# $\delta$-invariant amalgamation classes

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## 1 Introduction

A recent article by R. D. Aref'ev [1] classified all $\delta - invariant$ classes of finite graphs which are closed under amalgamation. A class is $\delta - invariant$ if membership of a graph in the class is determined by the dimension and cardinality of the graph, and dimension and cardinality of all its substructures. This paper also describes the countable generic models for these classes. The classification is expounded with different emphasis from [1] and the analysis of the generics is improved. This analysis provides $\omega$-stable generic structures with an array of saturation and model completeness properties which belies the similarity of their construction. Most of the definitions and notation are carried over from [1] or from Baldwin and Shi [4].

In Section 2, we review some of the basic properties of classes of finite structures with a strong submodel relation and introduce a number of special notations. We formally define $\delta$-invariance in Definition 2.8. The $\delta$-invariant amalgamation classes and described in terms of graph omitting in Section 3 and we prove the classification theorem. Sections 4,5, and 6 establish the saturation, categoricity and stability, and model completeness properties of the associated generics.

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## 2 Preliminaries

Here are some basic notions.

**Definition 2.1**

- For a finite graph $B$, let $e(B)$ denote the number of edges of $B$. Consider the function $\delta(u,v) = \alpha u - \beta v$ where $\alpha, \beta \in \Re^+$. Define $\delta(B) := \delta(\mid B \mid, e(B))$. $\delta(B)$ is called the dimension of structure $B$ and the function $\delta$ is called the dimension function.

- We say a finite structure $A$ is closed in an arbitrary structure $B$ (write $A \leq B$) if $A \subseteq B$ and for every finite $A'$ with $A \subseteq A' \subseteq B$, $\delta(A) \leq \delta(A')$.

- We say $A$ is n-strong in $B$, written as $A \leq_n B$, if for any $B'$ with $A \subseteq B' \subseteq B$ and $|B' \setminus A| \leq n$, $A \leq B'$.

- For finite structures $A$ and $B$, $\delta(A/B)$ denote $\delta(A \cup B) - \delta(B)$. Let $A$ and $B$ be disjoint subsets of structure $C$ then $r(A,B) := e(A \cup B) - e(A) - e(B)$.

Now we list a few simple results which are easily seen from definitions.

**Proposition 2.2** *a) If $A \cap B = \emptyset$ then $\delta(A/B) = \delta(A) - \beta r(A,B)$.*
*b) If $A_1, A_2, B$ are disjoint then $\delta(A_1 \cup A_2/B) = \delta(A_1/B) + \delta(A_2/A_1 \cup B)$.*
*c) If $B \subseteq B_1$ and $B_1 \cap A = \emptyset$ then $\delta(A/B) \geq \delta(A/B_1)$.*
*d) $\leq$ is transitive relation.*
*e) If $A \leq B$ and $C \subseteq b$ then $A \cap C \leq C$.*

Next we expand the concept of dimension in the usual manner to deal with infinite structures.

**Definition 2.3** For any structure $M$ and any finite $A \subseteq M$, we define the dimension $A$ in $M$,

$$d_M(A) := \inf\{\delta(A') : A \subseteq A' \subseteq M\}.$$

This yields:

**Proposition 2.4** *A finite set $A$ is closed in $M$ if $d_M(A) = \delta(A)$ (equivalently: $A \leq M$).*

**Definition 2.5** The infinite set $A$ is closed in $M$ if for every finite $A_0 \subseteq A$, there is a subset $A_0'$ with $A_0 \subseteq A_0' \subseteq A$ and $d_M(A_0) = \delta(A_0')$

**Definition 2.6** The class $\mathbf{K}$ is *cofinal* in a structure $A$ if for all finite substructures $A_0$ in $A$ there is a $A' \in \mathbf{K}_0$ such that $A_0 \subseteq A' \subseteq A$ .

**Definition 2.7** An infinite structure $M$ has *finite closures* if for every finite $A \subseteq M$ there is a finite $B \leq M$ with $A \subseteq B$. *A class* $\mathbf{K}$ *has finite closures if each* $M \in \mathbf{K}$ *does.*

Aref'ev [1] introduced the key notion of this paper stressing the region in the first quartile associated with the class $\mathbf{K}$¿

**Definition 2.8** Let $H$ be a subset of $\omega \times \omega$, and $\delta$ be a dimension function. $\mathbf{K}$ is a $\delta - invariant$ class if it is of the following form:

- Define $v(B) :=<\mid B \mid, \delta(B)>$.
- $\mathbf{K} := \{\text{finite } A : (\forall B \subseteq A) v(B) \in H\}$.

In this paper, we will write $\mathbf{K}_H$ to show that the class depends on the set H.

We can restate this definition, avoiding direct reference to the set $H$. Let $K$ be a class of finite structures. $K$ is *$\delta$-invariant* if for any finite $A$, $A \in \mathbf{K}$, if and only if for every subgraph $B$ of $A$, there is a $B' \in \mathbf{K}$ such that $v(B) = v(B')$

**Note: 2.9** Definition 2.8 does *not* say that a finite graph B is in a class $\mathbf{K}_H$ if $v(B) \in H$. One way to view this definition is to say that two graphs $A$ and $B$ are equivalent if and only if $\{v(C) : C \subseteq B\} = \{v(C) : C \subseteq A\}$. Place this equivalence relation on a collection of finite graphs and describe a special notion of substructure by: $A \sqsubseteq B$ if and only if $\{v(C) : C \subseteq A\} \subseteq \{v(C) : C \subseteq B\}$. Then the class of these equivalence classes of graphs is closed under this notion of substructures if and only if, the class, formed by the collection of finite graphs, is $\delta$-invariant. Notice then that the definition of $\delta$-invariance implies the class is closed under the conventional definition of substructures.

