UNCOUNTABLE CATEGORICITY OF LOCAL ABSTRACT ELEMENTARY CLASSES WITH AMALGAMATION

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ABSTRACT. We give a complete and elementary proof of the following upward categoricity theorem: Let $\mathcal{K}$ be a local abstract elementary class with amalgamation, arbitrarily large models, and countable Löwenheim-Skolem number. If $\mathcal{K}$ is categorical in $\aleph_1$ then $\mathcal{K}$ is categorical in every uncountable cardinal. In particular, this provides a new proof of the upward part of Morley's theorem in first order logic without any use of prime models or heavy stability theoretic machinery (dependence relations, Morley rank, etc.).

INTRODUCTION

Shelah's categoricity conjecture asserts that for any abstract elementary class $\mathcal{K}$, there is a cardinal κ such that if $\mathcal{K}$ is categorical in some $\lambda > \kappa$ then $\mathcal{K}$ is categorical in all larger cardinals. In general this question remains wide open. But under the additional hypothesis that $\mathcal{K}$ has the amalgamation property, Shelah proved an approximation [Sh394]: If $\mathcal{K}$ is categorical in cofinally many cardinals then it is eventually categorical. He shows this by showing that if $\mathcal{K}$ is categorical in some cardinal λ beyond an appropriate Hanf number H_2 (see [Ba2]), then $\mathcal{K}$ is categorical in every cardinal between H_2 and λ. This was a seminal paper. However it was difficult to read, contained many gaps, a few inaccuracies, and much material which was not needed for the most expeditious proof of the result. Shelah has circulated a substantially revised version. This paper has sparked a flood of work in the last few years ([Ba, Ba2, GrVD1, GrVD2, GrVD3, GrVV, Hy, Le2, VD2]). Baldwin clarified some of the role of Ehrenfeucht-Mostowski models in [Ba2] and the more sophisticated uses in [Ba]. Grossberg and VanDieren [GrVD2], abstracted the notion of tame from Shelah's argument and proved that if $\mathcal{K}$ is tame and $\mathcal{K}$ is categorical in λ and λ^+ with $\lambda > \mathsf{LS}(\mathcal{K})$ then $\mathcal{K}$ is categorical in all cardinals beyond λ^+. Fix for the moment the following terminology; a Galois type $p \in S(M)$ is *extendible* if it has a nonalgebraic extension to every N containing M; p is *fully minimal* if there is at most one such nonalgebraic extension to each N containing M. Now the moral we take from [GrVD2] is the following Theorem: If $\mathcal{K}$ is λ-categorical and there is a fully minimal extendible type in $S(M)$ (with $|M| = \lambda$) such that there is no (p, λ) Vaughtian pair, then $\mathcal{K}$ is categorical in all cardinals

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greater than λ. To get such a p which is fully minimal and extendible depends on tameness. There are several strategies to find such a p with no Vaughtian pair ([Sh394, GrVD2, Ba, Le2]); each paper uses its own variant on the notions that we dubbed 'fully minimal' and 'extendible' for this survey. We introduce another variant here. The upwards categoricity result is improved to assume categoricity in only a single cardinal λ^+, with $\lambda \geq \mathsf{LS}(\mathcal{K}) = \aleph_0$ in [Le2], and later $\lambda > \mathsf{LS}(\mathcal{K})$ in [Ba, GrVD3]. The most important tool for these extensions is the proof of the result sketched in [Sh394]: below the categoricity cardinals: chains of μ saturated models of length at most μ are saturated.

An important theme stemming from both [Sh394] and [GrVD1] is to study abstract elementary classes with strong 'compactness' condition on Galois types. The notion of a *local* abstract elementary class (AEC) is stronger than *tame*; we discuss the distinction in the text. In this paper, for countable languages we prove upward categoricity transfer from categoricity in $\aleph_1$ for local AEC without any reliance on the unions of saturated models lemma. In fact, the argument here is self-contained. The importing of 'quasiminimality' and 'big' from the study of atomic models to this more general context and the use of superlimits is due to Lessmann. With these techniques we avoid any reference to a notion of independence. This work and that of [HV] argue for the study of local AEC. The recent work of [GrVV] considers the case $\aleph_0 < \lambda = \mathsf{LS}(\mathcal{K})$ by making stronger 'model theoretic' hypotheses and employing much heavier machinery.

The paper is organized as follows. Section 0 contains some well-known facts (most of them due to Shelah) about abstract elementary classes with amalgamation, whose often simplified and complete proofs can be found in [Ba]. Section 1 is devoted to some facts about big and quasiminimal types. Section 2 contains the proof of the main theorem.

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0. Preliminaries

In this section, we recall some of the results of Shelah on abstract elementary classes. For more details and further context the readers are advised to consult Baldwin's online book [Ba], where all these facts and examples can be found, or Grossberg's expository paper [Gr].

We assume throughout that $(\mathcal{K}, \prec_{\mathcal{K}})$ is an *abstract elementary class* (AEC) in the language L, namely, $\mathcal{K}$ is a class of L-structures, equipped with a partial ordering $\prec_{\mathcal{K}}$ on the L-structures in $\mathcal{K}$ satisfying the following conditions:

(1) $\mathcal{K}$ is closed under isomorphism;

(2) If $M, N \in \mathcal{K}$ and $M \prec_{\mathcal{K}} N$ then $M \subseteq N$ *i.e.*, M is an L-substructure of N;
(3) There is a least cardinal $\mathsf{LS}(\mathcal{K})$ such that for all $N \in \mathcal{K}$ and $A \subseteq N$ there is $M \prec_{\mathcal{K}} N$ containing A of size at most $|A| + \mathsf{LS}(\mathcal{K})$.
(4) If $M, N, M^* \in \mathcal{K}$ with $M \subseteq N$ and $M, N \prec_{\mathcal{K}} M^*$ then $M \prec_{\mathcal{K}} N$;
(5) $\mathcal{K}$ is closed under Tarski-Vaught chains: Let $(M_i : i < \lambda)$ be a $\prec_{\mathcal{K}}$-increasing and continuous chain of models of $\mathcal{K}$. Then $\bigcup_{i<\lambda} M_i \in \mathcal{K}$. Also $M_0 \prec_{\mathcal{K}} \bigcup_{i<\lambda} M_i$ and further, if $M_i \prec_{\mathcal{K}} N \in \mathcal{K}$ for each $i < \lambda$, then $\bigcup_{i<\lambda} M_i \prec_{\mathcal{K}} N$.

The cardinal $\mathsf{LS}(\mathcal{K})$ in (3) is called the *Löwenheim-Skolem number*. We will assume throughout this paper that $\mathsf{LS}(\mathcal{K}) = \aleph_0$ and that $\mathcal{K}$ has no finite models, but neither of these requirements is necessary for the results in the first section. Since we do not have formulas, we cannot phrase the Tarski-Vaught test; (4) and (5) are the consequences we need from it. Notice that none of the conditions permit us to construct models of large cardinality. We list a few examples, as well as non-examples.

