Geometry and Proof

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Abstract

We discuss the relation between the specific axiomatizations, specifically Hilbert's [Hil71] reformulation of geometry at the beginning of the last century, and the way elementary geometry has been expounded in high schools in the United States. Further we discuss the connections among formal logic, the teaching of logic and the preparation of high school teachers. In part our goal is to describe how high school geometry instruction developed in the United States during the 20th century in hopes of learning of the development in other countries. We conclude with some recommendations concerning teaching reasoning to high school students and preparing future teachers for this task.

We view Hilbert's geometry as a critique of Euclid and focus on three aspects of it: a) the need for undefined terms, b) continuity axioms, c) the mobility postulate. (We are at the moment being historically cavalier and using 'Hilbert' as a surrogate for an analysis by many contributors including in particular Pasch and Dedekind. The problems that Hilbert addressed had been raised since classical times; Hilbert's simultaneous solution to many of them in the wake of the discovery of non-Euclidean geometry made this book seminal.)

a) Hilbert's recognition that Euclid's definition of e.g a point was meaningless and that the appropriate procedure is to consider a set of axioms that represents objects which "might as well be chairs, tables and beer mugs " plays a foundational role in the modern approach to mathematics and in particular to model theory. Two technical notions arise from the common notion of 'definition'. The basic notions are not defined; rather the system (geometry) is 'defined' by the axioms relating them; auxiliary notions are 'defined' as abbreviation for relations among the basic notions. This modern conception was influenced by Frege [Ste] as well as Dedekind.

b) Hilbert's introduction of continuity axioms meant that he was studying 'geometry over the reals'. This is a notion that was likely meaningless to Euclid. The distinction between the Greek conception of numbers and the modern view of number systems is another variant of insight a). More technically, Hilbert is pointing out are geometries over various fields that have to be considered and founding the modern understanding of the relation between the algebraic properties of the 'coordinatizing near-field' and the properties of the geometry. Note that the introduction of 'between' as a fundamental relation (rather than saying three points are collinear) presages that the coordinatizing ring will be ordered. Note that some analysis of this sort is necessary for a full explanation of similarity.

c) Hilbert recognized that the use of superposition vitiated Euclid's proof of the congruence theorems. His solution was to assume SAS and prove the other two.

We first discuss how the specific choice of the axiom system for geometry affects the high school course. The easiest way to distinguish the various formulations of geometry is to apply the insight of Hilbert which developed into the modern model theoretic notion of a formal language for studying a particular subject.

Reconstructing Euclid, we would say his undefined terms are points and lines and the basic relations are incidence and congruence. Hilbert has as undefined terms, points, lines, planes and the fundamental relations congruence and betweenness. Birkhoff [Bir32] developed an axiomatization of geometry building in the real field axioms. The undefined terms include points, lines, planes real numbers, and functions to the reals giving the length of line segments or the measure of an angle.

In the first half of the twentieth century most U.S. high school geometry were vaguely 'Euclidean'. Influenced by Moise, the 'new math' era developed geometry texts which used Birkhoff's approach of building in the real numbers. The obvious advantage of such a procedure is to unify the study of algebra and geometry and so streamline the curriculum. There are several disadvantages:

1. Basic properties of Euclidean geometry require real proofs; the formalization of algebra results in long derivations of trivialities (example to be provided).

2. This streamlining only works if the students have a good background in algebra; the algebra course in the United States was swamped in the 60's by the increasing percentage of less-prepared students taking algebra.

3. Faced with the inability of students to do (algebra) proofs, the publishers reaction was to remove (geometry) proofs from the curriculum.

Item 3) has resulted in texts which flatten out the geometry and destroy any notion that geometry is proved from a few basic principles. For example Glencoe [BCea05] has 24 postulates; these include SAS, SSS, ASA, and HL (if two right triangles have a hypoteneuse and a leg congruent then the triangles are congruent.) Three of the postulates (the ruler, protractor, and segment addition postulates) tie the geometry to the real numbers so the axioms for the reals are a suppressed additional set of hypotheses. (The derivation of the other 4 congruence theorems from SAS (with no reliance on the mobility postulate) is routine and I think part of my high school education.) The difficulty of teaching from such a text is exacerbated by the difficulty of trying to instill intuitions for geometric notions such as congruence while being hamstrung by locutions which require one to say ‘the measure of angle A equals the measure of angle B’ to express equality of angles.