**Definition 2.10** Given finite structures $B$ and $C$, with intersection $A$, a *free amalgamation* of $B$ and $C$ over $A$ is the structure $B \cup C$ with no edges from $B$ to $C$ without a vertex in $A$. This is written as: $B \otimes_A C$.

The following properties of the free amalgam play an important role later.

**Proposition 2.11** *Let* $A = B \cap C$*, then:*

1. *If* $A \leq C$ *then* $B \leq B \otimes_A C$.
2. *If* $A \leq C$*,* $f'$ *is an embedding from* $B' \subseteq B \setminus A$ *into* $C \setminus A$ *such that* $r(f'(B'), C \setminus A) = 0$*, and* $f = f' \cup id_{B \setminus B'}$ *then* $f(B) \leq (B \setminus B') \otimes_A C$.
3. *If* $A \leq_n C$ *then* $B \leq_n B \otimes_A C$.

Proof: 1) is well-known, e.g. [4]. 2) Let $f(B) \subseteq X \subseteq (B \setminus B') \otimes_A C$ and $X' = X \cap C$. Notice then, $r(X \setminus f(B), f(B)) = r(X' \setminus A, A)$ since $(B \setminus B') \otimes_A C$ is a free amalgam, and $r(f'(B'), C \setminus A) = 0$. Also note $X' \setminus A = X \setminus f(B)$. Now the proof follows directly from part 1.

3) Let $B \subseteq X \subseteq B \otimes_A C$, $X' = X \cap C$, and $|X \setminus B| \leq n$. The proof then follows from part 1. $\square$

**Definition 2.12** The model $M$ is called $(\mathbf{K}, \leq)$-*homogeneous-universal* (h.u.) or $(\mathbf{K}, \leq)$-*generic* if it satisfies following conditions:

1. *If $A \leq M, A \leq B \in \mathbf{K}$, then there exists $B \leq M$ such that $B \cong_A B$,*
2. *$M$ has finite closures.*

Finally we name some useful finite graphs and some of their properties.

**Definition 2.13**

1. A graph $C_n = \{a_0, ..., a_{n-1}\}$ in which only the vertices $a_i, a_{i+1}, i < n-2$ and $a_0, a_{n-1}$ are connected by edges, is called the *cycle* of length $n$.
2. A graph $K_n = \{a_0, ..., a_{n-1}\}$ in which only the vertices $a_i, a_{i+1}, i < n-2$ are connected by edges, is called a *chain* of length $n-1$.
3. A graph $A$ is called the *direct sum* of graphs $A_0, \ldots, A_n$ ( written $A = A_1 + ... + A_n$ or $A = \sum_i A_i$, or $A_n^k$ for the direct sum of k isomorphic copies of $A_n$) if $A = \cup_i A_i$ and $i \neq j$, implies $A_i \cap A_j = \emptyset$ and $r(A_i, A_j) = 0$.
4. A graph $A$ is called *connected* if it is not equal to any direct sum of proper nonempty subgraphs.
5. A graph $A$ is called a *minimal subgraph of graph* $B$ with property $\mathcal{P}$ if for any nonempty $C \subset A$, $C$ does not satisfy property $\mathcal{P}$.
6. A graph $A$ is called a *maximal subgraph of graph* $B$ with property $\mathcal{P}$ if for any $C$ with $A \subset C \subseteq B$, $C$ does not satisfy property $\mathcal{P}$.
7. A connected graph with no cycles is called a *tree* (this definition includes a single point). A *complete tree* is a tree in which every vertex has an infinite number of vertices attached to it. We denote a complete tree by $T$.
8. A discrete graph, that is a collection of points with no edges is denoted by $D_m$, where m is the number of points.

**Definition 2.14** An n-cycle is a *fuzzy* cycle if it has a complete tree attached to every vertex; we denote such a graph by $F_n$. An n-cycle which has a tree, not necessarily infinite but containing at least one point not on the cycle, connected to at least one vertex of the cycle is called an *unclean* cycle. A *clean* n-cycle then refers to $C_n$.

**Definition 2.15** A class is said to *omit* a graph $B$, if $B$ is not a member of the class. Similarly, a class allows or *admits* a graph $B$ if it is a member of the class

**Note: 2.16** In this paper, $\mathbf{K}$ will be $\delta$-invariant and so $\mathbf{K} = \mathbf{K}_H$ for some appropriate $H$. Also, the dimension function will be: $\delta(A) =\mid A \mid -e(A)$. So for example the above definitions yield: $\delta(C_n) = 0, \delta(K_n) = 1$, for any $n < \omega$.

# 3 $\leq$-Amalgamation property

In this section we will simply state some propositions which were proved in [1], and only offer a proof of something if the proof or the result is new.

**Proposition 3.1** *Let $A \in \mathbf{K}$ be a connected graph.*
$\delta(A) = 0$ *if and only if $A$ contains exactly one cycle.*

**Proposition 3.2** *If $A \leq B$ then for any $C_m \subseteq B$ either $C_m \subseteq A$ or $C_m \subseteq B \setminus A$.*

With the restrictions of Note 2.16, one can observe several interesting properties about possible $\delta$-invariant amalgamation classes. First, any connected graph can have at most one cyclic subgraph (because negative dimension is not allowed), in which case it has dimension 0. As Theorem 3.10 points out, if $K_3$ is allowed, then by $\delta-$invariance any tree is allowed in the class. Thus in a $\delta-$invariant class $\mathbf{K}$ containing $K_3$, one can only restrict which connected graphs containing a cycle lie in $\mathbf{K}$. On the other hand, if $K_3$ is not allowed, then the class is much simpler. Only direct sums of points, 3-cycles, and $K_2$'s are allowed. Before establishing these facts, we define the following classes:

