# How big should the monster model be?

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To Jouko Väänänen on his 60th birthday.

The 1970's witnessed two attempts to organize large areas of mathematics by introducing a structure in which all of that mathematics took place. Both of these programs explicitly invoke large cardinal hypotheses; advocates of both programs deny that these hypotheses are necessary. There were a few years ago no extant metatheorems justifying these conclusions (but see McLarty [25]). By a metatheorem, I mean a way to tell by looking at a particular proposition whether the purported use of large cardinals is necessary. The natural expectation is that this criteria would be syntactic. But the possibility of geometric criteria arise in [20] and the general model theoretic development suggests other 'contentual criteria' (Section 4.) We have been unable to develop any reasonable such theorem in the model theoretic case. This paper just explores the territory.

As we will see, in both cases, the role of replacement as an intermediary between Zermelo Set Theory and large cardinals arises in the analysis. This work was stimulated by some questions raised by Newelski in [27] concerning the 'absoluteness' of the existence of a bounded orbit and the discussions at about the same time concerning the possible use of large cardinals in Wiles' proof of Fermat's last theorem.

On the one hand Grothendieck wanted to provide a framework in which to organize large areas of algebraic geometry. For this he invented the notion of a universe, a large enough set to be closed under the usual algebraic operations. He explicitly developed cohomology theory using the existence of (a proper class of) universes[1].

At about the same time, Shelah introduced his version of a 'universal domain', later dubbed a *monster model*: $\mathfrak{C}$ is a saturated model of $T$ with cardinality $\overline{\kappa}$ a strongly inaccessible cardinal. He writes in [30], 'The assumption on $\overline{\kappa}$ does not, in fact, add any axiom of set theory as a hypothesis to our theorems.' This is a rather unclearly stated metatheorem. 'Which theorems?' An acceptable answer in the short run is, 'the theorems in this book'. And, apparently, the statement is true in that sense. We seek here to clarify what theorems are intended and which are not.

We give a brief comparison of the algebraic geometry side (in Section 1.2) relying heavily on [23, 25, 20] and the model theoretic issues (in Subsection1.3). Section 2

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[1]The issue we address here of the size of the universe is distinct from the organization via sheaves addressed in [19].

provides the connection between monster models and Hanf numbers. In Section 3 we examine the model theoretic question more carefully in three examples and pose some specific problems. We conclude with speculations about the ways one might sort out the properties which do not depend on a long universe in Section 4

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# 1 The two programs

## 1.1 Set Theoretic Background

We discuss four different set theories. The standard background theory for model theory is ZFC. While there has been some work lately on the necessary uses of set theory in model theory [33, 14, 3] they have focused on the role of the axiom of choice [2] or on principles near the GCH[3] and mostly on infinitary model theory. Our subject here is first order model theory and the role of axioms having to do with the length of the universe.

We consider in Subsection 3.1 whether even relatively small cardinals such as $\beth_{\omega_1}$ are necessary in model theory. In the section on foundations of cohomology we discuss ZC (Zermelo Set Theory), i.e. ZFC without replacement or foundation, but with the full axiom of separation. Maclane set theory (i.e. bounded Zermelo or MC in McLarty) weakens ZC still more by postulating only separation for $\Delta_0$-formulas (bounded quantification).

In the other direction, we consider whether certain uses of inaccessible cardinals are as transparently unnecessary as is usually claimed by model theorists. For this it is useful to extend from ZFC to NBG, (Von Neumann, Bernays, Gödel) set theory. This is a *conservative* extension of ZFC, which distinguishes between classes and sets. See [23] for a more detailed account.

## 1.2 Grothendieck Universes and Wiles proof

In formulating the general theory of cohomology Grothendieck developed the concept of a universe – a collection of sets large enough to be closed under any operation that arose. Grothendieck proved that the existence of a single universe is equivalent over ZFC to the existence of a strongly inaccessible cardinal [15, Vol. I, p. 196]. More precisely, $U$ is the set $V_\alpha$ of all sets with rank below $\alpha$ for some uncountable strongly inaccessible cardinal.

McLarty summarised the general situation in [23]:

> Large cardinals as such were neither interesting nor problematic to Grothendieck and this paper shares his view. For him they were merely

[2] See Shelah's proof in [33] that for countable theories Morley's categoricity theorem is provable in ZF.

[3] In particular many results in infinitary logic apparently require $2^\kappa < 2^{\kappa^+}$, at least for $\kappa < \aleph_\omega$. See [1] for background and more references.

legitimate means to something else. He wanted to organize explicit calculational arithmetic into a geometric conceptual order. He found ways to do this in *cohomology* and used them to produce calculations which had eluded a decade of top mathematicians pursuing the Weil conjectures [29]. He thereby produced the basis of most current algebraic geometry and not only the parts bearing on arithmetic. His cohomology rests on universes but weaker foundations also suffice at the loss of some of the desired conceptual order.