Examples 0.1. (1) The class $\mathcal{K}$ of models of a first order theory T in the language L with $M \prec_{\mathcal{K}} N$ if M is an elementary submodel of N forms an abstract elementary class with $\mathsf{LS}(\mathcal{K}) = |L| + \aleph_0$.
(2) The class of saturated models of a superstable first order theory under elementary substructure forms an abstract elementary class with $\mathsf{LS}(\mathcal{K})$ the first stability cardinal.
(3) The class of models of a first order theory in the language L omitting a prescribed set of L-types with elementary substructure forms an abstract elementary class with $\mathsf{LS}(\mathcal{K}) = |L| + \aleph_0$.
(4) More generally, let $\psi \in L_{\lambda,\omega}$ and consider a fragment $\mathcal{A}$ of $L_{\lambda,\omega}$ containing ψ. Then the class $\mathcal{K}$ of models of ψ with $M \prec_{\mathcal{K}} N$ if M is an $L_{\mathcal{A}}$-elementary substructure of N forms an abstract elementary class with $\mathsf{LS}(\mathcal{K}) = |\mathcal{A}|$.
(5) Let n be an integer above the arity of any relation or function in the language L. Then the class of L^n-theories with L^n-elementary submodel is an abstract elementary class with $\mathsf{LS}(\mathcal{K}) \leq |L| + \aleph_0$.
(6) The class of reducts to L of models of a theory T^* in an expanded language T^* under L-elementary substructure *does not* form in abstract elementary class in general, as (5) may fail. For example, the class of free groups in the language of groups does not form an abstract elementary class under L-elementary substructure (or even infinitary-elementary substructure). In fact, a famous example of Silver shows that such classes may be categorical in a cofinal sequence of cardinals, and not categorical in another cofinal sequence of cardinals.
(7) Any class of models closed under elementary equivalence with first order elementary submodel *does* form an abstract elementary class.

(8) The class of models of an $L_{\infty,\omega}$-theory with $L_{\infty,\omega}$-substructure *does not* form an abstract elementary class in general: it may not have a Löwenheim-Skolem number.
(9) The previous two examples have more concrete exemplars. The class of Artinian (descending chain condition) commutative rings with unit becomes an AEC under elementary submodel. (See [Ba1].) But the class of Noetherian (ascending chain condition) commutative rings with unit can never be an abstract elementary class. Hodges ([Ho] 11.5.5) shows such rings are not a PC_Δ class with omitting types and any AEC is such a class by Shelah's presentation theorem ([Sh88], [Ba]).

We say that $f : M \to N$ is a $\mathcal{K}$*-embedding* if f is an embedding and $im(f) \prec_{\mathcal{K}} N$.

Hypothesis 0.2. We assume that $\mathcal{K}$ satisfies the *amalgamation property* (AP): If $M_0 \prec_{\mathcal{K}} M_1, M_2$, there is a model M^* and $\mathcal{K}$-embedding $f_\ell : M_\ell \to M^*$ which are the identity on M_0. And we assume also that $\mathcal{K}$ *joint embedding property* for $\mathcal{K}$-embeddings, which is as AP except with $M_0 = \emptyset$. We also assume that $\mathcal{K}$ has arbitrarily large models. These properties imply immediately that $\mathcal{K}$ has no maximal models.

Let λ be a cardinal. By repeated use of AP and JEP, we can easily construct a λ*-model homogeneous* model N *i.e.*, if $M_1 \prec_{\mathcal{K}} M_2$ of size less than λ and there is a $\mathcal{K}$-embedding $f_1 : M_1 \to N$ then there exists a $\mathcal{K}$-embedding $f_2 : M_2 \to N$ extending f_1. We also 'allow' N_1 to be empty *i.e.*, any M of size less than λ $\mathcal{K}$-embeds inside N. We can further find a model which is *strongly* λ*-model homogeneous i.e.*, satisfies in addition that any isomorphism $f : M_1 \to M_2$ with $M_\ell \prec_{\mathcal{K}} N$ of size less than λ extends to an *automorphism* of N.

Let us now consider the problem of *types*. As we pointed out, we do not have formulas and hence no adequate syntactic notion of types. We therefore deal with a semantic notion; we consider a relation $\sim$ on triples of the form (a, M, N), where $M \prec_{\mathcal{K}} N$ and a an element of N. We say that

$$(a_1, M_1, N_1) \sim (a_2, M_2, N_2)$$

if $M_1 = M_2$ and there exists a model M^* amalgamating N_1 and N_2 over M_1 via $\mathcal{K}$-embeddings $g_\ell : M_\ell \to M^*$ fixing M_1 such that

$$g_1(a_1) = g_2(a_2).$$

It is an exercise using AP to check that $\sim$ is an equivalence relation on such triples. The equivalence class of (a, M, N) is the *Galois type of* a *over* M *(in* N*)* and will be denoted by $\mathsf{tp}(a/M, N)$. Since we consider no other types, we will simply say 'type' for 'Galois type' but we may choose to use the full phrase for emphasis. We denote by $S(M)$ the set of Galois types over M. We say that N' *realizes* $\mathsf{tp}(a/M, N)$ if $M \prec_{\mathcal{K}} N'$ and there exists $a' \in N'$ such that $\mathsf{tp}(a'/M, N') = \mathsf{tp}(a/M, N)$. We also write $\mathsf{tp}(a/M, N) \restriction M'$ for $M' \prec_{\mathcal{K}} M$ for $\mathsf{tp}(a/M', N)$.

We now examine these notions in some familiar classes of examples:

Examples 0.3. (1) *The first order case*: The class of models of a complete first order theory T with infinite models has AP, JEP, and arbitrarily large models. Galois types correspond to the usual notion of types over models.

(2) *The homogeneous case*: Let T be a complete first order theory and let D be a set of types over the empty set. D is *good* if for arbitrarily large λ, there exist λ-homogeneous models of size at least λ realizing, over the empty set, exactly those types in D. Then, the class of models omitting all types outside D under elementary submodel forms an abstract elementary class with AP, JEP, and arbitrarily large models. Galois types over models correspond to the usual syntactic notion of types in this context. This generalizes to logics other than first order with similar conclusions.

(3) *n-variable logic with amalgamation*: Under amalgamation over sets [Dj] (where it actually belongs to homogeneous model theory) we have AP and JEP, and the syntactic L^n-types are the Galois types. In [BaLe], a special kind of amalgamation is introduced (in addition to AP and JEP) precisely so that Galois types and syntactic L^n-type coincide.

(4) *The excellent case*: Let $\mathcal{K}$ be a class of models of a Scott sentence $\psi \in L_{\omega_1,\omega}$ under $L_{\mathcal{A}}$-elementary equivalence with a chosen countable fragment $\mathcal{A}$ of $L_{\omega_1,\omega}$ containing ψ. If $\mathcal{K}$ is *excellent* (see [Sh87a], [Sh87b], or [Le1] for the definition in the equivalent case of an atomic class), then $\mathcal{K}$ has AP, JEP, and arbitrarily large models. Again Galois types correspond to the syntactic notion of types over models there. Notice that excellence is the crucial reason why this is so. This is also a motivating reason for introducing the general context of abstract elementary classes: even in concrete cases, showing that Galois types are well-behaved is very difficult.

It is natural to make the following definition.

Definition 0.4. Let $\mu > \mathsf{LS}(\mathcal{K})$. We say that N is *μ-saturated* if N realizes each $q \in S(M)$ with $M \prec_{\mathcal{K}} N$ of size less than μ. We say that N is *saturated* if N is $\|N\|$-saturated.

Notice that, we only consider μ-saturation for $\mu > \mathsf{LS}(\mathcal{K})$; the notion of $\mathsf{LS}(\mathcal{K})$-saturation is problematic as there may not be any models of size less than $\mathsf{LS}(\mathcal{K})$ in general. It is an easy observation that if M is μ-model homogeneous, then M is μ-saturated (the converse also holds, see below).

Examples 0.5. If $\mathcal{K}$ is first order then a model N is μ-saturated in the above sense if and only if $\mathcal{K}$ is μ-saturated in the usual sense. In the homogeneous case, when $\mathcal{K}$ is the class of models of a good diagram D, then a D-model N is μ-saturated if and only if it is (D, μ)-homogeneous (see [Sh3] for definition). And, if $\mathcal{K}$ is an excellent class of models of a Scott sentence in $L_{\omega_1,\omega}$, then $N \in \mathcal{K}$ is μ-saturated if and only if N is μ-full (for

From now until the rest of this paper, we fix a suitably big cardinal $\overline{\kappa}$ and a model $\mathfrak{C}$ which is strongly $\overline{\kappa}$-model homogeneous. We will use $\mathfrak{C}$ as a *monster model*: Every model of size less than $\overline{\kappa}$ is isomorphic to a $\prec_{\mathcal{K}}$-submodel of $\mathfrak{C}$ and every type $p \in S(M)$ with $M \prec \mathfrak{C}$ of size less than $\overline{\kappa}$ is realized in $\mathfrak{C}$, as $\mathfrak{C}$ is $\overline{\kappa}$-saturated.