**Definition 3.3**

1. $\mathbf{K}_0$ is the class of discrete finite graphs
2. $\mathbf{K}_{1,m}$ is the class of finite graphs which omit $K_3$ and $C_3^k$ for all $k > m$.
3. Let the set $I \subset \omega$ have the following properties:
   (a) $m, n \in I$ and $m \neq n \Rightarrow m + n \in I$
   (b) $n, n+1 \in I \Rightarrow \forall m \geq 0(n + m \in I)$

   $\mathbf{K}_{2,I}$ contains all finite graphs of positive dimension and all finite graphs of dimension zero from the following set: $\{B$ :if $A \subseteq B$ and $\delta(A) = 0$ then $|A| \in I\}$.

**Note: 3.4** It can be easily checked that the above described classes are $\delta$-invariant classes. To do so is not hard, but describing the set "H" which defines the class is messy, and we do not reproduce Aref'ev's characterization.

**Theorem 3.5** *Let $(\mathbf{K}, \leq)$ be derived from $\delta$ as in Definition 3.3. Then there are exactly three possible types of amalgamation classes which are $\delta$-invariant. These classes are:*

1. *The class $\mathbf{K}_0$,*
2. *The classes $\mathbf{K}_{1,m}$,*
3. *The classes $\mathbf{K}_{2,I}$,*

This theorem will follow from the following lemmas. The first is obvious

**Lemma 3.6** *The class $\mathbf{K}_0$ contains exactly those finite structures which consist of vertices with no edges; it has the amalgamation property.*

**Lemma 3.7**

1. *A graph is a member of the class $\mathbf{K}_{1,m}$ if and only if it is of the following form: $D_l \oplus K_2^n \oplus C_3^k$, where $l, n < \omega$ and $k \leq m$.*
2. *The members of the set: $\{\mathbf{K}_{1,m} : m \leq \omega\}$ are all the possible $\delta$-invariant classes, which omit $K_3$ but not $K_2$, and have the amalgamation property.*

Proof. Part one of this lemma follows directly from the definition, by noting one fact: every connected graph except $D_1$, $K_2$, and $C_3$ embeds $K_3$.

In order to prove part two of the lemma, we argue that, since we insist the class has the amalgamation property and omits $K_3$, the only possible difference between two classes is the number of 3-cycles allowed in the class. First we claim there can not be a bound on the number of copies of $K_2$'s and $D_l$'s embeddable in a member of $\mathbf{K}_{1,m}$. For all n, $K_2^n \in \mathbf{K}_{1,m}$ follows from the following amalgamation. Let $A$ consist of 1 point labeled 1. Let $B$ consist of one $K_2$, with vertices labeled 1 and 2. It is easy to see $A \leq B$. Now define $C$ to be one $K_2$, with vertices 3,4 and the point 1. The amalgamation of $B$, $C$ over $A$ is one $K_2$ labeled 1,2, and another $K_2$, labeled 3,4. So, inductively the claim holds. Now since the class is closed under substructures, for all l, $D_l \in \mathbf{K}_{1,m}$ follows easily.

Finally we check that for each $m$, $\mathbf{K}_{1,m}$ is closed under amalgamation. Let $B, C \in \mathbf{K}_{1,m}, A = B \cap C, A \leq B, A \leq C$ and Suppose $E$ is the free amalgamation of $B$ and $C$ over $A$. From part *i)* of this lemma , we know $B$ and $C$ can be written in the form $D_l \oplus K_2^n \oplus C_3^k$, and $D_{l'} \oplus K_2^{n'} \oplus C_3^{k'}$ , respectively, and without loss of generality, $k' \geq k$. Since $A \leq B$ and $A \leq C$ no element of $(B \cup C) \setminus A$ can be connected to more than one element of $A$. Moreover each cycle embedded in $E$ is embedded in one of $A, B \setminus A, C \setminus A$. Since $A \in \mathbf{K}_{1,m}$, $A \simeq D_{l''} \oplus K_2^{n''} \oplus C_3^{k''}$. So $E \simeq D_{l'''} \oplus K_2^{n'''} \oplus C_3^{k'''} \oplus K_3^j$, where $l''' = l + l' - l''$ , $k''' = k + k' - k''$ and $n''' = n + n' - 2j$ (If $E$ contained a copy of $K_4$ one of $B, C$ would contain a copy of $K_3$.). To find an amalgam of $B$ and $C$ over $A$

which is in $\mathbf{K}_{1,m}$, identify the ends of any $K_3$ in $E$ and identify each copy of $C_3$ in $B \setminus A$ with a copy of $C_3$ in $C \setminus A$. $\square$

Finally we will look at the most complicated of the $\delta - invariant$ classes. In order to get some feel for what these classes might look like, consider the following example, which uses the next lemma to describe the class $\mathbf{K}_{2,I}$, for a particular $I$.

**Example 3.8** Define $I = \{3, 7, 10, 13, 16, 17, 18...\}$; first notice that I satisfies the properties in Definition 3.3. Now, $C_n$ is a subgraph of a graph in $\mathbf{K}_{2,I}$ if and only if $n \in I$. In particular $C_3, C_7, C_{10}, C_{13} \in \mathbf{K}_{2,I}$ and there are no unclean examples of these n-cycles in the class. Since $6, 14 \notin I$, we can see that $C_3$, and $C_7$ occur uniquely in any graph which is a member of $\mathbf{K}_{2,I}$, while there are direct sums of any finite number of $C_n$'s, where $n \in I$, and $n > 7$, in graphs belonging to this class. Finally, for any $n \geq 16$, there are both unclean and clean n-cycles in graphs of this class.

