Model Companions of T_{Aut} for stable T

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March 13, 2003

Let T be a complete first order theory in a countable relational language L. We assume relation symbols have been added to make each formula equivalent to a predicate. Adjoin a new unary function symbol σ to obtain the language L_σ; T_{Aut} is obtained by adding axioms asserting that σ is an L-automorphism.

The modern study of the model companion of theories with an automorphism has two aspects. One line, stemming from Lascar [6], deals with 'generic' automorphisms of arbitrary structures. A second, beginning with Chatzidakis and Hrushovski [1] and questions of Macintyre about the Frobenius automorphism is more concerned with specific algebraic theories. This paper is more in the first tradition: we find general necessary and sufficient conditions for a stable first order theory with automorphism to have a model companion.

Kikyo investigates the existence of model companions of T_{Aut} when T is unstable in [3]. He also includes an argument of Kudaibergenov showing that

*Partially supported by NSF grant DMS 9803496 and DMS-0100594.

†This is paper 759 in Shelah's bibliography. The authors thank Mittag-Leffler Institute, Rutgers University, and the The Israel Science Foundation for partial support of this research.

if T is stable with the finite cover property then T_{Aut} has no model companion. This argument was implicit in [2] and is a rediscovery of a theorem of Winkler [9] in the 70's. We provide necessary and sufficient conditions for T_{Aut} to have a model companion when T is stable. Namely, we introduce a new condition: T admits obstructions, and show that T_{Aut} has a model companion if and only if T does not admit obstructions. This condition is a weakening of the finite cover property: if a stable theory T has the finite cover property then T admits obstructions.

Kikyo also proved that if T is an unstable theory without the independence property, T_{Aut} does not have a model-companion. Kikyo and Shelah [5] have improved this by weakening the hypothesis to: T has the strict order property.

For p a type over A and σ an automorphism, $\sigma(p)$ denotes $\{\phi(\mathbf{x}, \sigma(\mathbf{a})) : \phi(\mathbf{x}, \mathbf{a}) \in p\}$. References of the form II.4.13. are to [8]. We thank Assaf Hasan for his comments on the manuscript. We also thank the referee for his insightful remarks and in particular forcing us to make explicit and prove Lemma 2.9, which may have independent interest.

Further related work is contained in [**?**] which investigates when T_{Aut} has a stable model completion.

1 Example

In the following example we examine exactly why a particular T_{Aut} does not have a model companion. Eventually, we will show that the obstruction illustrated here represents the reason T_{Aut} (for stable T) can fail to have a model companion. Let L contain two binary relation symbols E and R and unary predicates P_i for $i < \omega$. The theory T asserts that E and R are equivalence relations and that E has infinitely many infinite classes, which are refined by R into two-element classes. Moreover, each P_i holds only of elements from one E-class and contains exactly one element from each R class of that E-class. Thus, $x \neq y \wedge P_i(x) \wedge P_j(y)$ implies $\neg R(x, y)$ if $i \neq j$.

Now, T_{Aut} does not have a model companion. To see this, let $\psi(x, y, z)$ be the formula: $E(x, z) \wedge E(y, z) \wedge R(x, y) \wedge x \neq y$. Let Γ be the L_σ-type in the variables $\{z\} \cup \{x_i y_i : i < \omega\}$ which asserts $\psi(x_i, y_i, z)$ holds for each i, the sequence $\langle x_i y_i : i < \omega\rangle$ is L-indiscernible, the x_i are distinct and the y_i

are distinct, and for every $\phi(x, \mathbf{w}) \in L(T)$:

$$(\forall \mathbf{w}) \bigvee \{ \bigwedge_{i \in U} \phi(x_i, \mathbf{w}) \leftrightarrow \phi(y_i, \sigma(\mathbf{w})) : U \subseteq \lg(\mathbf{w}) + 3, |U| > (\lg(\mathbf{w}) + 3)/2 \}.$$

Thus if $\langle b_i c_i : i < \omega \rangle a$ realize Γ in a model M,

$$\sigma(\mathrm{avg}(\langle \mathrm{b_i} : \mathrm{i} < \omega \rangle / \mathrm{M})) = \mathrm{avg}(\langle \mathrm{c_i} : \mathrm{i} < \omega \rangle / \mathrm{M}).$$

For any finite $\Delta \subset L(T)$, let $\chi_{\Delta,k}(\mathbf{x}, \mathbf{y}, z)$ be the conjunction of the Δ-formulas satisfied by $\langle b_i c_i : i < k \rangle a$ where $\langle b_i c_i : i < k \rangle a$ are an initial segment of a realization of Γ. Let $\theta_{\Delta,k}$ be the sentence

$$(\forall x_0, \ldots x_{k-1}, y_0, \ldots y_{k-1}, z) \chi_{\Delta,k}(x_0, \ldots x_{k-1}, y_0, \ldots y_{k-1}, z) \rightarrow$$

$$(\exists x_0, y_0, x_1, y_1)[\psi(x_0, y_0, z) \wedge \psi(x_1, y_1, z) \wedge \sigma(x_1) = y_1].$$

We claim that if T_{Aut} has a model companion T^*_{Aut}, then for some k and Δ,

$$T^*_{\mathrm{Aut}} \vdash \theta_{\Delta,k}.$$

For this, let $M \models T^*_{\mathrm{Aut}}$ such that $\langle b_i c_i : i < k \rangle a$ satisfy Γ in M. Suppose $M \restriction L \prec N$ and N is an $|M|^+$-saturated model of T. In N we can find b, c realizing the average of $\langle b_i : i < \omega \rangle$ and $\langle c_i : i < \omega \rangle$ over M respectively. Then

$$\sigma(\mathrm{avg}(\langle \mathrm{b_i} : \mathrm{i} < \omega \rangle / \mathrm{M})) = \mathrm{avg}(\langle \mathrm{c_i} : \mathrm{i} < \omega \rangle / \mathrm{M})$$

and so there is an automorphism σ^* of N extending σ and taking b to c. Since (M, σ) is existentially closed (T^*_{Aut} is model complete), we can pull b, c down to M. By compactness, some finite subset Γ_0 of Γ suffices and letting Δ be the formulas mentioned in Γ_0 and k the number of x_i, y_i appearing in Γ_0 we have the claim.

But now we show that if (M, σ) is any model of T_{Aut}, then for any finite Δ and any k, $(M, \sigma) \models \neg\theta_{\Delta,k}$. For this, choose b_i, c_i for $i < k$ which are E-equivalent to each other and to an element a in a class P_j where P_j does not occur in Δ and with $R(b_i, c_i)$ and $b_i \neq c_i$. Then $\mathbf{b}, \mathbf{c}, a$ satisfy $\chi_{\Delta,k}$ but there are no b_k, c_k and automorphism σ which makes $\theta_{\Delta,k}$ true. As, for each j,

$$T \vdash (\forall x, y, z)(\psi(x, y, z) \wedge P_j(z) \rightarrow [P_j(x) \leftrightarrow \neg P_j(y)]).$$

To put this situation in a more general framework, recall some notation from [8]. Δ will denote a finite set of formulas: $\{\phi_i(\mathbf{x}, \mathbf{y}_i) : \lg(\mathbf{x}) = m, i <$

$|\Delta|\}$; p is a Δ-m-type over A if p is a set of formulas $\phi_i(\mathbf{x}, \mathbf{a})$ where $\mathbf{x} = \langle x_1, \ldots x_{m-1}\rangle$ (these specific variables) and $\mathbf{a}$ from A is substituted for $\mathbf{y}_i$. Thus, if A is finite there are only finitely many Δ-m-types over A.

Now let Δ_1 contain Boolean combinations of $x = y, R(x, y), E(x, y)$. Let Δ_2 expand Δ_1 by adding a finite number of the $P_j(z)$ and let Δ_3 contain $P_k(x)$ where P_k does not occur in Δ_2.

Now we have the following situation: there exists a set $X = \{b_0, b_1, c_0, c_1, a\}$, $P_j(a)$ holds, all 5 are E-equivalent and $R(b_i, c_i)$ for $i = 0, 1$ such that:

1. $\langle \mathbf{b}_i\mathbf{c}_i : i \leq 12\rangle$ is Δ_2-indiscernible over a.
2. $\langle b_0c_0, b_1c_1\rangle$ can be extended to an infinite set of indiscernibles $\overline{\mathbf{b}}\overline{\mathbf{c}}$ which satisfy the following.
   (a) $\psi(b_i, c_i, a)$.
   (b) $\sigma(\mathrm{avg}_{\Delta_2}(\overline{\mathbf{b}}/\mathrm{M})) = \mathrm{avg}_{\Delta_2}(\overline{\mathbf{c}}/\mathrm{M})$.
3. $\mathrm{tp}_{\Delta_1}(\mathbf{b}_2\mathbf{c}_2/\mathrm{X}) \vdash \sigma(\mathrm{tp}_{\Delta_3}(\mathbf{b}_2/\mathrm{X})) \neq \mathrm{tp}_{\Delta_3}(\mathbf{c}_2/\mathrm{X})$.