*As written* the great applications of cohomology theory (e.g. Wiles and Faltings) implicitly rely on universes. Most number theorists regard the applications as requiring much less than their 'on their face' strength and in particular believe the large cardinal appeals are 'easily eliminable'. An animated discussion on Math Overflow [37] and earlier on FOM[4] emphasizes this belief. This discussion is perhaps best summed up by the amusing remark of Pete Clark, 'If a mathematician of the caliber of Y.I. Manin made a point of asking in public whether the proof of the Weil conjectures depends in some essential way on inaccessible cardinals, is this not a sign that "Of course not; don't be stupid" may not be the most helpful reply?'

There are in fact two issues. McLarty[23] writes: 'Wiles's proof uses hard arithmetic some of which is on its face one or two orders above PA, and it uses functorial organizing tools some of which are on their face stronger than ZFC.'

There are two current programs for verifying in detail the intuition that the formal requirements for Wiles proof of Fermat's last theorem can be substantially reduced. On the one hand, McLarty's current work ([25, 24]) aims to reduce the 'on their face' strength of the results in cohomology from large cardinal hypotheses to finite order Peano. On the other hand Macintyre aims to reduce the 'on their face' strength of results in hard arithmetic[20] to Peano. These programs may be complementary or a full implementation of Macintyre's might avoid the first.

McLarty ([25]) reduces

1. ' all of SGA[5]' to Bounded Zermelo plus a Universe.
2. '"the currently existing applications" to Bounded Zermelo itself, thus the consistency strength of simple type theory.'

   The Grothendieck duality theorem and others like it become theorem schema.

The essential insight of the McLarty's papers on cohomology is the role of replacement in giving strength to the universe hypothesis. In [25], a $ZC$-*universe* is defined to be a transitive set U modeling such $ZC$ such that every subset of an element of $U$ is itself an element of $U$. He remarks that any $V_\alpha$ for $\alpha$ a limit ordinal is provable in $ZFC$ to be a $ZC$-universe. McLarty then asserts the essential use of replacement in the original Grothendieck formulation is to prove: For an arbitrary ring $R$ every module over $R$ embeds in an injective $R$-module and thus injective resolutions exist for all $R$-modules. But he gives a proof in a system with the proof theoretic strength of finite order arithmetic[6] that every sheaf of modules on any small site has an infinite

[4]Search 'universes' or Wiles on the Fom archive.

[5]SGA refers to a sequence of Grothendiecks works.

[6]In [24], McLarty reduces the strength to second order arithmetic for certain cohomology theories.

resolution. (Section 3.6 of [25].)

Macintyre [20] dismisses with little comment the worries about the use of'large-structure' tools in Wiles proof. He begins his appendix [20], 'At present, all roads to a proof of Fermats Last Theorem (henceforward FLT) pass through some version of a Modularity Theorem (generically MT) about elliptic curves defined over Q ... A casual look at the literature may suggest that in the formulation of MT (or in some of the arguments proving whatever version of MT is required) there is essential appeal to higher-order quantification, over one of the following'. He then lists such objects as $\mathbb{C}$, modular forms, Galois representations ... and summarises that a superficial formulation of MT would be $\Pi^1_m$ for some small $m$. But he continues, 'I hope nevertheless that the present account will convince all except professional sceptics that MT is really $\Pi^0_1$.' There then follows a 13 page highly technical sketch of an argument for the proposition that MT can be expressed by a sentence in $\Pi^0_1$ along with a less-detailed strategy for proving MT in PA.

Macintyre's complexity analysis is in traditional proof theoretic terms. But his remark that 'genus' is more useful geometric classification of curves that the syntactic notion of degree suggests that other criteria may be relevant.

From the standpoint of this paper the McLarty's approach is not really a metatheorem but a statement that there was only one essential use of replacement and it can be eliminated. In contrast, Macintyre argues that 'apparent second order quantification' can be replaced by first order quantification. But the argument requires deep understanding of the number theory for each replacement in a large number of situations. Again, there is no general theorem that this type of result is provable in PA. A battery of techniques is displayed for translating the statements to $\Pi^0_1$ and reducing the proof theoretic strength of the axioms.

## 1.3 Monster models in Model theory

Many model theory papers begin 'We work in a big saturated model' or slightly more formally, 'We are working in a saturated model of cardinality $\kappa$ for sufficiently large $\kappa$ (a monster model).'