**Notation: 3.9** We say $I$ contains an *interval* if there exists an $n \in I$ such that for $m \geq n$, m is in $I$. Also, we will denote the least such n as $i_0$.

**Lemma 3.10**

1. *Each member of the class $\mathbf{K}_{2,I}$ is a direct sum of a finite number of the following finite graphs:*
   - *(a) a graph of positive dimension, which is a finite direct sum of trees of arbitrary finite cardinality,*
   - *(b) $C_n$ if $n \in I$ and $2n \in I$,*
   - *(c) unclean cycles $C_m$, with any finite cardinality, for all $m \geq i_0$ if $I$ contains an interval,*

   *and at most one $C_n$ if $n \in I$ and $2n \notin I$.*
2. *The members of the set $\{\mathbf{K}_{2,I} : I \subset \omega$ and $I$ has the properties in Definition 3.3 $\}$ are all the possible $\delta - invariant$ classes, which admit $K_3$ and have the amalgamation property.*

Proof. Since all graphs of positive dimension are in each $\mathbf{K}_{2,I}$, statement 1) part a), follows easily. So now for the remaining part of 1) we determine exactly which 0-dimensional structures occur as members of $\mathbf{K}_{2,I}$. With Proposition 3.1 in mind, first consider clean cycles. Suppose $n \in I$ and $2n \in I$. Then from part 1) of the definition of $I$, $n + m(n) = (m+1)n \in I$, for all positive integers $m$. Now a subgraph of a direct sum of clean cycles is a graph of positive dimension, which is in $\mathbf{K}_{2,I}$ by definition, or a direct sum of a smaller number of clean cycles. Part b) of 1) follows by induction. Suppose in contrast, that $n \in I$, but

$2n \notin I$. Then there is at most one copy of $C_n$ embedded in any member of $\mathbf{K}_{2,I}$. For since $2n \notin I$, $C_n^2$ is omitted from $\mathbf{K}_{2,I}$.

It remains to verify part c) of 1). In order for an unclean cycle $C_n$ to be a member of $\mathbf{K}_{2,I}$, $n$ and $n+1$ must be in $I$. By the second part of the definition of $I$, $I$ must then contain an interval. Let m be in this interval. Then there exists a connected graph of dimension 0, and cardinality $m \in I$, so $C_m \in \mathbf{K}_{2,I}$ (Note a subgraph of a clean cycle is a positive dimension graph and thus is in $\mathbf{K}_{2,I}$). Thus $C_m$ with an edge attached is in $\mathbf{K}_{2,I}$, as $m+1 \in I$, and all of its substructures are in $\mathbf{K}_{2,I}$ too. So inductively any connected graph of dimension 0 of finite cardinality k greater than m, which extends $C_m$, is in $\mathbf{K}_{2,I}$. If $n \in I$ but $n+1 \notin I$, then $C_n$ can not have an additional edge attached to it. Thus every $C_k$ in the class, with k less than $i_0$, must be clean. $\square$

So, having completed part 1) of the lemma, we now deal with part 2). Next we show the class $\mathbf{K}_{2,I}$, has the amalgamation property. Let $C, B \in \mathbf{K}_{2,I}$, $A = B \cap C$, $A \leq B$, and $A \leq C$. Lemma 3.11 and Lemma 3.12 (below) imply that $B \otimes_A C \in \mathbf{K}_{2,I}$ if and only if there is only one copy of a clean cycle $C_m$, where $2m \notin I$, in $B \otimes_A C$. If there is more than one such cycle in the amalgam, we define an embedding $f' : B \setminus A \longrightarrow C \setminus A$ which identifies the unique clean cycle in $B \setminus A$ with the corresponding clean cycle in $C \setminus A$. Now by Proposition 2.11 we are done.

What remains to be shown is: a minimal $\delta-$invariant class $\mathbf{K}$ with AP, which admits $K_3$ and a finite set of 0 dimensional graphs, has the form $\mathbf{K}_{2,I}$ for some $I$ satisfying the conditions of Definition 3.3 part 3). We claim that, if $\mathbf{K}$ admits $K_3$, has AP and is $\delta$-invariant, then $\mathbf{K}$ admits all graphs of positive dimension. What we need to show then is: for all integers $i \geq j > 0$, $< i, j > \in H$, where $\mathbf{K} = \mathbf{K}_H$ as in Definition 2.8. We first show: if $K_3 \in \mathbf{K}_{2,I}$ then $K_n \in \mathbf{K}_{2,I}$ for all n. Consider the following amalgamation. Let $K_2 \in A$, with vertices 1,2. Let $K_3 \in B, C$ with vertices 0,1,2 and 1,2,3 respectively. Then an amalgamation of $B$ and $C$ over $A$ is $K_4$, with vertices 0,1,2,3. Notice, points 0 and 3, can not be identified, since this map of C into the amalgam would not an embedding. So inductively, for all n, $K_n \in \mathbf{K}$. Now, because the class is closed under substructures, for all $m < \omega$, $D_m$ is in the class. Finally to prove the claim. Choose $i, j > 0$ with $\omega > i \geq j$. We have $K_{2i} \in \mathbf{K}$ , so $E = D_{j-1} \oplus K_{i-j+1} \in \mathbf{K}$, since this is a subgraph of $K_i$. Thus $v(E) = < i, j > \in H$.