We call a sequence like $\langle \mathbf{b}_i\mathbf{c}_i : i \leq 2\rangle\mathbf{a}$ a $(\sigma, \Delta_1, \Delta_2, \Delta_3, n)$-obstruction over the empty set. In order to 'finitize' the notions we will give below more technical formulations of the last two conditions; we will have to discuss obstructions over a finite set A. In the example, the identity was the only automorphism of the prime model. We will have to introduce a third sequence $\mathbf{b}'$ to deal with arbitrary σ. But this example demonstrates the key aspects of obstruction which are the second reason for T_{Aut} to lack a model companion.

2 Preliminaries

In order to express the notions described in the example, we need several notions from basic stability theory. By working with finite sets of formulas in a stable theory without the finite cover property we are able to refine arguments about infinite sets of indiscernibles to arguments about sufficiently long finite sequences.

We may speak recklessly of indiscernible sequence but in the paper we deal exclusively with Δ-indiscernible sets which are defined just below. For infinite sequences in a stable theory such recklessness is without penalty (since infinite indiscernible sequences are indiscernible sets); since we are speaking of finite sequences, it is essential that we really mean indiscernible sets.

Definition 2.1 *Let Δ be a finite set of formulas which we will assume to be closed under permutation or identification of variables and under negation; $\neg\neg\phi$ is identified with ϕ.*

1. *We say that $E = \langle \mathbf{a}_i : i \in I\rangle$, where all $\mathbf{a}_i$ have the same length m, is (Δ, r) indiscernible if it satisfies the following conditions. Suppose $i_0 \dots i_{r-1}$ and $j_0 \dots j_{r-1}$ are distinct elements of I and for $i < t < r$, u_i is a subset of m with $\Sigma_{i<t}|u_i| = r$, $\lg(\mathbf{x}_i) = u_i$, and $\phi(\mathbf{x}_0, \dots \mathbf{x}_{t-1}) \in \Delta$. Then*

   $$\phi(\mathbf{a}_{i_0} \restriction u_0, \mathbf{a}_{i_1} \restriction u_1, \dots \mathbf{a}_{i_{t-1}} \restriction u_{t-1}) \leftrightarrow \phi(\mathbf{a}_{j_0} \restriction u_0, \mathbf{a}_{j_1} \restriction u_1, \dots \mathbf{a}_{j_{t-1}} \restriction u_{t-1}).$$

2. *E is Δ-indiscernible if it is (Δ, r)-indiscernible for all r, or equivalently for all r' with r' at most the maximum number of variables in a formula in Δ.*

3. *For any sequence $E = \langle \mathbf{a}_i : i \in I\rangle$ and $j \in I$ we write E_j for $\langle \mathbf{a}_i : i < j\rangle$.*

Remark 2.2 *Pedantically the formulas in Δ contain variables only among $x_0, \dots x_n$ for some n, but we will freely write $\phi(x)$, $\phi(y)$ to increase intelligibility. We do not distinguish strictly between an arbitrary finite set of formulas Δ and its closure described in Definition 2.1.*

We will rely on the following facts/definitions from [8] to introduce two crucial functions for this paper: $F(\Delta, n)$ and $f(\Delta, n)$.

Fact 2.3 *Recall that if T is stable, then for every finite $\Delta \subset L(T)$ and $n < \omega$ there is a finite $\Delta' = F(\Delta, n)$ with $\Delta \subseteq \Delta' \subset L(T)$ and a $k^* = f(\Delta, n)$ with the following properties.*

1. *Assume we have finite set A and a set $\mathbf{E} = \langle \mathbf{e}_i : i \in I\rangle$ of n-tuples such that for $i < j$,*

   $$\mathrm{tp}_{\Delta'}(\mathrm{\mathbf{e}_j}/\mathrm{\mathbf{E}_i A}) = \mathrm{tp}_{\Delta'}(\mathrm{\mathbf{e}_i}/\mathrm{\mathbf{E}_i A})$$

   and

   $$R_{(\Delta',2)}(\mathbf{e}_j/\mathbf{E}_j A) = R_{(\Delta',2)}(\mathbf{e}_i/\mathbf{E}_i A),$$

   (whence, $\mathrm{tp}_{\Delta'}(\mathrm{e_j}/\mathrm{\mathbf{E}_i A})$ is definable over A). Then $\mathbf{E}$ is a set of Δ-indiscernibles over A.

2. *For any set of Δ'-indiscernibles over the empty set, $\mathbf{E} = \langle \mathbf{e}_i : i < k\rangle$ with $\lg(\mathbf{e}_i) = n$ and $k \geq k^*$ for any $\phi(\mathbf{u}, \mathbf{v}) \in \Delta$ and any $\mathbf{d}$ with $\lg(\mathbf{d}) = \lg(\mathbf{v}) = m$ either $\{\mathbf{e}_i : \phi(\mathbf{e}_i, \mathbf{d})\}$ or $\{\mathbf{e}_i : \neg\phi(\mathbf{e}_i, \mathbf{d})\}$ has strictly less than $k^*/10$ elements. (II.4.13, II.2.20)*

3. *This implies that, for appropriate choice of k^*,*

   (a) *there is an integer $m = m(\Delta, n) \geq n$ such that for any set of Δ'-indiscernibles $\langle \mathbf{e}_i : i < k\rangle$ over A with $\lg(\mathbf{e}_i) = n$ and $k \geq k^*$ and any $\mathbf{a}$ with $\lg(\mathbf{a}) \leq m$ there is a $U \subseteq k$ with $|U| < k^*/2$ such that $\langle \mathbf{e}_i : i \in k - U\rangle$ is Δ-indiscernible over $A\mathbf{a}$;*

   (b) *moreover if $k \geq k^*$, for any set A, $\mathrm{avg}_\Delta(\langle \mathbf{e}_\mathrm{i} : \mathrm{i} < \mathrm{k}\rangle/\mathrm{A})$ is well-defined. Namely, $\mathrm{avg}_\Delta(\langle \mathbf{e}_\mathrm{i} : \mathrm{i} < \mathrm{k}\rangle/\mathrm{A}) =$*

$$\{\phi(\mathbf{x}, \mathbf{a}) : |\{\mathbf{e}_i : i < k, \phi(\mathbf{e}_i, \mathbf{a})\}| \geq \frac{k^*}{10}, \mathbf{a} \in A, \phi(\mathbf{x}, \mathbf{y}) \in \Delta\}.$$

In 3 a), m is the least $k \geq n$ such all $\phi \in \Delta$ have at most k free variables. But, in 3b) $\mathrm{avg}_\Delta(\langle \mathbf{e}_\mathrm{i} : \mathrm{i} < \mathrm{k}\rangle/\mathrm{A})$ need not be consistent. (Let A be all the members of one finite class in the standard fcp example.)

The closure conditions on Δ given in Definition 2.1 guarantee:

Fact 2.4 *If $\langle \mathbf{a}_i \mathbf{b}_i : i < \alpha\rangle$ is a set of Δ-indiscernibles over $\mathbf{a}$, with the length of the $\mathbf{a}_i$ equal to n, then $\langle \mathbf{a}_i \mathbf{a}_i \mathbf{b}_i : i < \alpha\rangle$ and $\langle \mathbf{b}_i : i < \alpha\rangle$ are Δ-indiscernible sets as well.*

Recall:

Definition 2.5 *The theory T does not have the finite cover property if for every finite Δ (considered with n-variable parameters) there is a k such that for any A and any Δ-type q over A, if q is k-consistent then q is consistent. We require $f(\Delta, n)$ greater than this k.*

Fact 2.6 *If T does not have the finite cover property, in addition to Fact 2.3, we can choose $k^* = f(\Delta, n)$ to satisfy the following conditions.*

1. *If $\mathbf{E} = \langle \mathbf{e}_i : i < k^*\rangle$ is a set of n-tuples, which is Δ'-indiscernible over the empty set, for any A, $\mathrm{avg}_\Delta(\mathbf{E}/\mathrm{A})$ is a* consistent *complete Δ-type over A.*

2. *Any set of Δ'-indiscernibles (of n-tuples) with length at least k^* can be extended to one of infinite length (II.4.6).*

3. *For any pair of $F(F(\Delta,n),n)$-indiscernible sets $\mathbf{E}^1 = \langle \mathbf{e}^1_i : i < k\rangle$ and $\mathbf{E}^2 = \langle \mathbf{e}^2_i : i < k\rangle$ over $\mathbf{a}$ with $\lg(\mathbf{e}^j_i) = n$ (for $j = 1, 2$) and $k \geq k^*$ such that*
$$\mathrm{avg}_{\mathrm{F}(\Delta,\mathrm{n})}(\mathbf{E}^1/\mathbf{a}\mathbf{E}^1\mathbf{E}^2) = \mathrm{avg}_{\mathrm{F}(\Delta,\mathrm{n})}(\mathbf{E}^2/\mathbf{a}\mathbf{E}^1\mathbf{E}^2),$$
there exists $\mathbf{J} = \langle \mathbf{e}_j : k < j < \omega\rangle$ such that both $\mathbf{E}^1\mathbf{J}$ and $\mathbf{E}^2\mathbf{J}$ are $F(\Delta,n)$-indiscernible over $\mathbf{a}$.