What does *sufficiently* mean? In every case I know such a declaration is not intended to convey a reliance on the existence of large cardinals. Rather, in Marker's phrase, it is a declaration of laziness, 'If the stakes were high enough I could write down a ZFC proof'. As we note below, in standard cases the author isn't being very lazy; but the route to formalizing a metatheorem expressing this intuition does not seem clear.

This was not a problem for the early history of classification theory. Work focussed on stable theories. And a stable theory has a saturated model in every $\lambda$ with $\lambda^{\kappa(T)} = \lambda$, where $\kappa(T)$ is an invariant that is less than $|T|^+$. Thus, there is a plentiful supply of monster models. But recently model theory has moved to the investigation of unstable theories and these issues become more acute, as we discuss in Subsection 3.2. We will see that the difficulty is just the lack of saturated models but lack of the control of structure provided by stability theory.

The fundamental unit of study is a particular first order theory. The need is for a monster model of the theory $T$. If $M$ is a $\kappa$-saturated model of $T$, then every model $N$

of $T$ with cardinality at most $\kappa$ is elementarily embedded in $M$ and every type over a set of size $< \kappa$ is realized in $M$. So every configuration of size less than $\kappa$ that could occur in any model of $T$ occurs in $M$. Then general theorems are asserted to hold for each theory.

In fact, the requirement that the monster model be saturated in its own cardinality is excessive. A more refined version of the 'monster model hypothesis' asserts: Any first order model theoretic properties of sets of size less than kappa can be proved in a $\kappa$-saturated strongly $\kappa$-homogenous model M (any two isomorphic submodels of card less than $\kappa$ are conjugate by an automorphism of M). Such a model exists (provably in ZFC) in some $\kappa'$ not too much bigger than $\kappa$. See Hodges [17] or my monograph on categoricity [1] for the refined version. (Hodges' (big model condition is ostensibly stronger and slightly more complicated to state; but existence is also provable in ZFC.) Buechler [6], Shelah [30], and Marker[22] the expound harmless nature of the fully saturated version. Ziegler [38] adopts a class approach that could be formulated in Gödel Bernays set theory. And we will adopt that approach below.

In order to clarify the problem, we will address several specific problems where some issues arise in calculating the size of the necessary 'monster'.

## 2 Hanf Numbers and Monster models

In this section we expand a bit on the arguments for the eliminability of large cardinal hypotheses in uses of the monster model. Then we connect the properties of a class monster model with the calculability of certain Hanf functions.

Buechler[7] argues that the apparent reliance can be removed by a sequence of applications of the same proof. To prove model theoretic statements about structures of size at most $\kappa$, use a $\kappa$-monster. If $\kappa$ increases, choose a larger monster. Note that the size of the monster was not used in the argument.

So, for example, to compute the spectrum function of a first order theory via the strategy of classification theory, theories are divided into categories by properties (stability class, DOP, OTOP, depth) which have no dependence on the size of the model. Then for each class $P$ a function $f_P$ is defined such that for $\kappa < \rho$ which is the size of a given choice of monster model $f_P(\kappa)$ is (or is at least an upper bound for) the number of models in $\kappa$. This function works for all $\kappa$ by just redoing the argument for a larger $\rho$ as $\kappa$ grows.

We work in Von Neumann, Bernays, Gódel set theory NBG, a *conservative* extension of ZFC, which admits classes as objects.

**Definition 2.0.1** *A* monster model *is a class model* $\mathbb{M}$ *which is a union of* $\kappa$*-saturated models for arbitrarily large* $\kappa$*.*

This definition (from [38]) is quite different from the usual usage in model theory. We connect with more standard usage by defining the notion of a $\kappa$-monster which formalizes monster set models as certain kinds of special models [17, 8, 34].

[7] I paraphrase an argument that Buechler says holds 'with few exceptions' on page 70 of [6].

**Definition 2.0.2** 1. *A structure $M$ of infinite cardinality $\kappa$ is* special *if $M$ is the union of an elementary chain $\langle M_\lambda : \lambda < \kappa, \lambda \text{ a cardinal} \rangle$, where each $M_\lambda$ is $\lambda^+$-saturated.*

2. *A structure $M$ is strongly $\kappa$-homogeneous if for every $A$ contained in $M$ with $|A| < \kappa$, every embedding of $A$ into $M$ can be extended to an automorphism of $M$.*

3. *A $\kappa$-*monster *model $\mathbb{C}_\kappa$ is a special model of cardinality $\mu = \beth_{\kappa^+}(\kappa)$.*

**Fact 2.0.3** *A $\kappa$-monster is unique up to isomorphism, $\mu^+$-universal and strongly $\kappa^+$-homogenous.*

Now the natural conjecture is:

**Conjecture 2.0.4** *For any property $P$, the class monster $\mathbb{M}$ satisfies $P$ if and only if all sufficiently large $\kappa$-monsters $\mathbb{C}_\kappa$ satisfy $P$.*

The main problem is to specify what is meant by a property. A too generous definition is 'a class in NBG'. But the issue is to refine this notion. And all we actually give here are some specific examples that should be considered in making a definition.