Furthermore, types over such small $\mathcal{K}$-submodels correspond to *orbits* of the automorphism group of $\mathfrak{C}$ *i.e.*, if $\mathsf{tp}(a/M, \mathfrak{C}) = \mathsf{tp}(b/M, \mathfrak{C})$ there exists an automorphism f of $\mathfrak{C}$ fixing M such that $f(a) = b$. We denote by $Aut(\mathfrak{C}/M)$ the group of automorphisms of $\mathfrak{C}$ fixing M pointwise.

We work inside $\mathfrak{C}$ and only consider models, sets, and types of size less than $\overline{\kappa}$. Since any $N \prec_{\mathcal{K}} \mathfrak{C}$, all types $\mathsf{tp}(a/M, N)$ are of the form $\mathsf{tp}(a/M, \mathfrak{C})$, so we will simply write $\mathsf{tp}(a/M)$. Observe that given a $\mathcal{K}$-embedding $f : M \to N$ and a type $p \in S(M)$, the type $f(p)$ is well-defined: Let $a_\ell \in \mathfrak{C}$ realise p and let $f_\ell \in Aut(\mathfrak{C})$ extending f, for $\ell = 1, 2$. Then, $f_1^{-1} \circ f_2 \in Aut(\mathfrak{C}/M)$, which we can apply to $\mathsf{tp}(a_1/M) = \mathsf{tp}(a_2/M)$, so $\mathsf{tp}(a_1/M) = \mathsf{tp}(f_1^{-1} \circ f_2(a_2)/f_1^{-1} \circ f_2(M))$, from which we obtain

$$\mathsf{tp}(f_1(a_1)/f(M)) = \mathsf{tp}(f_2(a_2)/f(M)),$$

since $f_2(M) = f(M) = f_1(M)$. We denote by $f(p)$ this common value.

The monster model point of view allows us to dispense with amalgamation diagrams in favour of more familiar first order monster model arguments but is entirely equivalent.

As Baldwin showed in [Ba, Ba2], this simplifies some arguments considerably. As an example, we leave the next proposition as an exercise. The trivial implication of (1) was already pointed out. The left to right is now easy using the monster model (see Proposition 0.12 for a hint). (2) is simply a back and forth construction using (1).

Proposition 0.6. (1) *N is μ-saturated if and only if N is μ-model homogeneous.*
(2) *Two saturated models of N_1, N_2 containing M such that $\|N_1\| = \|N_2\| > \|M\|$ are isomorphic over M.*

The main concept of this paper is that of *categoricity*.

Definition 0.7. Let λ be a cardinal. We say that $\mathcal{K}$ is λ-*categorical* (or *categorical in* λ) if all models of $\mathcal{K}$ of size λ are isomorphic.

As in the first order case, the key to understand categoricity is *stability*.

Definition 0.8. Let λ be a cardinal. We say that $\mathcal{K}$ is λ-*stable* (or *stable in* λ) if $|S(M)| \leq \lambda$ for each $M \in \mathcal{K}$ of size λ.

The first consequence of categoricity is stability. Shelah's presentation theorem [Sh88, Ba, Gr] asserts than any abstract elementary class can be represented as the class of reducts to L of models of a first order theory in an expanded language L^* of size $\mathrm{LS}(\mathcal{K})$ omitting a set of first order L^*-types. This implies that the Hanf number for abstract elementary classes is at most $\beth_{(2^{\mathrm{LS}(\mathcal{K})})^+}$ ([Sh] VII). The next fact is proved using Ehrenfeucht-Mostowski models in a similar way to the first order case. The argument takes several pages and involves a number of elements. First, Shelah's presentation theorem allows the representation of the AEC $\mathcal{K}$ as an pseudo-elementary class with omitting types. Second, since $\mathcal{K}$ has arbitrarily large models $\mathcal{K}$ has Ehrenfeucht-Mostowski models. Now a careful choice of a sufficiently homogeneous linear order as skeleton ($\omega_1^{<\omega}$), which realizes only countably many cuts over countable subsets, allows one to conclude ω-stability; this is the only fact quoted in the entire paper that doesn't appear in various model theory texts. A complete proof of the lemma can be found in Baldwin's online book [Ba].

Fact 0.9. *If $\mathcal{K}$ is $\aleph_1$-categorical then $\mathcal{K}$ is $\aleph_0$-stable.*

We can now prove the existence of saturated models in $\aleph_1$.

Proposition 0.10. *If $\mathcal{K}$ is $\aleph_0$-stable, then there exists a saturated model of size $\aleph_1$.*

Proof. Construct an $\prec_{\mathcal{K}}$-increasing and continuous chain $(M_i : i < \aleph_1)$ of countable models M_i such that M_{i+1} realizes every Galois type over M_i. This is possible by $\aleph_0$-stability. The regularity of $\aleph_1$ implies that $\bigcup_{i<\aleph_1} M_i$ is saturated. □

Definition 0.11. We say that N is *universal over* M if $M \prec_{\mathcal{K}} N$ and for each M' with $M \prec_{\mathcal{K}} M'$ and $\|M'\| \leq \|N\|$, there is a $\mathcal{K}$-embedding $f : M' \to N$ which is the identity on M.

By Proposition 0.6, if N is saturated and $M \prec_K N$ with $\|M\| < \|N\|$, then N is universal over M (and in particular, if there is a saturated model of size μ, then any model of size μ extends to a saturated model). The existence of universal models of the same size follows from stability. We will iterate the idea of the next proof a number of times, to build limit models from universal ones, and superlimits from limits. This is why we give a complete proof.

Proposition 0.12. *Let $\mathcal{K}$ be μ-stable. For each M of size μ there is a universal model M' over M of size μ.*

Proof. Let $(M_i : i < \mu)$ be an increasing and continuous sequence of models of size μ, with $M_0 = M$, such that M_{i+1} realizes every type in $S(M_i)$. This is possible by μ-stability. Let $M' = \bigcup_{i<\mu} M_i$. We claim that M' is universal over M. Let N be of model of size μ with $M \prec_{\mathcal{K}} N$. We will find $f' : N \to M'$, which is the identity on M as follows. Write $N = M \cup \{a_i : i < \mu\}$. We construct an increasing and continuous chain of models $(N_i : i < \mu)$ and an increasing and continuous chain of $\mathcal{K}$-embeddings $f_i : N_i \cong M_i$, with $f_i \restriction M = id_M$, such that $a_i \in N_{i+1}$. (Note that we do not require that $N_i \prec_{\mathcal{K}} N$.) For $i = 0$, simply let $N_0 = M$ and $f_0 = id_M$, and at limits, take unions. Now having constructed $f_i : N_i \cong M_i$, consider $p_i = \mathrm{tp}(a_i/N_i)$. Then $f_i(p_i)$ is a type over M_i, hence realized in M_{i+1} by construction, say by b. Choose an automorphism of $\mathfrak{C}$ extending f_i sending a_i to b. Let $N_{i+1} = h^{-1}(M_{i+1})$. Then $N_i \prec_{\mathcal{K}} N_{i+1}$ and $a_i \in N_{i+1}$. Furthermore, $f_{i+1} = h \restriction N_{i+1} : N_{i+1} \cong M_{i+1}$ is as desired. This is enough: The $\mathcal{K}$-embedding $f : \bigcup_{i<{}^{1}} f_i$ is an isomorphism between $\bigcup_{i<{}^{1}} N_i$ and M' which is the identity on M. Since $N \subseteq \bigcup_{i<{}^{1}} N_i$ (and hence $N \prec_{\mathcal{K}} \bigcup_{i<{}^{1}} N_i$), then $f' = f \restriction N : N \to M'$ is the desired $\mathcal{K}$-embedding. □

Now let us return to types. Let $p \in S(M)$ and $q \in S(N)$, with $M \prec_{\mathcal{K}} N$. We say that q *extends* p if some (equivalently any) realization of q realizes p. We will write $p \subseteq q$ if q extends p, in spite of the fact that types are not sets of formulas.