Klein had stressed the importance of studying transformation of points and seeing geometric properties as those invariant under certain groups of transformations. The ‘new math’ introduced ‘transformations’ into the high school curriculum. But in current texts, transformations (dilations, translations, reflections) are only described. They play no role in the logical development of the subject but are just one more in list words to be memorized.

There is another formulation of geometry that provides a different resolution for the gaps involving superposition. Explicitly introduce another sort of object: a transformation. Provide axioms to make the admitted transformations a group of rigid motions. The use of transformations to replace the mobility postulate has been suggested Weinzweig([Wei97]) but this has not been widely adopted in the United States. I have seen a high school text from the 40’s that used this notion.

Weinzweig axiomatizes the properties of rigid motions and defines congruence. Hartshorne [Har00] begins as Euclid with an undefined notion of congruence, but introduces rigid motions and make their transitivity properties an axiom. In either case, the basic structure is being expanded by adding a sort for transformations.

We have discussed the role of specific axiomatizations in providing an understandable account of basic geometry. In earlier days, one of the reasons to learn geometry is that Euclid provided the most widely accepted model of systematic reasoning. This learning was a pillar of ‘liberal education’. This role of geometry seems not to be emphasized as much in the current standards. But the opening paragraphs of the geometry section of Illinois Learning Standards [Ill06] include, ”Historically, geometry is a way to develop skill in forming convincing arguments and proofs. This goal of developing a means of argument and validation remains an important part of our reasons for studying geometry today.” More specifically Goal 9 C is: Construct convincing arguments and proofs to solve problems. Among the subgoals for various ages is: 9.C.4c Develop and communicate mathematical proofs (e.g., two-column, paragraph, indirect) and counter examples for geometric statements.

A crucial step in meeting this goal is to return to Hilbert’s first insight. The distinction between definition and theorem is lost in the high school text (example to be inserted). The teaching of inference will have to proceed through: a) concrete examples to instill concepts in student minds, b) clear distinction between definition and proof, and c) the study of proofs with content.

I briefly elaborate on a). *Learning* to reason requires a clear understanding of the concepts about which you are reasoning, even if the goal of abstract reasoning is to remove this requirement. Instilling this understanding in high school students requires much more careful attention than usually occurs to the transition between normal and academic vocabulary.

Point b) is an aspect of making clear to students the reasons for proving things. It must be emphasized that this is not merely the verification of truth by deduction from intuitively acceptable axioms. But, that reducing the number of 'memorized' notions and understanding the deductive relations among various propositions is a fundamental strategy for understanding and remembering the information.

The attempt to integrate algebra and geometry hinders the attempt to reach goal c). Rene Thom [Tho98] remarked, "...the contemporary trend to replace geometry with algebra is educationally baneful and should be abolished. There is a simple reason for this: while there are geometry problems, there are no algebra problems. A so-called algebra problem can be only a simple exercise requiring the blind application of algebraic rules ....." Thom overstates the claim. But, not if one restricts to the uses of algebra in proofs in high school geometry books. We have made two major claims:

Providing in high school an example of rigorous reasoning from a clear set of axioms, is the best way to prepare students for abstract reasoning in all fields for a more detailed and abstract investigation of logic. Geometric Sketchpad can be used to lead students to false conclusions and so motivate the need for proof.

As we have illustrated above high school geometry curriculum is most easily described using the vocabulary of modern logic and model theory. Our teacher preparation programs should provide our future secondary teachers with this vocabulary and with the historical context of the geometry course.

References

[BCea05] Boyd, Cummins, and et al. *Geometry*. Glencoe Mathematics. Glencoe, Chicago IL, 2005.

[Bir32] G.D. Birkhoff. A set of postulates for plane geometry. *Annals of Mathematics*, 33:329–343, 1932.

[Har00] Robin Hartshorne. *Geometry: Euclid and Beyond*. Springer-Verlag, 2000.

[Hil71] David Hilbert. *Foundations of geometry*. Open Court Publishers, 1971. German publication 1899.

[Ill06] Illinois Learning Standards: Mathematics. http://www.isbe.state.il.us/ILS/math/standards.htm, 2006.

[Ste] Susan G. Sterrett. Frege and Hilbert on the Foundations of Geometry. written 1994: http://philsci-archive.pitt.edu/archive/00000723/.

[Tho98] R. Thom. 'Modern' Mathematics: An Educational and Philosphic Error? In Thomas Tymoczko, editor, *New Directions in the Philosophy of Mathematics*, pages 67–78. Princeton, 1998.

[Wei97] A.I. Weinzweig. Geometry through transformations. book manuscript, 1997.