So we may concern ourselves with graphs of dimension 0. We now show $I$ satisfies the conditions described in Definition 3.3 part 3). Note that since all graphs of positive dimension are in $\mathbf{K}_{2,I}$, $C_n$ is in $\mathbf{K}_{2,I}$ if and only if n is in $I$. We ask this question: Given graphs $A$, $B$, of dimension 0, with $A$ or $B$ in the class, what other dimension 0 graphs are forced to be in the class by the amalgamation property and $\delta - invariance$? First, if $C_n, C_m \in \mathbf{K}_{2,I}$ and $n \neq m$, then $C_m \oplus C_n \in \mathbf{K}_{2,I}$. This follows if we just define $B$ to be $C_n$ and $C$ to be $C_m$. If we amalgamate over the empty set, the amalgam must be $B \cup C$. This gives us Definition 3.3 part 3a).

Notice that if $C_n, C_{n+1} \in \mathbf{K}_{2,I}$, then for all $m \geq n$, $C_m \in \mathbf{K}_{2,I}$. For this consider the following amalgamation. Let $A$ be the graph $C_n$ with vertices 1 through n. Define $B$ to be the connected graph $C_n$, with the point $\{b\}$ attached to a vertex $i \in C_n$. Define $C$ similarly, except instead of the point $\{b\}$ attach the point $\{c\}$ to the vertex $i+1 \in C_n$. Now, the amalgamation of $B$ and $C$ over $A$ can not identify b to c (else the resulting graph would have negative dimension), so the amalgam must be $C_n$ with points b and c attached to vertices i and i+1 respectively. This is a graph with cardinality n+2, and dimension 0, and by $\delta - invariance$, we must have then $C_{n+2} \in \mathbf{K}_{2,I}$. This gives Definition 3.3 part 3b).

**Lemma 3.11** *Let $A \leq B$, $A \subseteq C$, then $B \otimes_A C \in \mathbf{K}_{2,I}$ if and only if the following two statements hold:*

1. *if $C_k^2 \subseteq B \otimes_A C$ then $2k \in I$*
2. *if $E \subseteq B \otimes_A C$ is an unclean cycle then $|E| \in I$*

Proof: The proof that the free amalgam is in the class implies the conjunction of the two statements is obvious.

Let $E$ be a minimal graph in $B \otimes_A C$ such that $E \notin \mathbf{K}_{2,I}$. We show $E$ is not of negative dimension. Define $C_0 = E \cap (C \setminus A)$, $B_0 = E \cap (B \setminus A)$, and $A_0 = E \cap A$. First we notice the following relation: $\delta(E) = \delta(C_0) + (\delta(B_0) - \delta(A_0)) = 0$. This follows from the fact that $C_0, A_0, B_0 \in \mathbf{K}_{2,I}$ and $A_0 \leq B_0$.

Assume now that $E$ is not in $\mathbf{K}_{2,I}$. Suppose $E$ is not connected, say $E = E_1 \oplus E_2$. Then $E_1, E_2 \in \mathbf{K}_{2,I}$ (by minimality of $E$), but $E_1 \oplus E_2 \notin \mathbf{K}_{2,I}$. It follows from Lemma 3.10 part 1), that $E_1 = E_2 = C_n$ and $2n \notin I$.

Suppose $E$ is connected. Since $\delta(E) = 0$, $E$ contains exactly one cycle. Furthermore, since $B_0, C_0 \neq 0$ (otherwise $E$ would be a subgraph of a member of the class) and $A \leq B$, $E$ is an unclean cycle. By hypothesis $|E| \in I$, and by the minimality of $E$, all of $E$'s substructures are in $\mathbf{K}_{2,I}$, thus $E \in \mathbf{K}_{2,I}$. This contradiction yields the result. $\square$

The following lemma is independent of the previous two lemmas. It it more specific than is required in Lemma 3.10, but this stronger form is used in the following section.

**Lemma 3.12** *Suppose $A \leq B$ and $A \leq_n C$, $B, C \in \mathbf{K}_{2,I}$, and $n \geq i_0$. If $E \subseteq B \otimes_A C$ is an unclean cycle, then $|E| \in I$*

Proof. Let $C'$ be the unique clean cycle contained in $E$. If $|C'| > n$, then $|E| \geq |C'| \geq i_0$, so $|E| \in I$. If not, by Proposition 3.2, $C'$ is contained in one of $B \setminus A, C \setminus A$, or $A$, as $E$ is connected. In each case there is an $\{a\}$ in one of $B$ or $C$ such that $r(C', a) \neq 0$ and $C' \cup \{a\}$ is contained in one of $B$ or $C$. Either implies that $|C'|$ and $|C'|+1$ are in $I$. But $|E| \geq |C'|+1$ is in $I$ as required. $\square$

## 4 $\omega-saturation$

We now focus our attention on certain properties of the generics of these classes. The first task will be to determine whether the generic is $\omega-saturated$ or not. This will be done with the following propositions. First we report two definitions and a theorem from [4], reformulating observations by Poizat [5], which are the key to the proof of the propositions.

**Definition 4.1** A class $\mathbf{K}$ has the *uniform amalgamation property* (uap) if the following condition holds for every $A \leq B \in \mathbf{K}$. For every $m \in \omega$ there is an $n = f_B(m)$ such that if $A \leq_n C$ there there is a $D$, a strong embedding of C into D, and an m-strong embedding of B into D that completes a commutative diagram with the given embeddings of $A$ into $B$ and $C$.