Consider an $\subseteq$-increasing chain of types $(p_i : i < \delta)$, say with $p_i \in S(M_i)$. The first question (*existence*) is whether there is $a \in \mathfrak{C}$ such that a realizes p_i, for each $i < \delta$ (unions of types are really intersections of orbits). The second question (*uniqueness*) is whether when $a, b \in \mathfrak{C}$ such that a, b realize p_i for each $i < \delta$ and $M_{\pm} = \bigcup_{i<\pm} M_i$ do we necessarily have

$$\mathrm{tp}(a/M_{\pm}) = \mathrm{tp}(b/M_{\pm})?$$

The answer to both questions is *no* in general; concrete examples are provided in [BaSh]. In order to deal with the first question, we introduce the following definition.

Definition 0.13. An $\subseteq$-increasing chain of Galois types $(p_i : i < \delta)$ with $p_i \in S(M_i)$ is *coherent* if there exist elements $a_i \in \mathfrak{C}$ and $f_{i;j} \in Aut(\mathfrak{C}/M_i)$, for $i < j < \delta$, such that:

(1) $p_i = \mathrm{tp}(a_i/M_i)$;
(2) $f_{i;j}(a_j) = a_i$ for $i < j < \delta$.
(3) $f_{i;j} = f_{i;k} \circ f_{k;j}$ for any $i < k < j < \delta$.

The next proposition implies that the union of a coherent chain of Galois types is realized.

Proposition 0.14. *Let $(p_i : i < \delta)$ be a coherent chain of types, with $p_i \in S(M_i)$. Then there exists $p_{\pm} \in S(M_{\pm})$, with $M_{\pm} = \bigcup_{i<\pm} M_i$, such that $(p_i : i < \delta + 1)$ is a coherent chain of types.*

Proof. Let $a_i \models p_i$ and $f_{i;j} \in Aut(\mathfrak{C}/M_i)$, for $i < j < \delta$, witness the coherence of $(p_i : i < \delta)$. Let $M_{\pm} = \bigcup_{i<\pm} M_i$. We need to find $a_{\pm}$ so that for $p_{\pm} = \mathrm{tp}(a_{\pm}/M_{\pm})$ there are $f_{i;\pm}$ for $i < \delta$ demonstrating that $(p_i : i < \delta + 1)$ is coherent.

Let $g_i = f_{0,i} \upharpoonright M_i : M_i \to \mathfrak{C}$. Notice that the sequence $(g_i : i < \delta)$ of $\mathcal{K}$-embeddings is increasing and continuous. Hence we can find $g \in Aut(\mathfrak{C})$ extending $\bigcup_{i<\pm} g_i$. Let $a_\pm = g^{-1}(a_0)$ and define $f_{i,\pm} = f_{0,i}^{-1} \circ g$. Then $f_{i,\pm}$ fixes M_i since g extends $f_{0,i} \upharpoonright M_i$ and sends $a_\pm$ to a_i. Furthermore, $f_{i,j} \circ f_{j,\pm} = f_{i,j} \circ f_{0,j}^{-1} \circ g = f_{0,i} \circ g = f_{i,\pm}$. $\square$

Remark 0.15. Since any $\subseteq$-increasing chain of Galois types $(p_i : i < \omega)$ is coherent, the previous proposition shows that its union is realized. Since any countable ordinal is either a successor or has cofinality ω, we derive easily from this that the union of any countable chain of types is realized. Without further assumptions, this may fail for longer chains in general.

We now consider tameness. We will then consider a strengthening which is related to uniqueness. Baldwin [Ba] introduces two parameter versions of both notions. These will be needed in any attempt to extend the results here without making the 'global tameness' assumptions that we use here.

Definition 0.16. Let χ be an infinite cardinal. We say that $\mathcal{K}$ is χ-tame, if whenever $p \neq q \in S(N)$, there exists $M \prec_{\mathcal{K}} N$ of size χ such that $p \upharpoonright M \neq q \upharpoonright M$. We will say that $\mathcal{K}$ is *tame* if $\mathcal{K}$ is $\aleph_0$-tame.

In Remark 1.9 of [Sh394], Shelah refers to the question as to whether categoricity implies tameness as 'the main difficulty'.

Remark 0.17.
(1) If $\mathcal{K}$ is first order, homogeneous, L^n with amalgamation or excellent then $\mathcal{K}$ is χ-tame for $\chi = \mathsf{LS}(\mathcal{K})$.
(2) It follows from Shelah's result in [Sh394] that if $\mathcal{K}$ is categorical in arbitrarily large cardinals, then $\mathcal{K}$ is χ-tame for some χ less than the Hanf number. There is no argument deriving locality from a categoricity hypothesis.
(3) It is not clear at this stage, how strong tameness is. In the interesting particular cases considered by Zilber and Gavrilovich, tameness is established by proving 'excellence', though sometimes only an excellence-like condition, as the context is not strictly $L_{!_1,!}$. The advantage of excellence is that it is a condition involving only countable models, whereas tameness involves uncountable models also. The disadvantage is that it is far more complicated. Also, it follows from our upward categoricity theorem and Shelah's results on categorical sentences in $L_{!_1,!}$ [Sh48] that it is consistent with ZFC that any local $L_{!_1,!}$-class with AP and arbitrarily large models that is categorical in $\aleph_1$ is excellent.

And now the strengthening:

Definition 0.18. We say that $\mathcal{K}$ is *local* if whenever $p \neq q \in S(N)$ and $N = \bigcup_{i<¹} N_i$, for μ a cardinal, then there is $i < \mu$ such that $p \upharpoonright N_i \neq q \upharpoonright N_i$.

Notice that if $\mathcal{K}$ is first order, homogeneous, or excellent, then $\mathcal{K}$ is local. Baldwin calls this property ∞-*local* in [Ba].

Proposition 0.19. *If $\mathcal{K}$ is local then $\mathcal{K}$ is tame.*

Proof. We prove by induction on μ, that if $\mathsf{tp}(a/M) \neq \mathsf{tp}(b/M)$, for M of size μ, then there is a countable $M' \prec_{\mathcal{K}} M$ such that $\mathsf{tp}(a/M') \neq \mathsf{tp}(b/M')$.

For $\mu = \aleph_0$, there is nothing to show. Now assume that $\mu > \aleph_0$. Let M be given. Choose $(M_i : i < \mu)$ increasing and continuous such that $\|M_i\| = |i| + \aleph_0$ and $\bigcup_{i<\imath} M_i = M$. If $\mathsf{tp}(a/M) \neq \mathsf{tp}(b/M)$, then there is $i < \mu$ such that $\mathsf{tp}(a/M_i) \neq \mathsf{tp}(b/M_i)$, since $\mathcal{K}$ is local. But by induction hypothesis, there is $M' \prec_{\mathcal{K}} M_i$ countable such that $\mathsf{tp}(a/M') \neq \mathsf{tp}(b/M')$. So, we are done since $M' \prec_{\mathcal{K}} N$. □

We now show that the answer to both existence and uniqueness question is positive when $\mathcal{K}$ is local:

Proposition 0.20. *Assume that $\mathcal{K}$ is local. Let $(M_i : i \leq \delta)$ be an increasing and continuous sequence of models, and $(p_i \in S(M_i) : i < \delta)$ be an $\subseteq$-increasing sequence of types. Then there is a unique $p \in S(M_\pm)$ extending each $i < \delta$.*

Proof. Uniqueness follows easily: If δ is a successor, there is nothing to show, so we may assume that δ is a limit. By taking a cofinal subsequence if necessary, we may assume that δ is a cardinal, so uniqueness follows immediately from the fact that $\mathcal{K}$ is local.