4. *We express the displayed condition in 3) on $\mathbf{E}^1, \mathbf{E}^2$ by the formula: $\lambda_\Delta(\overline{\mathbf{e}}^1, \overline{\mathbf{e}}^2, \mathbf{a})$, where $\overline{\mathbf{e}}^i$ enumerates $\mathbf{E}^i$.*

5. *If $\mathbf{E}^1$ and $\mathbf{E}^2$ contained in a model M are $F(\Delta,n)$-indiscernible over $\mathbf{a} \in M$ and each have length at least k^*, then $M \models \lambda_\Delta(\overline{\mathbf{e}}^1, \overline{\mathbf{e}}^2, \mathbf{a})$ if and only if $\mathrm{avg}_\Delta(\mathbf{E}^2/\mathrm{M}) = \mathrm{avg}_\Delta(\mathbf{E}^2/\mathrm{M})$.*

Proof. For 1, make sure that k^* is large enough that every Δ-type which is k^*-consistent is consistent (II.4.4 3)). Now 3) follows by extending the common $F(\Delta,n)$-average of $\mathbf{E}^1$ and $\mathbf{E}^2$ over $\mathbf{a}\mathbf{E}^1\mathbf{E}^2$ by 2). Finally, condition 5) holds by adapting the argument for III.1.8 from the set of all L-formulas to Δ. $\square_{2.6}$

Note that both F and f can be chosen increasing in Δ and n.

The following observations culminate in a new consequence of nfcp that will be used to reduce from 'obstruction' to 'simple obstruction' in the next section. The first is III.3.4 of [8] or V.1.23 of [**?**]. Just choose u such that $\mathrm{tp}(\mathrm{b}_*/\mathrm{X})$ does not fork over u and $\mathrm{tp}(\mathrm{b}_*/\mathrm{u})$ is stationary. Note that we do not assume the existence of an $\mathbf{a}_*$.

Fact 2.7 *Suppose T is stable, and further that $\langle \mathbf{b}_i : i < |T|^+\rangle \cup \{\mathbf{b}_*\}$ and $\mathbf{X} = \langle \mathbf{a}_i\mathbf{b}_i : i < |T|^+\rangle$ are sequences of L-indiscernibles over $\mathbf{a}$. Then there is a $\mathbf{U} \subseteq X$, $\mathbf{U}$ indexed by an initial segment of cardinality $\leq |T|$, such that if $\mathbf{X}_{\mathbf{U},j}$ denotes $\mathbf{X} - (\mathbf{U} \cup \{\mathbf{a}_j\mathbf{b}_j\})$, for every $\mathbf{b}_j \notin \mathbf{U}$*
$$\mathrm{tp}(\mathbf{b}_*/\mathbf{X}_{\mathbf{U},\mathrm{j}}) = \mathrm{tp}(\mathbf{b}_\mathrm{j}/\mathbf{X}_{\mathbf{U},\mathrm{j}}).$$

Now we 'finitize' this fact.

Lemma 2.8 *Fix n and m. For every finite set of formulas Δ there exist $k_0, k_1 < \omega$ and a finite set of formulas Δ^+ such that if $k_2 \geq k_1$, $\mathbf{a}_i, \mathbf{b}_i$ are sequences of length n, $\mathbf{a}$ is set of parameters of size m, while $\mathbf{X} = \langle \mathbf{a}_i \mathbf{b}_i : i < k_2 \rangle$ and $\langle \mathbf{b}_i : i < k_2 \rangle \cup \mathbf{b}_*$ are sets of Δ^+-indiscernibles over $\mathbf{a}$, then there is a $\mathbf{U} \subseteq \mathbf{X}$, $|\mathbf{U}| < k_0$, such that if $\mathbf{X}_{\mathbf{U},j}$ denotes $\mathbf{X} - (\mathbf{U} \cup \mathbf{a}_j \mathbf{b}_j)$, for every $\mathbf{a}_j, \mathbf{b}_j \notin \mathbf{U}$*

$$\mathrm{tp}_\Delta(\mathbf{b}_*/\mathbf{X}_{\mathbf{U},\mathrm{j}}) = \mathrm{tp}_\Delta(\mathbf{b}_\mathrm{j}/\mathbf{X}_{\mathbf{U},\mathrm{j}}).$$

Proof. If not, for every $t = \langle k_0, k_1, \Delta_1 \rangle$ with Δ_1 a finite set of formulas containing Δ there is a $k_2^t \geq k_1$, a structure M^t, and a sequence $X^t = \langle \mathbf{a}_i^t \mathbf{b}_i^t : i < k_2 \rangle$ and $\mathbf{b}_*^t$ such that $\mathbf{X}$ and $\langle \mathbf{b}_i^t : i < k_2 \rangle \cup \{\mathbf{b}_*\}$ are sequences of Δ^+-indiscernibles over $\mathbf{a}$ contained in M^t but for every $\mathbf{U} \subseteq X$, with $|\mathbf{U}| < k_0^t$, for some $\mathbf{b}_j \notin \mathbf{U}$

$$\mathrm{tp}_\Delta(\mathbf{b}_*/\mathrm{X}_{\mathbf{U},\mathrm{j}}) \neq \mathrm{tp}_\Delta(\mathbf{b}_\mathrm{j}/\mathrm{X}_{\mathbf{U},\mathrm{j}}).$$

Now expand L by adding constants for $\mathbf{a}, \mathbf{b}_*$ and a new $2n$-ary predicate symbol P. Then (M^t, P), where P holds of each tuple $\langle \mathbf{a}_i^t \mathbf{b}_i^t \rangle$, satisfies the first order sentence expressing this failure. By compactness and saturation we obtain a contradiction to Fact 2.7. $\square_{2.8}$

Lemma 2.9 *Suppose T is stable without the finite cover property. For every finite Δ, n there are k_0, k_1 and Δ_1 such that if $\mathbf{X} = \langle \mathbf{a}_i \mathbf{b}_i : i < k_1 \rangle$, $\mathbf{X}' = \langle \mathbf{b}_i : i < k_1 \rangle$, and $X' \cup \mathbf{b}_*$ are sequences of Δ_1-indiscernibles (of sequences of length n there is $V \subseteq k_1$ with $|V| < k_0$ and there is an $\mathbf{a}_*$ such that $\langle \mathbf{a}_i \mathbf{b}_i : i \in k_1 - V \rangle \cup \{\mathbf{a}_* \mathbf{b}_*\}$ is a sequence of Δ-indiscernibles.*

Proof. Choose Δ^+, k_0, k_1 according to Lemma 2.8. Choose $\mathbf{a}_{k_1}, \mathbf{b}_{k_1}$ to realize the average of $\mathbf{X}$ and let $q(\mathbf{x}, \mathbf{y})$ denote the Δ-type of $\mathbf{a}_{k_1}, \mathbf{b}_{k_1}$ over $\mathbf{X}$. Since the finite cover property fails there is an r such that the consistency of q is determined by its r-element subsets. Let Δ_1 contain Δ^+ and all formulas of the form $\bigwedge_{i<r} (\exists \mathbf{x}) \phi_i(\mathbf{x}, \mathbf{y}, \mathbf{u}_1, \mathbf{v}_1, \ldots \mathbf{u}_s, \mathbf{v}_s)$ where $\phi_i(\mathbf{a}_{k_1}, \mathbf{b}_{k_1}, \mathbf{a}_{i_1}, \mathbf{b}_{i_1}, \ldots \mathbf{a}_{i_s}, \mathbf{b}_{i_s})$ holds and $\phi_i \in \Delta$. Now if $\mathbf{X}$ and $\mathbf{X}'$ satisfy the hypotheses for this choice of Δ_1, by Lemma 2.8, $q(\mathbf{x}, \mathbf{b}_*)$ is consistent as required. $\square_{2.9}$

3 Obstructions

In this section we introduce the main new notion of this paper: obstruction.

We are concerned with a formula $\psi(\mathbf{x}, \mathbf{y}, \mathbf{z})$ where $\lg(\mathbf{x}) = \lg(\mathbf{y}) = n$ and $\lg(\mathbf{z}) = m$. We will apply Facts 2.3 and 2.6 with $\mathbf{e}_i = \mathbf{b}_i \mathbf{b}_i' \mathbf{c}_i$ where each of

$\mathbf{b}_i$, $\mathbf{b}'_i$, and $\mathbf{c}_i$ has length n. Thus, our exposition will depend on functions $F(\Delta, 3n)$, $f(\Delta, 3n)$. In several cases, we apply Fact 2.3 with $\phi(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{v})$ as $\theta(\mathbf{u}_2, \mathbf{v}) \leftrightarrow \theta(\mathbf{u}_3, \mathbf{v})$ for various θ.

The following notation is crucial to state the definition.