This conjecture would follow if 'all sufficiently large $\kappa$-monsters $\mathbb{C}_\kappa$ satisfy $P$.' were replaced by a 'uniform proof' that 'for all sufficiently large $\kappa$-monsters that $\mathbb{C}_\kappa$ satisfies $P$.' This is the strategy that works successfully for the spectrum problem. But I don't see how to get this claim in general; we examine a specific problem where a uniform argument is not apparent in Subsection 3.2.

Finding such a uniform argument seems related to Hanf numbers. Hanf [16] introduced the following extremely general and soft argument. $P(\boldsymbol{K}, \lambda)$ ranges over such properties as: $\boldsymbol{K}$ has a model in cardinality $\lambda$, $\boldsymbol{K}$ is categorical in $\lambda$, or the type $q$ is omitted in some model of $\boldsymbol{K}$ of cardinality $\lambda$. We will see some more novel examples below.

**Theorem 2.0.5 (Hanf)** *Fix a* set *of classes $\boldsymbol{K}$ of a given kind (e.g. the classes of models defined by sentences of $L_{\mu,\nu}$ for some fixed $\mu, \nu$ of a given similarity type). For any property $P(\boldsymbol{K}, \lambda)$ there is a cardinal $\kappa$ such that if $P(\boldsymbol{K}, \lambda)$ holds for some $\lambda > \kappa$ then $P(\boldsymbol{K}, \lambda)$ holds for arbitrarily large $\lambda$.*

Proof. Let

$$\mu_{\boldsymbol{K}} = \sup\{\lambda : P(\boldsymbol{K}, \lambda) \text{ holds }\}$$

where $\mu_{\boldsymbol{K}} = \infty$ if there is no bound on the cardinality of models of $\boldsymbol{K}$ satisfying $P$. Then

$$\kappa = \sup\{\mu_{\boldsymbol{K}} : \mu_{\boldsymbol{K}} < \infty\}.$$

$\square_{2.0.5}$

**Definition 2.0.6** *$P$ is* downward closed *if there is a $\kappa_0$ such that if $P(\boldsymbol{K}, \lambda)$ holds with $\lambda > \kappa_0$, then $P(\boldsymbol{K}, \mu)$ holds if $\kappa_0 < \mu \leq \lambda$.*

The following is obvious.

**Theorem 2.0.7** *If a property $P$ is downward closed then for any $\kappa$ there is a cardinal $\mu$ such that for any class of models $\boldsymbol{K}$ with vocabulary*[8] *of size $\kappa$, if some model in $\boldsymbol{K}$ with property $P$ has cardinality greater than $\mu$, then there is a model in $\boldsymbol{K}$ with property $P$ in all cardinals greater than $\mu$.*

That is, if each of a collection of classes is downward closed for a property $P$ there is a Hanf Number for $P$ in the following stronger sense.

**Definition 2.0.8 (Hanf Numbers)** *The* Hanf number *for $P$, among classes $\boldsymbol{K}$ with vocabulary of cardinal $\kappa$, is $\mu$ if: if there is a model in $\boldsymbol{K}$ with cardinality $> \mu$ that has property $P$, then there is a model with property $P$ in all cardinals greater than $\mu$. In this situation $\mu$ is the Hanf number of $P$ (for classes with vocabulary of cardinal $\kappa$).*

**Definition 2.0.9** *A function $f$ is* calculable *if it can (provably in ZFC) be defined in terms of cardinal addition, multiplication, exponentiation, and iteration of the $\beth$-function.*

Here are several examples of properties of first order $T$ where Hanf numbers may or not be calculable.

1. $T$ has a model in $\kappa$.
2. $T$ has a saturated (or special) model in $\kappa$.
3. $T$ has a model that is a group with a bounded orbit in the sense of Subsection 3.2.
4. The Hanf number for omission and saturation (Subsection 3.3).