For existence, assume that $(p_i : i < \delta)$ is given. We show by induction on $i < \delta$ that $(p_j : j \leq i)$ is coherent. For $i = 0$ or a successor, this is easy. Assume that i is a limit and that $(p_j : j < i)$ is coherent. Then by Proposition 0.14, there exists $p'_i \in S(M_i)$ such that $(p_j, p'_i : j < i)$ is coherent. But $p'_i \upharpoonright M_j = p_i \upharpoonright M_j$, for each $j < i$ by definition. Hence by uniqueness, we must have $p'_i = p_i$, which shows that $(p_j : j \leq i)$ is coherent. Thus $(p_i : i < \delta)$ is coherent, and so there exist $p \in S(M_\pm)$ extending each p_i by another application of Proposition 0.14. □

1. BIG AND QUASIMINIMAL TYPES

In this section, we assume that $\mathcal{K}$ is an abstract elementary class with AP, JEP, and arbitrarily large models. We assume further that $\mathsf{LS}(\mathcal{K}) = \aleph_0$ and that $\mathcal{K}$ is $\aleph_0$-stable.

With amalgamation, any type has an extension, but a non-algebraic type may have a bounded number of solutions and thus no non-algebraic extension to a model that contains all of them. The next definition is a strengthening of nonalgebraicity to avoid these types. We begin by discussing only countable models.

Definition 1.1. Let $p \in S(M)$. We say that p is *big* if p has a nonalgebraic extension to any M' with $M \prec_{\mathcal{K}} M'$ and $\|M\| = \|M'\|$.

Notice that if $p \in S(N)$ is big and $M \prec_{\mathcal{K}} N$ then $p \restriction M$ is big. The next proposition will allow us to find big types.

Proposition 1.2. *Let $p \in S(M)$ and M countable. The following conditions are equivalent:*

(1) *p is big;*
(2) *p has a nonalgebraic extension to some M' universal over M;*
(3) *p is realised uncountably many times in $\mathfrak{C}$.*

Proof. (1) implies (2) by definition, since there exists a countable universal model M' over M by Proposition 0.12. (2) implies (3): Let M' be a universal model over M and let $p' \in S(M')$ be a nonalgebraic extension of p. Suppose that $A \subseteq \mathfrak{C}$ is a countable set of realizations of p. Let N be countable containing $A \cup M$. By universality of M' over M, we may assume that $N \prec_{\mathcal{K}} M'$. Since p is realised outside N (any realization of p'), then A does not contain all the realizations of p in $\mathfrak{C}$, so p must be realised uncountably many times in $\mathfrak{C}$. Finally (3) implies (1) is clear, as p must be realised outside any countable model containing M. □

We now show that big types exist.

Proposition 1.3. *There exists a big type $p \in S(M)$, for each countable M. Moreover, if $p \in S(M)$ is big and M' is countable containing M, then there is a big $p' \in S(M')$ extending p.*

Proof. Let M be given. Choose N countable universal over M. Then any nonalgebraic $q \in S(N)$ is such that $q \restriction M \in S(M)$ is big by the previous proposition. Moreover, if $p \in S(M)$ is big and M' is countable containing M, we can choose N countable universal over M'. Since p is big, p has a nonalgebraic extension $q \in S(N)$; again $q \restriction M'$ is big by the previous proposition. □

We now consider the simplest big types.

Definition 1.4. A type $p \in S(M)$ is *quasiminimal* if p is big and has exactly one big extension in $S(M')$ for any $M \prec_{\mathcal{K}} M'$ with $\|M\| = \|M'\|$.

We will primarily be interested in quasiminimal types over countable models. The name quasiminimal is consistent with Zilber's usage, since each quasiminimal type is realised uncountably many times but has at most one extension which is realised uncountably many times. We can now show that quasiminimal types exist by using the usual tree argument:

Proposition 1.5. *There exists a quasiminimal type over some countable model. Moreover, if $p \in S(M)$ is big and M is countable, then there is a countable M' extending M and a quasiminimal $p' \in S(M')$ extending p.*

Proof. Since big types exist by the previous proposition, it is enough to show the second sentence. Let $p \in S(M)$ be big and suppose, for a contradiction, that p has no quasiminimal extension over a countable model. Since p has a big extension over any model by the previous proposition, this means that each big extension of p has at least two big extensions. We can therefore construct a tree of types $(p_\eta : \eta \in {}^{\omega>}2)$ with $p_\eta \in S(M_\eta)$ and M_η countable, such that

(1) $M_{\langle\rangle} = M$ and $p_{\langle\rangle} = p$;
(2) $\langle p_{\eta\restriction n} : n < \ell(\eta)\rangle$ is $\subseteq$-increasing;
(3) p_η is big;
(4) $M_{\eta\frown 0} = M_{\eta\frown 1}$ but $p_{\eta\frown 0} \neq p_{\eta\frown 1}$.

But this contradicts $\aleph_0$-stability: Let $\eta \in {}^{\omega}2$. Since $(p_{\eta\restriction n} : n < \omega)$ is countable and increasing, there is p_η extending each $p_{\eta\restriction n}$ by Remark 0.15. Let N be countable containing $\bigcup_{\eta \in {}^{\omega>}2} M_\eta$. Each p_η for $\eta \in {}^{\omega}2$ has an extension in $S(N)$, so there are $2^{\aleph_0}$ types over N, a contradiction. □

We finish this section with a result on uniqueness of nonalgebraic extensions over certain countable models: *limit models* and over saturated models of cardinality $\aleph_1$.

Definition 1.6. Let M be a countable model. Let $\alpha < \omega_1$ be a limit ordinal. A countable model N is an *α-limit model over M* if there exists an increasing and continuous chain $(M_i : i < \alpha)$ such that $M_0 = M$, each M_{i+1} is universal over M_i, and $N = \bigcup_{i<\alpha} M_i$. We say that $(M_i : i < \alpha)$ is an *α-tower for N over M*.

Observe that if N is an α-limit over M, then N is an ω-limit over M: If $(M_i : i < \alpha)$ is an α-tower for N over M, choose $(\alpha_n : n < \omega)$ a cofinal sequence for α with $\alpha = 0$. Then $(M_{\alpha_n} : n < \omega)$ is an ω-tower for N over M as $M_{\alpha_{n+1}}$ is universal over M_{α_n}. Observe also that for any countable M and any limit ordinal $\alpha < \omega_1$, there exists an α-limit N over M by repeated applications of Proposition 0.12.

We now prove two facts about limit models, which are adapted from Shelah's Lemma 2.2 in [Sh394]; they are stated and proved in [Sh88]. Analogous arguments for uncountable cardinalities are much more difficult; compare [VD1, GrVV].

Proposition 1.7. *Let $\alpha_1, \alpha_2 < \omega_1$ be limit ordinals. Let M be countable and assume that N_ℓ is an α_ℓ-limit over M. Then $N_1 \cong_M N_2$.*

Proof. Without loss of generality, we may assume that $\alpha_1 = \omega = \alpha_2$. Let $(M_n^\ell : n < \omega)$ be an ω-tower for N^ℓ over M, for $\ell = 1, 2$. Proving the isomorphism between N_1 and N_2 is now a standard back-and-forth construction using the universality of M_{n+1}^ℓ over M_n^ℓ: We construct an increasing sequence of

$\mathcal{K}$-embeddings f_n such that $\mathrm{dom}(f_{2n})$ contains M_n^1 and $im(f_{2n+1})$ contains M_n^2, with $f_0 = id_M$. This is possible, since each M_{n+1}^ℓ is universal over M_n^ℓ, for $\ell = 1, 2$, and is enough, as the union of the f_n is an isomorphism between N_1 and N_2 which is the identity on M. □

Since the value of the ordinal α is immaterial, we will simply say that N is a *limit* over M, when N is an α-limit over M. The next proposition is simply proved by pasting the towers witnessing the limits together.