Notation 3.1 *If $\overline{\mathbf{d}} = \langle \mathbf{d}_i : i < r \rangle$ is a sequence of $3n$-tuples, which is Δ-indiscernible over a finite sequence $\mathbf{f}$, and $r \geq k^* = f(\Delta, 3n)$, then $\tau_\Delta(\mathbf{z}, \overline{\mathbf{d}}\mathbf{f})$ is the formula with free variable $\mathbf{z}$ of length $3n$ and parameters $\overline{\mathbf{d}}\mathbf{f}$ which asserts that there is a subsequence $\overline{\mathbf{d}}'$ of $\overline{\mathbf{d}}$ with length $f(\Delta, 3n)$ so that $\overline{\mathbf{d}}'\mathbf{z}$ forms a set of Δ-indiscernibles over $\mathbf{f}$.*

Now we come to the main notion. Intuitively, $\langle \mathbf{b}_i \mathbf{b}'_i \mathbf{c}_i : i < k \rangle \mathbf{a}$ is a $(\Delta_1, \Delta_2, \Delta_3, n)$-obstruction over A if $\langle \mathbf{b}_i \mathbf{b}'_i \mathbf{c}_i : i < k \rangle$ is an indefinitely extendible sequence of Δ_2-indiscernibles over $\mathbf{a}$ such that the $\mathbf{b}_i$'s, $\mathbf{b}'_i$'s and $\mathbf{c}_i$'s each have length n and the Δ_2-average of the $\mathbf{b}'_i$'s and the $\mathbf{c}_i$'s is the same (over any set) but any realizations of the Δ_1-type of the $\mathbf{b}'_i$ and the Δ_1-type of the $\mathbf{c}_i$ over $\mathbf{a}$ and the sequence $\langle \mathbf{b}_i \mathbf{b}'_i \mathbf{c}_i : i < k \rangle$ have different Δ_3-types over A. More formally, we define:

Definition 3.2 *Fix a finite $\Delta_1 \subseteq \Delta_2 \subseteq L(T)$ and $\Delta_3 \subseteq L(T)$, finite $\mathbf{a} \subseteq A \subset M \models T$ with $\lg(\mathbf{a}) \leq m(\Delta_2, n)$ (as in Fact 2.3), σ an automorphism of M, and a natural number n. We say $\langle \mathbf{b}_i \sigma(\mathbf{b}_i) \mathbf{c}_i : i < k \rangle \mathbf{a}$ is a $(\sigma, \Delta_1, \Delta_2, \Delta_3, n)$* obstruction *over A if the following conditions hold.*

1. *$\langle \mathbf{b}_i \sigma(\mathbf{b}_i) \mathbf{c}_i : i < k \rangle$ is $F(\Delta_2, 3n)$-indiscernible over $\mathbf{a}$.*

2. *$k \geq f(\Delta_2, 3n)$; $\lg(\mathbf{b}_i) = \lg(\mathbf{b}'_i) = \lg(\mathbf{c}_i) = n$.*

3. *$\mathrm{avg}_{\Delta_2}(\overline{\mathbf{e}}^1/\mathrm{M}) = \mathrm{avg}_{\Delta_2}(\overline{\mathbf{e}}^2/\mathrm{M})$ where $\overline{\mathbf{e}}^1 = \langle \sigma(\mathbf{b}_i) : i < k \rangle$ and $\overline{\mathbf{e}}^2 = \langle \mathbf{c}_i : i < k \rangle$.*

4. *Using the formula τ_{Δ_1} from Notation 3.1 with $\mathbf{x}, \mathbf{x}', \mathbf{y}$ representing the free variable $\mathbf{z}$ there, we have*

$$M \models (\forall \mathbf{x}, \mathbf{x}', \mathbf{y})[\tau_{\Delta_1}(\mathbf{x}, \mathbf{x}', \mathbf{y}, \langle \mathbf{b}_i \sigma(\mathbf{b}_i) \mathbf{c}_i : i < k \rangle \mathbf{a}) \rightarrow$$

$$\bigvee \{\phi(\mathbf{x}', \mathbf{f}) \wedge \neg \phi(\mathbf{y}, \mathbf{f}) : \mathbf{f} \in A, \phi \in \Delta_3\}]$$

By Fact 2.6, Condition 3) is expressed by a formula of $\mathbf{e}^1, \mathbf{e}^2$ and $\mathbf{a}$. Crucially, the hypothesis of the fourth condition in Definition 3.2 is an L-formula with parameters $\langle \mathbf{b}_i \sigma(\mathbf{b}_i) \mathbf{c}_i) : i < k \rangle \mathbf{a}$; the conclusion is an L-formula with parameters from A as well. Δ_1 and Δ_2 have $3n$ type-variables; Δ_3 has n type-variables.

Fact 3.3 *Note that (a) if $\langle \mathbf{b}_i \mathbf{b}'_i \mathbf{c}_i : i \leq k \rangle \mathbf{a}$ is a $(\sigma, \Delta_1, \Delta_2, \Delta_3, n)$ obstruction over A and $\Delta_1 \subseteq \Delta'_2 \subseteq \Delta_2$, then $\langle \mathbf{b}_i \mathbf{b}'_i \mathbf{c}_i : i \leq k \rangle \mathbf{a}$ is a $(\sigma, \Delta_1, \Delta'_2, \Delta_3, n)$ obstruction over A. Further, (b) if $\langle \mathbf{b}_i \mathbf{b}'_i \mathbf{c}_i : i \leq k \rangle \mathbf{a}$ is a $(\sigma, \Delta_1, \Delta_2, \Delta_3, n)$ obstruction over A and $A \subseteq A'$, where A' is finite, $\Delta_1 \subseteq \Delta'_2 \subseteq \Delta_2$, then $\langle \mathbf{b}_i \mathbf{b}'_i \mathbf{c}_i : i \leq k \rangle \mathbf{a}$ is a $(\sigma, \Delta_1, \Delta'_2, \Delta_3, n)$ obstruction over A'. Finally, (c) if $W \subseteq k$ is large enough then $\langle \mathbf{b}_i \mathbf{b}'_i \mathbf{c}_i : i \in W \rangle \mathbf{a}$ is a $(\sigma, \Delta_1, \Delta'_2, \Delta_3, n)$ obstruction over A.*

Definition 3.4 1. *We say $(M, \sigma) \models T_{\mathrm{Aut}}$* has no σ-obstructions *when there is a function $G(\Delta_1, n)$ with $F(\Delta_1, 3n) \subseteq G(\Delta_1, n) \subset_{<\omega} L(T)$ such that if Δ_1 is a finite subset of $L(T)$ and $G(\Delta_1, n)$ is contained in the finite $\Delta_3 \subset L(T)$, then for every finite subset A of M, there is no $(\sigma, \Delta_1, G(\Delta_1, n), \Delta_3, n)$ obstruction over A.*

2. *We say T* has no obstructions *when there is a function $G(\Delta_1, n)$ (which does not depend on (M, σ)) such that for each $(M, \sigma) \models T_{\mathrm{Aut}}$, if Δ_1 is a finite subset of $L(T)$, A is finite subset of M, and Δ_3 is a finite subset of $L(T)$, there is no $(\sigma, \Delta_1, G(\Delta_1, n), \Delta_3, n)$ obstruction over A.*

Definition 3.5 *A simple obstruction is an obstruction where the automorphism σ is the identity. The notions of a theory or model having a simple obstruction are the obvious modifications of the previous definition.*

Lemma 3.6 *T has obstructions if and only if T has simple obstructions.*

Proof. Suppose T has obstructions; we must find simple obstructions; the other direction is obvious. So, suppose for some Δ_1, and n, and for every finite $\Delta_2 \supseteq F(\Delta_1, 3n)$, there is a finite Δ_3 and a tuple $(M^{\Delta_2}, \sigma^{\Delta_2}, A^{\Delta_2}, k^{\Delta_2})$ such that: $(M^{\Delta_2}, \sigma^{\Delta_2}) \models T_{\mathrm{Aut}}$, A^{Δ_2} is a finite subset of M^{Δ_2} and $\mathbf{b}^{\Delta_2}, \sigma(\mathbf{b}^{\Delta_2}), \mathbf{c}^{\Delta_2}, \mathbf{a}^{\Delta_2}$ contained in M^{Δ_2} are a $(\sigma^{\Delta_2}, \Delta_1, \Delta_2, \Delta_3, n)$ obstruction of length k^{Δ_2} over A^{Δ_2}. Choose Δ_2 so that it contains the set of formulas Δ_1^+ associated to Δ_1 as Δ_1 is associated to Δ in Lemma 2.9. Without loss of generality $\lg(\mathbf{a}) = m = m(\Delta_1, 3n)$ and we can choose an appropriate Δ_3 depending on Δ_2. Now, define a family of simple obstructions by replacing each component of the given sequence of obstructions by an appropriate object with left prefix sim.

$$^{\mathrm{sim}}A^{\Delta_2} = A^{\Delta_2}$$

$$^{\mathrm{sim}}(\mathbf{b}_i^{\Delta_2}) =^{\mathrm{sim}} (\mathbf{b}_i'^{\Delta_2}) = \sigma(\mathbf{b}_i^{\Delta_2})$$

$$^{\mathrm{sim}}\mathbf{c}_i^{\Delta_2} = \mathbf{c}_i^{\Delta_2}.$$

$$^{\mathrm{sim}}\mathbf{a}^{\Delta_2} = \mathbf{a}^{\Delta_2}.$$

$$^{\mathrm{sim}}\Delta_1 = \Delta_1^+.$$

We use the same sets of formulas for the Δ_2 and Δ_3.

