by the analytic j-function defines a *non*-$\aleph_0$-categorical strongly minimal set with trivial forking geometry.

Some 1-ample but not 2-ample geometries

Geometries not just combinatorial geometries

We consider structures that are geometries of dimension 2 in a roughly Tarskian sense in their natural language.
In addition they have a combinatorial geometry of infinite dimension given model theoretically.

Projective Planes: B (1994/95)

There is an almost strongly minimal (rank 2) projective plane.
It has the Lenz-Barlotti class with the least structure.
In particular, the ternary function of the coordinatizing field cannot be decomposed into an 'addition' and a 'multiplication'.

The geometry was some what flukish. A graph with no 4-cycles is constructed. It can be interpreted as two sorted projective geometry

Designs and planes

Definition

A *pairwise balanced design, or PBD*, is a set P of points with a collection B of subsets of P (called blocks) such that any two points lie in a unique block.
Note that in any PBD, two blocks can intersect in at most one point.

Definition

A *plane* is collection of points and lines such that 2 points determine a line; consequently two lines intersect in at most one point.

So planes are incidence structures that correspond to PBD.
PBDs where all blocks have the same size $|A| = k$ are known as balanced incomplete block designs (BIBDs) of index $\lambda = 1$, as $2-(v, k, 1)$ designs, and as Steiner systems $S(2, k, v)$.

2-sorted vrs 1-sorted

In a two-sorted formulation, i.e. points and lines, clearly no strongly minimal theory has both infinitely many points and infinitely many lines.

Even in 1-sort, there cannot be two lines with infinitely many points.

Note that this does not preclude bi-interpretability between 1-sorted and 2-sorted descriptions. Because, interpretations do not need to preserve Morley rank; they can change arity.

Strongly minimal planes

Let the vocabulary τ contain a single ternary predicate R.
Our axioms T will imply that in any $M \models T$, interpreting R as collinearity gives a plane (PBD).

Constraints on a strongly minimal plane

If each line in π has at least three points.

i) There is no infinite line in π.
ii) The plane π is neither affine nor projective.
iii) There is a finite bound N on the number of points on a line.

That is, every 1-sorted strongly minimal plane is a PBD.

Work in progress (B-Paolini

Theorem

For each $k \geq 4$, there is a strongly minimal plane such that each line has size k.
The theory is 1-ample but not 2-ample.

Not quite standard Hrushovski construction

Definition

For $A \in \mathbf{K}_0^*$, let:

$$\delta(A) = |A| - \sum_{\ell \in L(A)} \mathbf{n}_A(\ell).$$

,

Not all primitives have a unique base. There are special cases distinguishing extending lines for larger primitives.

The design theory comes in to try to find $2^{\aleph_0}$ theories.