Proposition 1.8. *Let $\alpha < \omega_1$ be an ordinal, not necessarily a limit. Assume that $(M_i : i < \alpha)$ is increasing and continuous such that M_{i+1} is a limit over M_i, for $i < \alpha$. Then $\bigcup_{i<®} M_i$ is a limit over M.*

We now consider nonalgebraic extensions of quasiminimal types. At this point we need locality/tameness hypotheses.

Proposition 1.9. *Suppose $\mathcal{K}$ is local. Let $p \in S(M)$ be quasiminimal, with M countable, and let N be a limit over M. Then there is a unique nonalgebraic extension of p in $S(N)$.*

Proof. Let $(N_n : n < \omega)$ be an ω-tower for N over M. Let $q \in S(N)$ be the unique big type extending p in $S(M)$. Then q is nonalgebraic, which proves existence. Now assume that $q' \in S(N)$ be a nonalgebraic extension of p. Let $n < \omega$. Then $q \upharpoonright N_n$ and $q' \upharpoonright N_n$ are two big extension of p; the first by restriction, and the second by Proposition 1.2. Hence, by quasiminimality of p, we have $q \upharpoonright N_n = q' \upharpoonright N_n$. Since this holds for any $n < \omega$, we have that $q = q'$, since $\mathcal{K}$ is local. □

We can extend the previous result to the saturated model of size $\aleph_1$ (which exists by Proposition 0.10).

Proposition 1.10. *Suppose $\mathcal{K}$ is local. Let $p \in S(M)$ be quasiminimal, with M countable. There is a unique nonalgebraic extension of p to any saturated model N of size $\aleph_1$ containing M.*

Proof. First, there can be at most one nonalgebraic extension of p over the model saturated model of size $\aleph_1$, since it is saturated: If $q_1 \neq q_2 \in S(N)$, with N of size $\aleph_1$ both extend p, then, since locality implies tameness, there is M' countable, with $M \prec_{\mathcal{K}} M' \prec_{\mathcal{K}} N$, such that $q_1 \upharpoonright M' \neq q_2 \upharpoonright M'$. Since N is saturated, it is universal over M', and hence $q_1 \upharpoonright M'$ and $q_2 \upharpoonright M'$ are big, by Proposition 1.2. This contradicts the quasiminimality of p.

We now turn to existence and notice that by Proposition 0.6, it is enough to prove it for some saturated model of size $\aleph_1$. Consider an increasing and continuous chain of models $(M_i : i < \aleph_1)$, such that $M_0 = M$, and M_{i+1} is universal over M_i. This is possible by Proposition 0.12. Notice that each M_{i+1} realizes every type over M_i by universality. Hence the model $N = \bigcup_{i<\aleph_1}$ is saturated, and by Proposition 1.8, each M_i is a limit over M. Let $p_0 = p$. By Proposition 1.9, there is a unique nonalgebraic $p_i \in S(M_i)$ extending p_0. By uniqueness, the sequence $(p_i : i < \aleph_1)$ is $\subseteq$-increasing, and so there is $q \in S(N)$ extending each $i < \omega_1$ by Proposition 0.20. Then, q is clearly nonalgebraic, as each p_i is. □

2. UPWARD CATEGORICITY: GOING UP INDUCTIVELY

In this section, we assume that $\mathcal{K}$ is a local abstract elementary class, with AP, JEP, and arbitrarily large models. We assume that $\mathsf{LS}(\mathcal{K}) = \aleph_0$ and that $\mathcal{K}$ is categorical in $\aleph_1$. Notice that $\mathcal{K}$ is tame by Proposition 0.19 and $\aleph_0$-stable by Fact 0.9, so the results of the previous section apply.

The idea is to prove by induction on $\mu \geq \aleph_1$ that every model of size μ is saturated. This implies categoricity in μ by Proposition 0.6. This is the reason why the assumption that all the uncountable models of size at most μ are saturated will appear as an assumption in two of the following propositions.

We first show that we can extend quasiminimal types to larger models, provided all the intermediate models are saturated:

Proposition 2.1. *Let $p \in S(M)$ be quasiminimal, with M countable. Let $\mu \geq \aleph_1$ and assume that every model of size κ is saturated, with $\aleph_1 \leq \kappa \leq \mu$. Then p has a unique nonalgebraic extension to any model of size μ.*

Proof. We prove inductively that there exists a unique nonalgebraic extension of p in $S(N)$ by induction on $\mu = \|N\| \geq \aleph_1$. For $\mu = \aleph_1$ this is Proposition 1.10. Now assume that $\mu > \aleph_1$. By assumption, we can find $(N_i : i < \mu)$ an increasing and continuous chain of saturated models of size $\|N_i\| = |i| + \aleph_1$. By induction hypothesis, there exists a unique nonalgebraic $p_i \in S(N_i)$ extending p. By uniqueness, the sequence $(p_i : i < \mu)$ is $\subseteq$-increasing, so there exists $q \in S(N)$ extending each p_i by Proposition 0.20. Now the uniqueness of q is as in Proposition 1.10, since N is saturated. □

We now introduce Vaughtian pairs:

Definition 2.2. Let $p \in S(M)$ be quasiminimal, with M countable. A (p, μ)-*Vaughtian pair* is a pair of models N_1, N_2 of size μ with $M \prec_{\mathcal{K}} N_1 \prec_{\mathcal{K}} N_2$, $N_1 \neq N_2$, such that there is a nonalgebraic extension of p in $S(N_1)$ which is not realised in N_2.

Let $p \in S(M)$ be quasiminimal with M countable. The goal is to prove that there are no (p, μ)-Vaughtian pairs for any uncountable μ. In order to extend the traditional Vaught argument, we will need to find a countable substitute for the notion: N *is saturated over* M. In the excellent case [Le1], one can use countable *full* models over M: Two countable full models over M are isomorphic over M and the union of any countable chain of full models over M is full over M. Here, the key will be to use certain kinds of limits, introduced in [Sh88], the *superlimits*, which have good uniqueness properties (Proposition 2.5), and will behave well under unions (Proposition 2.6).

Definition 2.3. Let $\alpha < \omega_1$ be a limit ordinal. Let M be a countable model. A countable model N is an *α-superlimit over M* if there exists an increasing and continuous chain $(N_i : i < \alpha)$ such that $N_0 = M$, N_{i+1} is a limit over N_i, and $\bigcup_{i<®} N_i = N$. We call $(N_i : i < \alpha)$ as above an *α-supertower for N over M*.

Clearly, a superlimit is a limit, since if M_{i+1} is a limit over M_i then M_{i+1} is universal over M_i. But an α-superlimit is also an $(\omega \cdot \alpha)$-limit, by unravelling the definitions. So, we clearly have the uniqueness property, but we also have a stronger one. First, let us use a convenient piece of notation: Given N an α-superlimit over M, we denote by $\bar{N}$ some α-supertower $(N_i : i < \alpha)$ for N over M.

The next proposition shows that it is enough to understand ω-superlimits.

Proposition 2.4. *Let $\alpha < \omega_1$ be a limit ordinal. Let M be countable. Every α-superlimit over M is an ω-superlimit over M.*

Proof. As α is a countable limit ordinal, there is $(\alpha_n : n < \omega)$ cofinal in α, with $\alpha_0 = 0$. Then if $(N_i : i < \alpha)$ is an α-supertower for N over M, then $(N_{®_n} : n < \omega)$ is clearly an ω-supertower for N over M, as $N_{®_{n+1}}$ is a limit over $N_{®_n}$ by Proposition 1.8. □

So we focus on ω-superlimits. The proof of the next proposition is simply an iteration of Proposition 1.7.