We now have an obstruction with respect to the identity. For condition 1) this follows from Assumption 2.4; conditions 2) and 3) are immediate; we check condition 4). Since the following argument is uniform Δ_2, we omit the superscript Δ_2. Examining the implication in condition 4) for both the original obstruction and the simple obstruction we see it suffices to show, if we write $\tau^1(\mathbf{x}', \mathbf{x}', \mathbf{y})$ for $\tau_{\Delta_1^+}(\mathbf{x}', \mathbf{x}', \mathbf{y}, \langle \sigma(\mathbf{b}_i)\sigma(\mathbf{b}_i)\mathbf{c}_i : i < k\rangle \mathbf{a})$ and $\tau^2(\mathbf{x}, \mathbf{x}', \mathbf{y})$ for $\tau_{\Delta_1^+}(\mathbf{x}, \mathbf{x}', \mathbf{y}, \langle \mathbf{b}_i\sigma(\mathbf{b}_i)\mathbf{c}_i : i < k\rangle \mathbf{a})$, that

$$(\forall \mathbf{x}')(\forall \mathbf{y})[\tau^1(\mathbf{x}', \mathbf{x}', \mathbf{y}) \rightarrow (\exists \mathbf{x})\tau^2(\mathbf{x}, \mathbf{x}', \mathbf{y})].$$

We verify the implication. Lemma 2.9(taking the $\mathbf{b}_i$ there as $\sigma(\mathbf{b}_i), \mathbf{c}_i$ here) implies for any $\mathbf{b}', \mathbf{c}$ such that $\langle \sigma(\mathbf{b}_i), \mathbf{c}_i : i < k\rangle \cup \{\mathbf{b}', \mathbf{c}\}$ is a sequence of Δ_1^+-indiscernibles, there is a $\mathbf{b}$ so that $\langle \mathbf{b}_i\sigma(\mathbf{b}_i), \mathbf{c}_i : i < k\rangle \cup \{\mathbf{b}, \mathbf{b}', \mathbf{c}\}$ is a sequence of Δ_1-indiscernibles. Thus $[\tau^1(\mathbf{b}', \mathbf{b}', \mathbf{c}) \rightarrow \tau^2(\mathbf{b}, \mathbf{b}', \mathbf{c})]$ and we finish.

$\square_{3.6}$

Lemma 3.7 *If T is a stable theory with the finite cover property then T has a simple obstruction.*

Proof. By II.4.4 of [8], there is a formula $E(\mathbf{x}, \mathbf{y}, \mathbf{z})$ such that for each $\mathbf{d}$, $E(\mathbf{x}, \mathbf{y}, \mathbf{d})$ is an equivalence relation and for arbitrarily large n there is a $\mathbf{d}_n$ such that $E(\mathbf{x}, \mathbf{y}, \mathbf{d}_n)$ has exactly n classes. Let Δ_1 be $\{E(\mathbf{x}, \mathbf{y}, \mathbf{z}), \neg E(\mathbf{x}, \mathbf{y}, \mathbf{z})\}$ and consider any Δ_2. Fix $\mathrm{lg}(\mathbf{x}) = \mathrm{lg}(\mathbf{y}) = r$. There are arbitrarily long sequences $\mathbf{b}_n = \langle \mathbf{b}_j^n : j < n\rangle$ such that for some $\mathbf{d}_n$, $\mathbf{b}_n$ is a set of representatives for distinct classes of $E(\mathbf{x}, \mathbf{y}, \mathbf{d}_n)$. So by Ramsey, for any Δ_2 we can find such $\mathbf{b}_k$ where $k = f(\Delta_2, 3r)$ and $\mathbf{b}_k$ is a sequence of length k of

$F(\Delta_2, 3n)$-indiscernibles over the empty set and $\mathbf{b}_k$ is a set of representatives for distinct classes of $E(\mathbf{x}, \mathbf{y}, \mathbf{d}_n)$. Now, if Δ_3 contains formulas which express that the number of equivalence classes of $E(\mathbf{x}, \mathbf{y}, \mathbf{z})$ is greater than n and A contains representatives of all equivalence classes of $E(\mathbf{x}, \mathbf{y}, \mathbf{d}_n)$ we have an (identity, $\Delta_1, \Delta_2, \Delta_3, r$) obstruction over A. (Let $\mathbf{b} = \mathbf{b}' = \mathbf{c} = \mathbf{b}_k$; the hypothesis of clause 4 of Definition 3.2 is trivially false so the condition is satisfied.) $\square_{3.7}$

4 Model Companions of T_{Aut}

In this section we establish necessary and sufficient conditions on stable T for T_{Aut} to have a model companion. First, we notice when the model companion, if it exists, is complete.

Note that $\mathrm{acl}(\emptyset) = \mathrm{dcl}(\emptyset)$ in C_T^{eq} means every finite equivalence relation $E(\mathbf{x}, \mathbf{y})$ of T is defined by a finite conjunction: $\bigwedge_{i<n} \phi_i(\mathbf{x}) \leftrightarrow \phi_i(\mathbf{y})$.

Fact 4.1 *1. If T is stable, T_{Aut} has the amalgamation property.*

2. If, in addition, $\mathrm{acl}(\emptyset) = \mathrm{dcl}(\emptyset)$ in C_T^{eq} then T_{Aut} has the joint embedding property.

Proof. The first part of this Lemma was proved by (Theorem 3.3 of [6]) using the definability of types. For the second part, the hypothesis implies that types over the empty set are stationary and the result follows by similar arguments.

Lemma 4.2 *Suppose T is stable and T_{Aut} has a model companion T_σ^*.*

1. *Then T_σ^* is complete if and only if $\mathrm{acl}(\emptyset) = \mathrm{dcl}(\emptyset)$ in C_T^{eq}.*
2. *If $(M, \sigma) \models T_{\mathrm{Aut}}$ then the union of the complete diagram of M (in L) with the diagram of (M, σ) and T_σ^* is complete.*

Proof. 1) We have just seen that if $\mathrm{acl}(\emptyset) = \mathrm{dcl}(\emptyset)$ in C_T^{eq}, then T_{Aut} has the joint embedding property; this implies in general that the model companion is complete. If $\mathrm{acl}(\emptyset) \neq \mathrm{dcl}(\emptyset)$ in C_T^{eq}, let $E(\mathbf{x}, \mathbf{y})$ be a finite equivalence relation witnessing $\mathrm{acl}(\emptyset) \neq \mathrm{dcl}(\emptyset)$. Because E is a finite equivalence relation,

$$T_1 = T_{\mathrm{Aut}} \cup \{(\forall \mathbf{x}) E(\mathbf{x}, \sigma(\mathbf{x}))\}$$

is a consistent extension of T_{Aut}. But since

$$T_{\mathrm{Aut}} \cup \{\neg E(\mathbf{x}, \mathbf{y})\} \cup \{\phi(\mathbf{x}) \leftrightarrow \phi(\mathbf{y})\colon \phi \in L(T)\}$$

is consistent so is

$$T_2 = T_{\mathrm{Aut}} \cup \{(\exists \mathbf{x})\neg E(\mathbf{x}, \sigma(\mathbf{x}))\}.$$

But T_1 and T_2 are contradictory, so T^*_σ is not complete.

2) Since we have joint embedding (from amalgamation over any model) the result follows as in Fact 4.1.

We now prove the equivalence of three conditions. The first is a condition on a pair of models. The second is given by an infinite set of L_σ sentences (take the union over all finite Δ_2) and the average requires names for infinitely many elements of M. The third is expressed by a single first order sentence in L_σ. The equivalence of the first and third suffices (Theorem 4.8) to show the existence of a model companion. In fact, 1 implies 2 implies 3 requires only stability; the nfcp is used to prove 3 implies 1.