**Notation: 4.2** Define $\overline{\mathbf{K}}$ to be the class of structures which are cofinal in $\mathbf{K}$

**Lemma 4.3** *The class* $\overline{\mathbf{K}}$ *has finite closures with respect to* $(\mathbf{K}, \leq)$ *if the range of* $\delta$ *is well ordered.*

Proof. Let $A$ be in $\mathbf{K}_0$, and $M$ in $\overline{\mathbf{K}}$. Suppose $A \not\leq M$, then there is a finite $B_1$ in $M$ containing $A$ such that $\delta(B_1) \leq \delta(A)$. If $B_1 \leq M$ we are done; otherwise as before there is a $B_2$ in $M$ containing $B_1$ with lower dimension. Since the range of $\delta$ is well ordered, in a finite number of steps, say $n$, there will not exist a finite $C$ in $M$ which contains, and has lower dimension than $B_n$. Checking definitions we conclude, $A \subseteq B_n \leq M$.$\square$

We rely on Theorem 2.28 of [4] which asserts:

**Theorem 4.4** *If a class* $\overline{\mathbf{K}}$ *has finite closures, then the generic model of* $\overline{\mathbf{K}}$ *is* $\omega-saturated$ *if and only if* $\mathbf{K}$ *has uap*

Before going on, we describe up to isomorphism the unique generic model of each of the above defined classes. We outline a proof for the most complicated case; the simpler cases can be proved in a similar manner.

**Notation: 4.5** If the set $I$ contains an interval we denote it by $\overline{I}$. The set of elements $n \in I$ such that $2n \notin I$ will be denoted by $I'$.

**Theorem 4.6** *The unique generic up to isomorphism of the class:*

- $\mathbf{K}_0$ *is:* $D_\omega$,
- $\mathbf{K}_{1,m}$ *is:* $K_2^\omega \oplus C_3^m$,
- $\mathbf{K}_{1,\omega}$ *is:* $K_2^\omega \oplus C_3^\omega$,
- $\mathbf{K}_{2,I}$ *is:* $T^\omega \oplus \sum_{k\in I\setminus(I'\cup\overline{I})} C_k^\omega \oplus \sum_{k\in\overline{I}} F_k^\omega \oplus \sum_{i\in I'} C_i$.

Proof. We predict that the generic model, $M$, for $\mathbf{K}_{2,I}$ is in the form given in this lemma. Then we use the fact that a countable $(\mathbf{K}_{2,I}, \leq)$-homogeneous and $(\mathbf{K}_{2,I}, \leq)$-universal model is unique up to isomorphism to show $M$ is indeed the generic for $\mathbf{K}_{2,I}$. The fact $M$ is $(\mathbf{K}_{2,I}, \leq)$-universal is clear.

We need to show $M$ is $(\mathbf{K}_{2,I}, \leq)$-homogeneous. We will show that a local isomorphism between finite closed sets extends to an automorphism of $M$. Any permutation of $M$, which takes a component to an isomorphic counterpart (possibly to itself), is an automorphism of $M$. Also, any strong finite subgraph of $M$ consists of connected components which intersect distinct components of $M$. Thus, any finite number of isomorphisms between distinct finite closed connected subgraphs of $M$ can be jointly extended to an automorphism of $M$. So we may restrict our attention to closed connected subgraphs.

Note then, for example, any copy of $T$ will always be sent to another copy of $T$, or itself. Let $A \leq T$, then $\delta(A) = 1$. Thus the only component of $M$ in which an isomorphic copy of $A$, say $A'$, can be strong, is another copy of $T$, say $T'$. So an isomorphism of $A$ to $A'$ will extend to an automorphism of $M$ taking $T$ to $T'$. Similarly, if $C$ is finite connected strong submodel of a fuzzy cycle $F$, then $C$ must be a cycle, or an unclean cycle. Now the clean cycle in $C$, say $C_n$, must be sent to another copy of $C_n$ in $M$. But all copies of $C_n$ are in copies of $F$. Suppose then $C$ gets mapped to $C'$ in $F'$, then an isomorphism of $C$ to $C'$ will extend to an automorphism of $M$ taking $F$ to $F'$. Finally if $B$ is strong in $H$, a copy of $C_n$, which is a component of $M$, then $B = H$. Since the only cycles in $M$ which have cardinality $n$ are clean, $H$ must be sent to another copy of the component $C_n$ in $M$. So we conclude $M$ is $(\mathbf{K}_{2,I}, \leq)$-homogeneous. $\square$

The first proposition is clear, and is stated without a proof.

**Proposition 4.7** *The generic model of the class $\mathbf{K}_0$ is $\omega-$ saturated.*

**Proposition 4.8** *The generic model of the class $\mathbf{K}_{1,m}$ is $\omega-$ saturated for all $m \leq \omega$.*

Proof. We make the following claim: $A \leq_3 C$ implies $A \leq C$. Suppose $A \not\leq C$ then $\delta(C) < \delta(A)$. Now clearly there is at least one edge between $C \setminus A$ and $A$ (else $A \leq C$). So there exists a minimal subgraph $X$ of $C$ with the property that it has the least dimension among all subgraphs $B$, where $B$ intersects both $A$ and $C \setminus A$. The characterization of members of $\mathbf{K}_{1,m}$ implies the set $X$ will be isomorphic to $C_3$. So we have $A \not\leq A \cup X$, which implies $A \not\leq_3 X \cup A$, and finally $A \not\leq_3 C$. Notice this argument was independent from the value of $m$ in the statement of the lemma. So the proposition easily follows, from the claim and Theorem 4.4 $\square$

**Proposition 4.9** *The generic model of the class $\mathbf{K}_{2,I}$ where $I$ contains an interval is $\omega-$ saturated.*