Proposition 2.5. *Let N, N' be ω-superlimits over M. Then $N \cong_M N'$. Furthermore, if $(N_n : n < \omega)$ and $(N'_n : n < \omega)$ are ω-supertowers for N and N' (respectively) over M, then there exists an isomorphism $f : N \cong_M N'$ such that*

$$f \restriction N_n : N_n \cong N'_n, \quad \textit{for each } n < \omega.$$

We write $f : \bar{N} \cong \bar{N}'$ for an isomorphism between the two supertowers of two superlimits N and N' as in the previous proposition.

We will show that countable unions of ω-superlimits are ω-superlimits under the right circumstances. We will need to consider sequences of supertowers $\bar{N}_i$, so it is natural to order them. The most natural choice is to consider the following partial order $\leq$ between ω-supertowers:

$$(N_n : n < \omega) \leq (N'_n : n < \omega),$$

if for each $n < \omega$, N'_n is a limit over N_n. Unfortunately, this is too strong for our purposes, so we consider the weakening $\leq^*$, where the * serves, as usual, to denote *eventual* domination: We say that

$$(N_n : n < \omega) \leq^* (N'_n : n < \omega),$$

if for each $n < \omega$, there exists $m \geq n$ such that N'_m is a limit over N_n. Notice that N'_k is a limit over N_n for each $k \geq m$ by Proposition 1.8. It is clear that $\leq^*$ is transitive, and if $\bar{N} \leq \bar{N}'$ then $\bar{N} \leq^* \bar{N}'$.

The proof that there are no Vaughtian pairs requires the analysis of arbitrary sequences of models, where unlike towers there is no guarantee that each model is universal over its predecessor. For this, we consider unions of superlimits. The notation is a bit cumbersome, but the proof is a straightforward diagonal argument.

Proposition 2.6. *Let $\alpha < \omega_1$ be a limit ordinal. Let $(N^i : i < \alpha)$ be an $\prec_{\mathcal{K}}$-increasing and continuous chain of ω-superlimits over M with ω-supertowers $\bar{N}^i$, for $i < \alpha$. Suppose, in addition, that*

$$\bar{N}^i \leq^* \bar{N}^j, \quad \textit{for } i < j < \alpha.$$

Then $\bigcup_{i<®} N^i$ is an ω-superlimit over M. Moreover, there exists $(N^®_n : n < \omega)$ an ω-supertower for $\bigcup_{i<®} N^i$ over M such that

$$\bar{N}^i \leq^* (N^®_n : n < \omega), \quad \textit{for each } i < \alpha.$$

Proof. It is enough to prove the last sentence. In addition, by choosing a cofinal sequence $(\alpha_n : n < \omega)$ for α with $\alpha_0 = 0$, and using the transitivity of $\leq^*$, we may assume that $\alpha = \omega$. So we consider an $\leq^*$-increasing sequence $(\bar{N}^i : i < \omega)$ of ω-supertowers $\bar{N}^i$ for N^i over M. We will construct a strictly increasing function $f : \omega \to \omega$ such that $f(0) = 0$ and for each integer $n > 0$

$$N^n_{f(n)} \text{ is a limit over } N^i_k, \quad \text{for each } i, k < n.$$

This is enough: Let $N^!_n := N^n_{f(n)}$, for each $n < \omega$, Then $(N^!_n : n < \omega)$ is an ω-supertower for $\bigcup_{i<!} N^i$ over M since $N^!_0 = M$ and $N^!_{n+1}$ is a limit over $N^!_n$ by the definitions. Furthermore,

$$\bar{N}^i \leq^* (N^!_n : n < \omega), \quad \text{for each } i < \omega :$$

Let $i, k < \omega$ be given and consider $n := \max(i, k) + 1$. Then $N^!_n$ ($= N^n_{f(n)}$) is a limit over N^i_k by definition.

It remains to show that such an f can be found. By definition of $\leq^*$, for each $i < \omega$ there exists a strictly increasing function $f_i : \omega \to \omega$ such that

$$N^{i+1}_{f_i(n)} \text{ is a limit over } N^i_n, \quad \text{for each } n < \omega.$$

We define $f(n)$ by induction on n. Let $f(0) = 0$. Having constructed $f(n)$, we define $f(n+1)$ by taking the maximum of the following three numbers:

$$f_n(f(n)), \quad f_n \circ f_{n-1} \circ \cdots \circ f_0(n), \quad f(n) + 1.$$

Then $f(n+1)$ is as required: The fact that $f(n+1)$ is at least the first number ensures that $N^{n+1}_{f(n+1)}$ is a limit over N^i_k for $i \leq n$ and $k < n$. That $f(n+1)$ is at least the second number ensures that $N^{n+1}_{f(n+1)}$ is a limit over each N^i_n, for $i \leq n$, since

$$f_n \circ f_{n-1} \circ \cdots \circ f_0(n) \geq f_n \circ f_{n-1} \circ \cdots \circ f_i(n).$$

And finally, $f(n+1) > f(n)$ since $f(n+1)$ is at least the third number. This finishes the proof. □

We prove a simple result which will be used in the proof that there are no Vaughtian pairs:

Proposition 2.7. *Suppose that M_0, M_1 are countable and A is a countable set. There exists a countable N containing $M_0 \cup M_1 \cup A$ which is a limit over both M_0 and M_1.*

Proof. It is enough to find a countable model N which is universal over M_0 and M_1 and contains A. But this is clear: Choose first N' containing A which is universal over M_0. Now choose N'' containing $N' \cup M_1$ which is universal over M_1. Since $N' \prec_{\mathcal{K}} N''$ and N' is universal over M_0, then so is N''. □

We now prove that there are no Vaughtian pairs.

Proposition 2.8. *Let $p \in S(M)$ be quasiminimal with M countable. Then there are no (p, μ)-Vaughtian pairs, with $\mu \geq \aleph_1$.*

Proof. Suppose that $N_0 \prec_{\mathcal{K}} N_1$ is a (p, μ)-Vaughtian pair, for $\mu \geq \aleph_1$. By the usual ω-chain argument, we may assume that $\mu = \aleph_1$, and hence that N_0 and N_1 are saturated by Proposition 0.10.

We now construct a $(p, \aleph_0)$-Vaughtian pair $N^0 \prec_{\mathcal{K}} N^1$ such that N^ℓ is an ω-superlimit over M, with ω-supertower $\bar{N}^\ell$, for $\ell = 0, 1$, and such that

$$\bar{N}^0 \leq^* \bar{N}^1.$$

Let $N^\ell_0 = M$ for $\ell = 0, 1$. Choose a limit N^0_1 over M such that $N^0_1 \prec_{\mathcal{K}} N_0$ (this is possible since N_0 is saturated). Now choose $N^1_1 \prec_{\mathcal{K}} N_1$ a limit over N^1_0 containing an element $a \in N_1 \setminus N_0$. Now having constructed $N^0_n \prec_{\mathcal{K}} N_0$ and $N^1_n \prec_{\mathcal{K}} N_1$ countable with N^1_n a limit over N^0_n, choose $N^0_{n+1} \prec_{\mathcal{K}} N_0$ a limit over N^0_n containing all the realizations of the unique big extension of p to N^0_n in N^1_n (this is possible since this set is countable). Now choose countable $N^1_{n+1} \prec_{\mathcal{K}} N_1$ a limit over both N^1_n and N^0_{n+1} (this is possible by Proposition 2.7, N^1_{n+1} can be chosen inside N_1 by the saturation of N_1). Let $N^\ell = \bigcup_{n<\omega} N^\ell_n$, for $\ell = 0, 1$. Then

N^ℓ is an ω-superlimit over M with ω-supertower $\bar{N}^\ell$ such that $\bar{N}^0 \leq^* \bar{N}^1$ (even $\bar{N}^0 \leq \bar{N}^1$). Furthermore, $N^0 \prec_{\mathcal{K}} N^1$ forms a $(p, \aleph_0)$-Vaughtian pair. Let p_0 be the unique big type extending p in $S(N^0)$, which exists by countability of N^0 and quasiminimality of p.