Lemma 4.3 *Suppose T is stable without the fcp. Let $(M, \sigma) \models T_{\mathrm{Aut}}$, $\mathbf{a} \in M$ and suppose that (M, σ) has no σ-obstructions. Fix $\psi(\mathbf{x}, \mathbf{y}, \mathbf{z}) \in L_T$ with $\mathrm{lg}(\mathbf{x}) = \mathrm{lg}(\mathbf{y}) = n$ and $\mathrm{lg}(\mathbf{z}) = \mathrm{lg}(\mathbf{a}) = m$.*

The following three assertions are equivalent:

1. *There exists (N, σ), $(M, \sigma) \subseteq (N, \sigma) \models T_{\mathrm{Aut}}$ and*

$$N \models (\exists \mathbf{x}\mathbf{y})[\psi(\mathbf{x}, \mathbf{y}, \mathbf{a}) \wedge \sigma(\mathbf{x}) = \mathbf{y}].$$

2. *Fix $\Delta_1 = \{\psi(\mathbf{x}, \mathbf{y}, \mathbf{z})\}$ and without loss of generality $\mathrm{lg}(\mathbf{z}) \leq m(\Delta_1, n)$. For $k \geq 5 \cdot f(\Delta_1, 3n)$ and any finite $\Delta_2 \supseteq F(\Delta_1, 3n)$ (Fact 2.3), there are $\mathbf{b}_i\sigma(\mathbf{b}_i)\mathbf{c}_i \in {}^{3n}M$ for $i < k$ such that*

   (a) *$\langle \mathbf{b}_i\sigma(\mathbf{b}_i)\mathbf{c}_i : i < k\rangle$ is $F(\Delta_2, 3n)$-indiscernible over $\mathbf{a}$,*

   (b) *for each $i < k$, $\psi(\mathbf{b}_i, \mathbf{c}_i, \mathbf{a})$ holds,*

   (c) *For every $\mathbf{d} \in {}^{m}M$ and $\phi(\mathbf{u}, \mathbf{v}) \in \Delta_2$ we have*

$$|\{i < k : \phi(\sigma(\mathbf{b}_i), \mathbf{d}) \leftrightarrow \phi(\mathbf{c}_i, \mathbf{d})\}| \geq f(\Delta_2, 3n)/2.$$

3. *Let $\Delta_2 = G(\Delta_1, n)$. Then there are $\mathbf{b}_i\sigma(\mathbf{b}_i)\mathbf{c}_i \in {}^{3n}M$ for $i < k = 5 \cdot f(\Delta_2, 3n)$ such that (for λ_{Δ_2} from Fact 2.6 4):*

(a) $\langle \mathbf{b}_i\sigma(\mathbf{b}_i)\mathbf{c}_i : i < k\rangle$ *is* $F(G(\Delta_1, n), 3n)$*-indiscernible over* $\mathbf{a}$*,*

(b) for each $i < k$*,* $\psi(\mathbf{b}_i, \mathbf{c}_i, \mathbf{a})$*,*

(c) $\lambda_{\Delta_2}(\langle\sigma(\mathbf{b}_i) : i < k\rangle, \langle\mathbf{c}_i : i < k\rangle, \mathbf{a})$.

Proof. First we show 1) implies 2). Fix $\mathbf{b}, \mathbf{c} \in N$ with $N \models \psi(\mathbf{b}, \mathbf{c}, \mathbf{a}) \wedge \sigma(\mathbf{b}) = \mathbf{c}$. For Δ_2, let $\Delta_2^+ = F(F(\Delta_2, 3n), 3n)$. For each Δ_2, choose a finite $p \subseteq \mathrm{tp}_{\mathrm{L(T)}}(\mathbf{b}, \mathbf{c}/\mathrm{M})$ with the same $(\Delta_2^+, 2)$ rank as $\mathrm{tp}_{\Delta_2}(\mathbf{b}, \mathbf{c}/\mathrm{M})$ (so $\mathrm{tp}_{\Delta_2}(\mathbf{b}, \mathbf{c}/\mathrm{M})$ is definable over dom p). Now inductively construct an $F(\Delta_2, 3n)$-indiscernible sequence (by Fact 2.3) $\langle\mathbf{b}_i, \mathbf{c}_i : i < \omega\rangle$ by choosing $\mathbf{b}_i, \mathbf{c}_i$ in M realizing the restriction of $\mathrm{tp}_{\Delta_2^+}(\mathbf{b}, \mathbf{c}/\mathrm{M})$ to dom p along with the points already chosen. Let $\mathbf{b}'_i = \sigma(\mathbf{b}_i)$. For some infinite $U \subseteq \omega$, $\langle\mathbf{b}_i, \mathbf{b}'_i, \mathbf{c}_i : i \in U\rangle$ is $F(\Delta_2, 3n)$-indiscernible over $\mathbf{a}$; renumbering let $U = \omega$. Now conditions a) and b) of assertion 2) are clear. For clause c),

$$\mathrm{avg}_{\Delta_2}(\langle\mathbf{c}_{\mathrm{i}} : \mathrm{i} < \omega\rangle/\mathrm{M}) = \mathrm{tp}_{\Delta_2}(\mathbf{c}, \mathrm{M})$$

$$= \sigma(\mathrm{tp}_{\Delta_2}(\mathbf{b}, \mathrm{M})) = \sigma(\mathrm{avg}_{\Delta_2}(\langle\mathbf{b}_{\mathrm{i}} : \mathrm{i} < \omega\rangle/\mathrm{M}))$$

The first and last equalities hold by the choice of the $\mathbf{b}_i, \mathbf{c}_i$ and the middle since $\sigma(\mathbf{b}) = \mathbf{c}$. So, for each $\phi \in \Delta_2$ and each $\mathbf{d} \in M$ of appropriate length,

$$\phi(\mathbf{x}, \mathbf{d}) \in \mathrm{avg}_{\Delta_2}(\langle\mathbf{c}_{\mathrm{i}} : \mathrm{i} < \omega\rangle/\mathrm{M})$$

if and only if

$$\phi(\mathbf{x}, \sigma^{-1}(\mathbf{d})) \in \mathrm{avg}_{\Delta_2}(\langle\mathbf{b}_{\mathrm{i}} : \mathrm{i} < \omega\rangle/\mathrm{M}).$$

So for some $S_1, S_2 \subset \omega$ with $|S_1|, |S_2| < f(\Delta_2, 3n)/2$, we have for all $i \in \omega - (S_1 \cup S_2)$, $\phi(\mathbf{c}_i, \mathbf{d})$ if and only if $\phi(\mathbf{b}_i, \sigma^{-1}(\mathbf{d}))$. Since σ is an automorphism of M this implies for $i \in \omega - (S_1 \cup S_2)$, $\phi(\mathbf{c}_i, \mathbf{d})$ if and only if $\phi(\sigma(\mathbf{b}_i), \mathbf{d})$ which gives condition c) by using the first k elements of $\langle\mathbf{b}_i, \sigma(\mathbf{b}_i)\mathbf{c}_i : i \in \omega - S_1 \cup S_2\rangle$.

3) is a special case of 2). To see this, note that 3c) is easily implied by the form analogous to 2c): For every $m \leq m(\Delta_1, n)$ and $\mathbf{d} \in {}^mM$ and $\phi(\mathbf{u}, \mathbf{v}) \in G(\Delta_1, n)$ we have

$$|\{i < k : \phi(\sigma(\mathbf{b}_i), \mathbf{d}) \leftrightarrow \phi(\mathbf{c}_i, \mathbf{d})\}| \geq f(\Delta_2, 3n)/2.$$

If T does not have f.c.p. 3c) implies 2c) holds and we use that fact implicitly in the following argument. It remains only to show that 3) implies 1) with

$\Delta_1 = \{\psi\}$ and $\Delta_2 = G(\Delta_1, n)$. Without loss of generality we may assume N is $\aleph_1$-saturated. We claim the type

$$\Gamma = \{\psi(\mathbf{x}, \mathbf{y}, \mathbf{a})\} \cup \{\phi(\mathbf{x}, \mathbf{d}) \leftrightarrow \phi(\mathbf{y}, \sigma(\mathbf{d})) : \mathbf{d} \in M, \phi \in L(T)\} \cup \text{diag } (M)$$

is consistent. This clearly suffices.

Let $k = f(\Delta_2, 3n)$. Suppose $\langle \mathbf{b}_i \sigma(\mathbf{b}_i) \mathbf{c}_i : i < k \rangle \mathbf{a}$ satisfy 3). Let Γ_0 be a finite subset of Γ and suppose only formulas from the finite set Δ_3 and only parameters from the finite set A appear in Γ_0. Write $\mathbf{b}'_i$ for $\sigma(\mathbf{b}_i)$.

Now $\langle \mathbf{b}_i \mathbf{b}'_i \mathbf{c}_i : i \leq f(\Delta_2, 3n) \rangle \mathbf{a}$ easily satisfy the first two conditions of Definition 3.2 for being a $(\Delta_1, \Delta_2, \Delta_3, n)$-obstruction over A and, in view of Fact 2.6 3), 4), the third is given by condition 3c). Since there is no obstruction the 4th condition must fail. So there exist $\mathbf{b}^*, (\mathbf{b}^*)', \mathbf{c}^*$ so that

$$M \models \tau_{\Delta_1}(\mathbf{b}^*, (\mathbf{b}^*)', \mathbf{c}^*, \langle \mathbf{b}_i, \mathbf{b}'_i, \mathbf{c}_i : i < k \rangle \mathbf{a}).$$

and $\mathrm{tp}_{\Delta_3}((\mathbf{b}^*)'/\mathrm{A}) = \mathrm{tp}_{\Delta_3}(\mathbf{c}^*/\mathrm{A})$ so Γ_0 is satisfiable. $\square$4.3

As we'll note in Theorem 4.8, we have established a sufficient condition for T_{Aut} to have a model companion. The next argument shows it is also necessary.