Proof. We will show that the class $\mathbf{K}_{2,I}$ has uap, and with this, using Theorem 4.4 conclude that the generic model of this class is $\omega$-saturated. Choose graphs $A, B \in \mathbf{K}_{2,I}$ such that $A \leq B$. For any $B$, let $f_B(x) = max(i_0, x)$. Fix m, and $C$ with $A \leq_{f_B(m)} C$. We must find $D \in \mathbf{K}_{2,I}$ with $B \leq_m D$ and $C \leq D$. Now let $C \in \mathbf{K}_{2,I}$, if $f_B(m) \geq |C \setminus A|$, then we have that $A \leq C$. Thus in this case, the proof that the class exhibits uap, is the same as that for the class having $AP$, which is done in the last part of Lemma 3.10. So for the rest of the proof assume $f_B(m) < |C \setminus A|$,

Now, we use Proposition 3.11 to argue that there exist an amalgam of $B$ and $C$ over $A$ which is in the class. Suppose not, then statement 1) or 2) of Proposition 3.11 fails. Let us first focus on statement 1). For each $l \in I'$, $l < f_B$ so if $C_l$ is embedded in $B$ or $C$ the image is contained in exactly one of $A, B \setminus A$ or $C \setminus A$ by Proposition 3.2. Let $C''$ be the subgraph of $B$ which is isomorphic to $\sum_{l \in F} C_l$, where $F \subseteq I'$ and $l \in F$ if and only if $C_l$ embeds into both $B \setminus A$ and $C \setminus A$. Let $B' = B \setminus C''$ and $D = B' \otimes_A C$.

Clearly $D$ satisfies Lemma 3.11 part 1). By Lemma 3.12, if $E \subseteq B' \otimes_A C$ is an unclean cycle, $|E| \in I$. So by Lemma 3.11 $D \in \mathbf{K}_{2,I}$. Since $f_B \geq m$ by Lemma 2.11 $B' \leq_m B' \otimes_A C$ and clearly $C \leq B' \otimes_A C$. It remains to show that there is an $m$ - strong embedding of $B$ into $D$.

If $B' = B$ we are done. If not, then we construct $f' : B \longrightarrow B' \otimes_A C$. Define an embedding $g$, of $C''$ into $C$ in the obvious way, and define $f'$ to be: $f' = id_{B'} \cup g$. Clearly $f'$ is an embedding, and since $\delta(B) = \delta(B')$ and $r(f'(B), B' \otimes_A C) = r(B', B' \otimes_A C)$, it is an $m$ strong embedding as desired. □

**Proposition 4.10** *The generic model of the class $\mathbf{K}_{2,I}$ where $I$ consists of exactly one element is $\omega$ − saturated.*

Proof. From Theorem 4.6, $M$ is isomorphic to an infinite collection of complete trees direct summed with one clean cycle. Any 1-type of $M$ says, "I am on a tree", or "I am on the cycle". The result follows. □

**Proposition 4.11** *The generic model of the class $\mathbf{K}_{2,I}$ where $I$ does not contain an interval but contains more than one element, is not $\omega$ − saturated.*

Proof: Fix a set $I$. Define the graph $A$ to be a point, and the graph $B$ to be as in Fig. 2. Now suppose for example $f_B(x) = 12$. Now $A \leq_{12} C_{13}$, so define the graph $C$ to be $C_{13}$. The free amalgam of $B \otimes_A C$ is shown in Fig. 1.

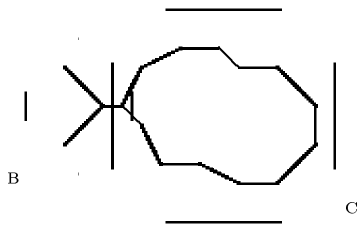


Fig. 1

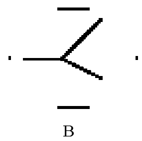


Fig. 2

From the figure it is clear an amalgam of $B$ and $C$ over $A$ must contain $C_{13}$. It should also be clear that since the embeddings must be the identity on $A$, the graph $B$ must be connected to the graph $C$. However, there is no way of embedding $B$ into $C$, and $B \otimes_A C \notin \mathbf{K}_{2,I}$ (since there are no unclean cycles in the class).

We can observe then, $f_B(x)$ needs to be greater then the cardinality of any clean cycle in $C$ to prevent counterexamples similar to the one given. However, if $I$ contains more than two elements, $I$ does not have an upper bound. Thus a function $f_B$ with this property does not exist. $\square$

# 5 Categoricity and Stability

In this section we state the categoricity properties of these classes. In some cases the proofs are clear. Only in the case of $\mathbf{K}_{2,I}$ does some work have to be done.

**Notation: 5.1** Let M be the theory of the generic of $\mathbf{K}$. We denote $Th(M, \mathbf{K})$ as the theory of the generic M for the class $\mathbf{K}$.

The following two results are evident by inspection of the generic as determined in Theorem 4.6.

**Proposition 5.2** *$Th(M, \mathbf{K}_0)$ is both $\aleph_0$ and $\aleph_1$ categorical.*

**Proposition 5.3** *$Th(M, \mathbf{K}_{1,m})$ is both $\aleph_0$ and $\aleph_1$ categorical.*

**Proposition 5.4** *$Th(M, \mathbf{K}_{1,\omega})$ is $\aleph_0$ categorical, but not $\aleph_1$ categorical.*

Proof. There is a model $M$ of $Th(M, \mathbf{K}_{1,\omega})$ of size $\aleph_1$ such that there are $\aleph_0$ copies of $C_3$ and $\aleph_1$ copies of $K_2$. There is also a model $N$ of $Th(M, \mathbf{K}_{1,\omega})$ with $\aleph_0$ copies of $K_2$ and $\aleph_1$ copies of $C_3$. Clearly $M \not\cong N$. $\square$

**Proposition 5.5** *$Th(M, \mathbf{K}_{2,I})$ is neither $\aleph_0$ nor $\aleph_1$ categorical.*

Proof. For $\aleph_0$-categoricity we observe that since an infinite tree is embedded in each model of $Th(M, \mathbf{K}_{2,I})$, the number of two types in this structure is not finite, and therefore the theory can not be $\aleph_0$ categorical.