To contradict categoricity in $\aleph_1$, we construct an increasing and continuous chain $(N^i : i < \aleph_1)$ of ω-superlimits over M, such that $N^i \neq N^{i+1}$ with a big extension $p_i \in S(N^i)$ of p which is not realised in N^{i+1}, and such that the sequence of limits $(\bar{N}^i : i < \omega_1)$ is $\leq^*$-increasing: We do this by induction on $i < \omega_1$. For $i = 0$, this is given. At limit $i < \omega_1$, let $N^i = \bigcup_{j<i} N^j$ with ω-supertower $\bar{N}^i$ over M as in Proposition 2.6. Now having constructed the ω-superlimit model N^i with ω-supertower $\bar{N}^i$ over M, for i limit or successor, choose an isomorphism $f_i : \bar{N}^0 \cong \bar{N}^i$ as in Proposition 2.5. Then f_i extends to an automorphism $g_i \in Aut(\mathfrak{C}/M)$ and we let $p_i = g_i(p_0)$, $N^{i+1} = g_i(N^1)$, and $\bar{N}^{i+1} = g(\bar{N}^1)$. Then $p_i \in S(N_i)$ is a big extension of p which is not realised in N^{i+1}, and $\bar{N}^i \leq^* \bar{N}^{i+1}$, since g_i is an automorphism respecting levels and $\bar{N}^0 \leq^* \bar{N}^1$.

Let $N^* = \bigcup_{i<\omega_1} N^i$. Then N^* has size ω_1 but omits p_0: Otherwise, there is $a \in N^*$ realizing p_0. Since $a \notin N^0$, there is $i < \omega_1$ such that $a \in N^{i+1} \setminus N_i$. Then $\text{tp}(a/N^i)$ is nonalgebraic and extends p. Hence, $\text{tp}(a/N^i) = p_i$ by Proposition 1.9 since N^i is a (super)limit over M, but this is a contradiction since $a \in N^{i+1}$ and p_i is not realised in N^{i+1}. So, p_0 is not realised in N^*, which implies that N^* is not saturated, contradicting Proposition 0.10. □

The key to carry out the induction in the main theorem is the successor case. We use the absence of Vaughtian pairs to show this. This argument is inspired by the final argument in [Sh394] and Theorem 4.1 of [GrVD3].

Proposition 2.9. *Let $\mu \geq \aleph_1$. Assume that all models of size κ are saturated, with $\aleph_1 \leq \kappa \leq \mu$. Then all models of size μ^+ are saturated.*

Proof. Fix $p \in S(M)$ be quasiminimal and M countable, by Proposition 1.5. Let N be a model of size μ^+, with $\mu \geq \aleph_1$. By assumption, N is μ-saturated, so we may assume that $M \prec_{\mathcal{K}} N$. Observe that since there are no (p, μ)-Vaughtian pair by Proposition 2.8, every nonalgebraic extension of p to a submodel of $N' \prec_{\mathcal{K}} N$ of size μ must be realised in N, otherwise by choosing any N'' of size μ, with $N' \prec_{\mathcal{K}} N'' \prec_{\mathcal{K}} N$, and $N' \neq N''$, we have a (p, μ)-Vaughtian pair. We now prove:

Claim. Let $M \prec_{\mathcal{K}} M'$, with M' of size μ, and a $\mathcal{K}$-embedding $f : M' \to N$ which is the identity on M. Let $a \in \mathfrak{C}$ realise p. Then there exist M'' of size μ, with $M' \prec_{\mathcal{K}} M''$ and $a \in M''$, and a $\mathcal{K}$-embedding $g : M'' \to N$ extending f.

Proof. If a is already in M', there is nothing to do. Otherwise the type $p' = \text{tp}(a/M')$ is a nonalgebraic extension of p, so $f(p')$ is a nonalgebraic extension of

p over a submodel of N of size μ. Thus $f(p')$ must be realised by some $b \in N$, by the observation of the first paragraph. Choose an automorphism h of $\mathfrak{C}$ extending f sending a to b, and choose $N' \prec_{\mathcal{K}} N$ of size μ containing b such that $f(M') \prec_{\mathcal{K}} N'$. Let $M'' = h^{-1}(N')$ and $g = h \restriction M''$. Then $g : M'' \to N$ extends f and M'' contains a and extends M', as desired. □

We now show that N is saturated. Fix $M_0 \prec_{\mathcal{K}} N$ of size μ and $q \in S(M_0)$. We will show that q is realised in N. First, we may assume that $M \prec_{\mathcal{K}} M_0$, since M_0 is saturated (and $\mu \geq \aleph_1$). We construct two increasing chains of models

$$(M_n : n < \omega) \text{ and } (M'_n : n < \omega), \text{ with } M_n \prec_{\mathcal{K}} M'_n,$$

such that each model is of size μ, M_0 is as given above, M'_0 realizes q, every realization of p in M'_n is in M_{n+1}. We also construct an increasing chain of $\mathcal{K}$-embeddings

$$f_n : M_n \to N, \text{ such that } f_n \restriction M_0 = id_{M_0}.$$

This is easy to do: Let M_0 be as above, and choose M'_0 of size μ extending M_0 and realizing q. Let $f_0 = id_{M_0} : M_0 \to N$. Now having constructed $f_n : M_n \to N$ and $M_n \prec_{\mathcal{K}} M'_n$, there are at most μ realizations of p in $M'_n \setminus M_n$, so by applying the previous claim μ-times, we can find M_{n+1} of size μ extending M_n containing every realization of p in M'_n, as well as a $\mathcal{K}$-embedding $f_{n+1} : M_{n+1} \to N$, extending f_n. Choose M'_{n+1} any model of size μ containing $M_{n+1} \cup M'_n$.

This is enough: Let $M_! = \bigcup_{n<!} M_n$ and $M'_! = \bigcup_{n<!} M'_n$. Then $M_! \prec_{\mathcal{K}} M'_!$ are models of size μ. By Proposition 2.1 there is a nonalgebraic extension of p in $S(M_!)$, and by construction, this nonalgebraic extension is not realised in $M'_! \setminus M_!$. Since there are no (p, μ)-Vaughtian pairs by Proposition 2.8, this implies that $M_! = M'_!$. Hence, $\bigcup_{n<!} f_n$ is a $\mathcal{K}$-embedding from $M'_!$ into N fixing M_0, and so sends a realization of q in $M'_!$ to a realization of q in N. This shows that q is realised in N. □

We can now prove upward categoricity from $\aleph_1$.

Theorem 2.10. *Let $\mathcal{K}$ be a local abstract elementary class with AP, JEP, arbitrarily large models, and $\mathsf{LS}(\mathcal{K}) = \aleph_0$. Assume that $\mathcal{K}$ is categorical in $\aleph_1$. Then $\mathcal{K}$ is categorical in every uncountable cardinal.*

Proof. We prove that uncountable models are saturated, which shows categoricity in every uncountable cardinal by Proposition 0.6. Suppose, for a contradiction, that there is $\chi \geq \aleph_1$ and a model of size χ which is not saturated. Choose χ minimal with this property. Then $\chi > \aleph_1$ (by Proposition 0.10) and cannot be a limit cardinal. Hence $\chi = \mu^+$, for some $\mu \geq \aleph_1$. By minimality of χ, every model of size κ, with $\aleph_1 \leq \kappa \leq \mu < \mu^+ = \chi$, is saturated. Hence, by Proposition 2.9, every model of size μ^+ is saturated. This contradicts the choice of χ. □

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