Lemma 4.4 *Suppose T is stable; if T has an obstruction then T_{Aut} does not have a model companion.*

Proof. We may assume T does not have f.c.p., since if it does we know by Winkler [9] and Kudaibergenov [3] that T_{Aut} does not have a model companion. By Lemma 3.6, we may assume T has a simple obstruction. So it suffices to prove:

> Suppose for some Δ_1, and n, and for every finite $\Delta_2 \supseteq F(\Delta_1, 3n)$, there is a finite Δ_3 and a tuple $(M^{\Delta_2}, \mathrm{id}_{\mathrm{M}^{\Delta_2}}, \mathrm{A}^{\Delta_2}, \mathrm{k}^{\Delta_2})$ such that: $(M^{\Delta_2}, \mathrm{id}_{\mathrm{M}^{\Delta_2}}) \models \mathrm{T}_{\mathrm{Aut}}$, A^{Δ_2} is a finite subset of M^{Δ_2}, $\mathbf{b}^{\Delta_2}, \sigma^{\Delta_2}(\mathbf{b}^{\Delta_2}) = \mathrm{id}_{\mathrm{M}^{\Delta_2}}(\mathbf{b}^{\Delta_2}) = \mathbf{b}^{\Delta_2}, \mathbf{c}^{\Delta_2}, \mathbf{a}^{\Delta_2}$ contained in M^{Δ_2} are an $(\mathrm{id}_{\mathrm{M}^{\Delta_2}}, \Delta_1, \Delta_2, \Delta_3, \mathrm{n})$ obstruction of length k^{Δ_2} over A^{Δ_2}.
>
> Then the collection $\mathbf{K}_\sigma$ of existentially closed models of T_{Aut} is not an elementary class.

Without loss of generality $\lg(\mathbf{a}) = m = m(\Delta_1, 3n)$ and we can write $\Delta_3 = \Delta_3(\Delta_2)$. By the usual coding we may assume $\Delta_1 = \{\psi(\mathbf{x}, \mathbf{y}, \mathbf{z})\}$ with $\lg(\mathbf{x}) = \lg(\mathbf{y}) = n$, $\lg(\mathbf{z}) = m$, $k = f(\Delta_1, 3n)$. Again, without loss of generality each

$(M^{\Delta_2}, \mathrm{id}_{\mathrm{M}^{\Delta_2}})$ can be expanded to an existentially closed model $(M^{\Delta_2}, \sigma_{M^{\Delta_2}})$ such that $\sigma_{M^{\Delta_2}}$ fixes A^{Δ_2}, $\mathbf{a}^{\Delta_2}$, $\mathbf{b}^{\Delta_2}$, and $\mathbf{c}^{\Delta_2}$ pointwise. Let $\mathcal{D}$ be a nonprincipal ultrafilter on $Y = \{\Delta_2 : F(\Delta_1, 3n) \subseteq \Delta_2 \subset_\omega L(T)\}$ such that for any $\Delta \in Y$ the family of supersets in Y of Δ is in $\mathcal{D}$. Expand the language L to L^+ by adding a unary function symbol σ, a new unary predicate symbol P, a $3n$-ary relation symbol Q and constants $\mathbf{a}$. Expand each of the M^{Δ_2} to an L^+-structure N_{Δ_2} by interpreting P as A^{Δ_2}, $\mathbf{a}$ as $\mathbf{a}^{\Delta_2}$ σ as $\sigma^{\Delta_2} = \mathrm{id}_{\mathrm{M}^{\Delta_2}}$ and Q as the set $\{\mathbf{b}_i^{\Delta_2}\mathbf{b}_i^{\Delta_2}\mathbf{c}_i^{\Delta_2} : i < k^{\Delta_2}\}$ of $3n$-tuples. Let N^* be the ultraproduct of the N_{Δ_2} modulo $\mathcal{D}$. Let A denote $P(N^*)$, $\mathbf{a}^*$ denote the ultraproduct of the $\mathbf{a}^{\Delta_2}$, and $\langle \mathbf{b}_i\mathbf{b}_i\mathbf{c}_i : i \in I\rangle$ enumerate $Q(N^*)$.Now we claim

Claim 4.5 *1.* $\lg(\mathbf{b}_i) = \lg(\sigma(\mathbf{b}_i)) = \lg(\mathbf{c}) = n$; $\lg(\mathbf{a}) = m$.

2. *$\langle \mathbf{b}_i\mathbf{b}_i\mathbf{c}_i : i \in I\rangle$ is a sequence of $L(T)$-indiscernibles over $\mathbf{a}^*$.*

3. *For each finite $\Delta_2 \subseteq L(T)$ with $\Delta_2 \supseteq F(\Delta_1, 3n)$ and each finite subsequence from $\langle \mathbf{b}_i\mathbf{b}_i\mathbf{c}_i : i \in I\rangle$ indexed by J of length at least $k = f(\Delta_2, 3n)$ the Δ_2-type of $\langle \mathbf{b}_i\mathbf{b}_i\mathbf{c}_i : i \in J\rangle\mathbf{a}$ is the Δ_2-type of some $(\mathrm{id}, \Delta_1, \Delta_2, \Delta_3, \mathrm{n})$-obstruction $\langle \mathbf{b}^{\Delta_2}, \mathbf{b}^{\Delta_2}, \mathbf{c}^{\Delta_2}\rangle\mathbf{a}^{\Delta_2}$ in M^{Δ_2} over A^{Δ_2}.*

4. $(\mathrm{avg_L}(\langle \mathbf{b}_\mathrm{i} : \mathrm{i} \in \mathrm{I}\rangle/\mathrm{N}^*) = \mathrm{avg_L}(\langle \mathbf{c}_\mathrm{i} : \mathrm{i} \in \mathrm{I}\rangle/\mathrm{N}^*)$.

This claim follows directly from the properties of ultraproducts. (For item 3, apply Fact 3.3 and the definition of the ultrafilter $\mathcal{D}$.) $\square_{4.5}$

Let Γ be the L-type in the variables $\langle \mathbf{x}_i\mathbf{x}'_i\mathbf{y}_i : i \in I\rangle \cup \{\mathbf{z}\}$ over the empty set of $\langle \mathbf{b}_i, \mathbf{b}_i, \mathbf{c}_i : i \in I\rangle\mathbf{a}^*$. For any finite $\Delta \subset L(T)$, let $\chi_{\Delta,k}(\overline{\mathbf{x}}\overline{\mathbf{x}'}\overline{\mathbf{y}}, \mathbf{z})$ be the conjunction of the Δ-type over the empty set of a subsequence of k elements from $\langle \mathbf{b}_i\mathbf{b}_i\mathbf{c}_i : i \in I\rangle$ and $\mathbf{a}^*$ from a realization of Γ with $\mathbf{z}$ for $\mathbf{a}^*$.

Notation 4.6 *Recall the definition of τ_{Δ_1} from Notation 3.1. Let $r = f(\Delta_1, n)$ and let $\theta_{\Delta_1}(\mathbf{x}_0, \ldots \mathbf{x}_{r-1}, \mathbf{x}'_0, \ldots \mathbf{x}'_{r-1}, \mathbf{y}_0, \ldots \mathbf{y}_{r-1}, \mathbf{z})$ be the formula:*

$$(\exists \mathbf{x}, \mathbf{x}', \mathbf{y})[\chi_{\Delta_1,r}(\mathbf{x}_0, \ldots \mathbf{x}_{r-1}, \mathbf{x}'_0, \ldots \mathbf{x}'_{r-1}, \mathbf{y}_0, \ldots \mathbf{y}_{r-1}, \mathbf{z})$$
$$\wedge \tau_{\Delta_1}(\mathbf{x}, \mathbf{x}', \mathbf{y}, \mathbf{x}_0, \ldots \mathbf{x}_{r-1}, \mathbf{x}'_0, \ldots \mathbf{x}'_{r-1}, \mathbf{y}_0, \ldots \mathbf{y}_{r-1}, \mathbf{z}) \wedge \sigma(\mathbf{x}') = \mathbf{y}].$$

Without loss of generality we assume $0, 1 \ldots r-1$ index disjoint sequences. Now we claim:

Claim 4.7 *If $\mathbf{K}_\sigma$, the family of existentially closed models of T_{Aut}, is axiomatized by T^*_{Aut}, then*

$$T^*_{\mathrm{Aut}} \cup \Gamma \cup \{\sigma(\mathbf{x}_i) = \mathbf{x}'_i : i \in I\} \vdash \theta_{\Delta_1}(\mathbf{x}_0, \ldots \mathbf{x}_{r-1}, \mathbf{x}'_0, \ldots \mathbf{x}'_{r-1}, \mathbf{y}_0, \ldots \mathbf{y}_{r-1}, \mathbf{z}).$$

(Abusing notation we write this with the $\mathbf{x}_i, \mathbf{x}'_i, \mathbf{y}_i$ for $i \in I$ free.)