For $\aleph_1$-categoricity we note there are no means by which a first order theory can determine the number of neighbors a vertex has in a tree, if the number is not finite. Thus in one model a single tree could exist. This tree could have every vertex with $\aleph_1$ neighbors except one which has $\aleph_0$ neighbors. Another model of $Th(M, \mathbf{K}_{2,I})$ could contain only one tree such that every vertex of this tree has $\aleph_0$ neighbors except one which has $\aleph_1$ neighbors. These two models are clearly not isomorphic. □

We can show that all these theories are $\omega$-stable. The following theorem follows from Lemma 2.21 and Theorem 3.34 in [4].

**Theorem 5.6** *All theories which arise from $\omega$-saturated generic models of $\delta$-invariant classes (where the range of $\delta$ is well ordered), are $\omega$-stable.*

This leaves out only one theory where the generic was not saturated. It is easy to check that there are only countable many types over the countable saturated model of $Th(M, \mathbf{K}_{2,I})$ so we have:

**Proposition 5.7** *$Th(M, \mathbf{K}_{2,I})$ where $I$ does not contain an interval, and $|I| = \omega$, is $\omega$-stable.*

# 6 Model Completeness

In this section we determine the model completeness of the theory of the generics of the various classes. The notation used in this section for the theory of the generic is carried over from the previous section. By inspection we see:

**Proposition 6.1** *$Th(M, \mathbf{K}_0)$ is model complete.*

**Proposition 6.2** *$Th(M, \mathbf{K}_{1,m})$ is model complete.*

Proof. Any models $A$ and $B$ of $Th(M, \mathbf{K}_{1,m})$ must have the same number of $C_3$'s. If $A \subseteq B$, then $B$ must have at least as many $K_2$'s as $A$ does. It follows there is an elementary embedding of $A$ into $B$. □

**Proposition 6.3** *$Th(M, \mathbf{K}_{1,\omega})$ is not model complete.*

Let $A$ be the generic of $Th(M, \mathbf{K}_{1,\omega})$. Construct $B$ by extending a $K_2$ in $A$ to a $C_3$ in $B$. Clearly $B$ is a model of $Th(M, \mathbf{K}_{1,\omega})$ and $A$ is a substructure of $B$. However $A$ is not an elementary substructure of $B$, and thus $Th(M, \mathbf{K}_{1,\omega})$ is not model complete. □

**Proposition 6.4** *$Th(M, \mathbf{K}_{2,I})$ is not model complete.*

Proof. Let $A$ be the generic of $Th(M, \mathbf{K}_{2,I})$. Let $T \subset A$ be a complete tree. Choose any vertex $a$ in $T$ with two neighbors, say $b$ and $c$. Construct $B$, a model of $Th(M, \mathbf{K}_{2,I})$, by removing $a$ from $A$. Consider the formula $\sigma : \exists x(e(x, b) \wedge e(x, c))$. Now we notice, if there was a path from $b$ to $c$, not through $a$, then $T$ would have a cycle. Therefore $B \subset A$ and $A \models \sigma(x; b, c)$ but $B \not\models \sigma(x; b, c)$. □

A slight extension of the argument for Proposition 6.3 shows that the class of models of $Th(M, \mathbf{K}_{1,\omega})$ is not closed under unions of chains and so the theory is $\Pi_2$-axiomatizable. In contrast, Baldwin and Kueker noted (with different notation) [2] that $\mathbf{K}_{2,\emptyset})$ is $\Pi_2$-axiomatizable and not model complete.

The following generalization of model complete was defined in [3].

**Definition 6.5** A theory $T$ is said to be *nearly model complete* if every formula is equivalent in $T$ to a Boolean combination of $\Sigma_1$-formulas.

**Theorem 6.6** *Let* $\mathbf{K}$ *be a* $\delta$*-invariant amalgamation class.* $Th(M, \mathbf{K})$ *is nearly model complete.*

Proof. Defining $(\mathbf{K}, \leq)$ (as in Definition 2.1) by $A \leq B$ if $\delta(B'/A) \geq 0$ for all $B'$ with $A \subseteq B' \subseteq B$ determines a relation which satisfies the hypotheses of Theorem 1.29 of [3]. By Corollary 1.31 of the same paper the theory of a $(\mathbf{K}, \leq)$-generic structure is nearly model complete. □

Note that this result is obvious when the generic is $\omega$-saturated; so we rely on [3] only for $\mathbf{K}_{2,I}$ where is $I$ is infinite but does not contain an interval.

# 7 Conclusion

Below is a table which lists the properties the theory of the generic for each $\delta$-invariant amalgamation class. The column for Model completeness is headed by M.C., and that for $\omega$-saturation of the generic is headed by Generic Sat.

| | $\omega$-stable | $\aleph_0$-cat. | $\aleph_1$-cat. | M. C. | Generic Sat. |
|---|---|---|---|---|---|
| $\mathbf{K}_0$ | Yes | Yes | Yes | Yes | Yes |
| $\mathbf{K}_{1,m}$ | Yes | Yes | Yes | Yes | Yes |
| $\mathbf{K}_{1,\omega}$ | Yes | Yes | No | No | Yes |
| $\mathbf{K}_{2,I}$ $\overline{I} = 0$ and $\|I\| = \omega$ | Yes | No | No | No | No |
| $\mathbf{K}_{2,I}$ $\|I\| = 1$ | Yes | No | No | No | Yes |
| $\mathbf{K}_{2,I}$ $\overline{I} \neq 0$ | Yes | No | No | No | Yes |

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