Note for each i, $\mathbf{x}_i = \mathbf{x}'_i$ is in Γ. For this, let $(M', \sigma') \models T^*_{\mathrm{Aut}}$ such that $\langle \mathbf{b}_i, \mathbf{b}_i, \mathbf{c}_i : i \in I\rangle \mathbf{a}$ satisfy Γ in M'. Suppose $M' \prec M''$ and M'' is an $|M'|^+$-saturated model of T. In M'' we can find $\mathbf{b}, \mathbf{b}', \mathbf{c}$ realizing the average of $\langle \mathbf{b}_i\mathbf{b}_i\mathbf{c}_i : i \in I\rangle$ over M'. Then

$$\sigma'(\mathrm{tp}(\mathbf{b}/\mathrm{M}')) = \sigma'(\mathrm{avg}(\langle \mathbf{b}_{\mathrm{i}} : \mathrm{i} \in \mathrm{I}\rangle/\mathrm{M}')) = \mathrm{avg}(\langle \sigma'(\mathbf{b}_{\mathrm{i}}) : \mathrm{i} \in \mathrm{I}\rangle/\mathrm{M}')$$
$$= \mathrm{avg}(\langle \mathbf{c}_{\mathrm{i}} : \mathrm{i} \in \mathrm{I}\rangle/\mathrm{M}') = (\mathrm{tp}(\mathbf{c}/\mathrm{M}')$$

(The first and last equalities are by the choice of $\mathbf{b}, \mathbf{c}$; the second holds as σ' is an automorphism, and the third follows from clause 4 in the description of the ultraproduct, Claim 4.5.) Now since M'' is $|M'|^+$-saturated there is an automorphism σ'' of M'' extending σ' and taking $\mathbf{b}$ to $\mathbf{c}$.

As $(M', \sigma') \models T^*_{\mathrm{Aut}}$, it is existentially closed. So we can pull $\mathbf{b}, \mathbf{c}$ down to M'. Thus, $(M', \sigma') \models \theta_{\Delta_1}(\mathbf{b}_0, \ldots \mathbf{b}_{r-1}, \mathbf{b}_0, \ldots \mathbf{b}_{r-1}, \mathbf{c}_0, \ldots \mathbf{c}_{r-1}, \mathbf{a})$. But (M', σ') was an arbitrary model of $T^*_{\mathrm{Aut}} \cup \Gamma \cup \{\sigma(\mathbf{x}_i) = \mathbf{x}'_i : i \in I\}$; so

$$T^*_{\mathrm{Aut}} \cup \Gamma \cup \{\sigma(\mathbf{x}_i) = \mathbf{x}'_i : i \in I\} \vdash$$
$$\theta_{\Delta_1}(\mathbf{x}_0, \ldots \mathbf{x}_{r-1}, \mathbf{x}'_0, \ldots \mathbf{x}'_{r-1}, \mathbf{y}_0, \ldots \mathbf{y}_{r-1}, \mathbf{z}).$$

$\square_{4.7}$

By compactness, some finite subset Γ_0 of Γ and a finite number of the specifications of σ suffice; let Δ^* be the formulas mentioned in Γ_0 along with those in $F(\Delta_1, 3n)$ and k the number of x_i, y_i appearing in Γ_0 and let $\Delta_2 = F(\Delta^*, n)$. Without loss of generality, $k \geq f(\Delta_1, 3n)$. Then, $T^*_{\mathrm{Aut}} \vdash$

$$(\forall \mathbf{x}_0 \ldots \mathbf{x}_{k-1}, \mathbf{x}'_0 \ldots \mathbf{x}'_{k-1}, \mathbf{y}_0, \ldots \mathbf{y}_{k-1})[(\chi_{\Delta_2,k}(\mathbf{x}_0, \ldots \mathbf{x}_{k-1}, \mathbf{x}'_0 \ldots \mathbf{x}'_{k-1}, \mathbf{y}_0, \ldots \mathbf{y}_{k-1}, \mathbf{z})$$
$$\wedge \bigwedge_{i<k} \sigma(\mathbf{x}_i) = \mathbf{x}'_i) \rightarrow \theta_{\Delta_1}(\mathbf{x}_0, \ldots \mathbf{x}_{r-1}, \mathbf{x}'_0 \ldots \mathbf{x}'_{r-1}, \mathbf{y}_0, \ldots \mathbf{y}_{r-1}, \mathbf{z})].$$

By item 3) in Claim 4.5, fix a Δ_2 and $\Delta_3 = \Delta_3(\Delta_2, m)$ containing Δ_2 and $\langle \mathbf{b}^{\Delta_2}_i, \mathbf{b}^{\Delta_2}_i \mathbf{c}^{\Delta_2}_i : i < k\rangle \mathbf{a}^{\Delta_2}$ which form a $(\mathrm{id}, \Delta_1, \Delta_2, \Delta_3, \mathrm{n})$-obstruction over A^{Δ_2} and so that:

$$M^{\Delta_2} \models \chi_{\Delta_2,k}(\langle \mathbf{b}^{\Delta_2}_i \mathbf{b}^{\Delta_2}_i \mathbf{c}^{\Delta_2}_i : i < k\rangle \mathbf{a}^{\Delta_2}).$$

So by the choice of Γ_0,

$$(M^{\Delta_2}, \sigma^{\Delta_2}) \models \theta_{\Delta_1}(\langle \mathbf{b}_i^{\Delta_2} \mathbf{b}_i^{\Delta_2} \mathbf{c}_i^{\Delta_2} : i < r \rangle \mathbf{a}^{\Delta_2}).$$

Now, let $\mathbf{b}, \mathbf{b}', \mathbf{c} \in M^{\Delta_2}$ witness this sentence; then $\sigma^{\Delta_2}(\mathbf{b}') = \mathbf{c}$. Then

$$(M^{\Delta_2}, \sigma^{\Delta_2}) \models \tau_{\Delta_1}(\mathbf{b}, \mathbf{b}', \mathbf{c}, \langle \mathbf{b}_i^{\Delta_2} \mathbf{b}_i^{\Delta_2} \mathbf{c}_i^{\Delta_2} : i < k \rangle \mathbf{a}^{\Delta_2}).$$

By the definition of obstruction,

$$\sigma^{\Delta_2}(\mathrm{tp}_{\Delta_3}(\mathbf{b}'/\mathrm{A}^{\Delta_2} \cup \mathbf{a}^{\Delta_2})) \neq \mathrm{tp}_{\Delta_3}(\mathbf{c}/\mathrm{A}^{\Delta_2} \cup \mathbf{a}^{\Delta_2}).$$

Since σ^{Δ_2} fixes $A^{\Delta_2} \cup \mathbf{a}^{\Delta_2}$ pointwise, this contradicts that $\sigma^{\Delta_2}(\mathbf{b}') = \mathbf{c}$ and we finish. $\square$4.4

Finally we have the main result.

Theorem 4.8 *If T is a stable theory, T_{Aut} has a model companion if and only if T admits no obstructions.*

Proof. We showed in Lemma 4.4 that if T_{Aut} has a model companion then there is no obstruction. If there is no obstruction, Lemma 3.7 implies T does not have the finite cover property. By Lemma 4.3 for every formula $\psi(\mathbf{x}, \mathbf{y}, \mathbf{z})$ there is an L_σ-formula $\theta_\psi(\mathbf{z})$ (Write out condition 3 of Lemma 4.3.) which for any $(M, \sigma) \models T_{\mathrm{Aut}}$ holds of any $\mathbf{a}$ in M if and only if there exists (N, σ), $(M, \sigma) \subseteq (N, \sigma) \models T_{\mathrm{Aut}}$ and

$$(N, \sigma) \models (\exists \mathbf{x}\mathbf{y})[\psi(\mathbf{x}, \mathbf{y}, \mathbf{a}) \wedge \sigma(\mathbf{x}) = \mathbf{y}].$$

Thus, the class of existentially closed models of T_{Aut} is axiomatized by the sentences: $(\forall \mathbf{z})\theta_\psi(\mathbf{z}) \rightarrow (\exists \mathbf{x}\mathbf{y})[\psi(\mathbf{x}, \mathbf{y}, \mathbf{a}) \wedge \sigma(\mathbf{x}) = \mathbf{y}]$. (We can restrict to formulas of the form $\psi(\mathbf{x}, \sigma(\mathbf{x}), \mathbf{a})$ by the standard trick ([4, 1]). $\square_{4.8}$

Kikyo and Pillay [4] note that if a strongly minimal theory has the definable multiplicity property then T_{Aut} has a model companion. In view of Theorem 4.8, this implies that if T has the definable multiplicity property, then T admits no obstructions. Kikyo and Pillay conjecture that for a strongly minimal set, the converse holds: if T_{Aut} has a model companion then T has the definable multiplicity property. They prove this result if T is a finite cover of a theory with the finite multiplicity property. It would follow in general from a positive answer to the following question.

Question 4.9 *If the ω-stable theory T with finite rank does not have the definable multiplicity property, must it admit obstructions?*

Pillay has given a direct proof that if a strongly minimal T has the definable multiplicity property, then T admits no obstructions. Pillay has provided an insightful reworking of the ideas here in a note which is available on his website [7].

Here is a final question:

Question 4.10 *Can T_{Aut} for an $\aleph_0$-categorical stable T admit obstructions?*

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