For 1), the Hanf number is the cardinality of the vocabulary, so it is calculable. For 2) the Hanf number is the first stability cardinal for stable theories and again this is calculable. But for unstable theories there is considerable not yet determinative research on the existence of saturated models so the Hanf number has not been calculated. See, e.g., [31, 12]. Note that 2) is not downward closed. We explore some cases of 3) where the Hanf number is calculated and some where it remains an open question in Subsection 3.2. And we note in Subsection 3.3 that the Hanf number for case 4) is incalculable in general but it is for superstable $T$.

If Conjecture 2.0.4 holds, the natural size for a monster model for studying a property $P$ is the Hanf number of $P$. Unfortunately, as our discussion in Subsection 3.2 shows, this equivalence is not obvious.

Most examples in the literature of Hanf numbers are variants on the Morley's omitting types theorem and the Hanf number[9] is $\beth_{\omega_1}$. There are more complicated examples in [32, 1].

[8] This gets a bit more technical; see page 32 of [1]

[9] This is for countable vocabularies; for a vocabulary with cardinality $\kappa$, the relevant Hanf number $\beth_{(2^\kappa)^+}$.

# 3 Three Examples

## 3.1 Is replacement needed?

One of the fundamental tools of model theory constructs indiscernibles realizing types from a prescribed set.

**Theorem 3.1.1** *Let $M$ be a big saturated model. For every* large enough *set $I \subset M$, there exists an infinite sequence of order indiscernibles $J \subset M$ such that for every finite* $\mathbf{b} \in J$ *there is an* $\boldsymbol{a} \in I$ *with* $\mathrm{tp}(\mathbf{b}/\emptyset) = \mathrm{tp}(\boldsymbol{a}/\emptyset)$.

The crux here is the requirement that the complete types of the sequences in $J$ are types realized in $I$. With no requirement on the types appearing in $J$, only Ramsey's theorem and compactness is needed in the standard Ehrenfeucht-Mostowski proof.

We can guarantee this result only if $|M| \geq \beth_{\omega_1}$. This example makes the question, 'What set theory is used in model theory?' a little sharper. Friedman proved [13] that Borel determinacy required the existence of $\beth_{\omega_1}$. Are there such examples of necessary uses of replacement in first order model theory? Morley [26] showed both that $\beth_{\omega_1}$ sufficed for the cardinality of $I$ and that it was necessary. But this necessity argument itself uses replacement.

In some sense Theorem 3.1.1 and Hanf numbers for omitting types require the existence of $\beth_{\omega_1}$ even to be stated. Those notions are about size or about 'logics'. But here is a theorem clearly stated in ZC, but for which known proofs use replacement. Byunghan Kim[18] proved:

**Theorem 3.1.2 (Kim)** *For a simple first order theory non-forking is equivalent to non-dividing.*

The usual easily applicable descriptions of simple theories involve uncountable objects. But definitions of simple, non-forking, and non-dividing are equivalent in ZC to statements about countable sets of formulas. Indeed we quote below such formulations which were given as the definitions in Casanovas' recent exposition [7]. Nevertheless, the argument for Kim's theorem employs Morley's technique for omitting types; that is: The standard argument uses the Erdos-Rado theorem on cardinals less than $\beth_{\omega_1}$.

Our goal here is simply to state this proposition clearly enough to show that it is properly formulated without any use of replacement. For this, we simply repeat the basic definitions from [7] where the exact result we are after is given a short complete proof. We work in a complete first order theory in a countable vocabulary.

**Definition 3.1.3** *Let $\boldsymbol{a}_i$ be a sequence of finite tuples in a model of a first order theory $T$. A set of formulas $X = \{\phi(\mathbf{x}, \boldsymbol{a}_i) : i < \omega\}$ is $k$-inconsistent if every $k$ element subset of $X$ is inconsistent.*

With this notion in hand we can define forking and dividing.

**Definition 3.1.4** *Let $A \cup \{\boldsymbol{a}\}$ ( $A \cup \{\boldsymbol{a}\} \cup \{\boldsymbol{a}_j : j < n\}$ ) be a subset of a model of $T$.*

1. *The formula $\phi(\mathbf{x}, \boldsymbol{a})$ $k$-divides over $A$ if there is an infinite set $I = \{a_i : i < \omega\}$ such that $\{\phi(\mathbf{x}, \boldsymbol{a}_i : i < \omega\}$ is $k$-inconsistent and all the $\boldsymbol{a}_i$ realize $\mathrm{tp}(\boldsymbol{a}/A)$. $\phi$* divides *if it $k$-divides for some $k$.*

2. *The set of formulas $\pi(\mathbf{x}, \boldsymbol{a})$ forks over $A$ if for some finite set of formulas $\psi_j(\mathbf{x}, \boldsymbol{a}_j)$ with $j < n$, $\pi(\mathbf{x}, \boldsymbol{a}) \vdash \bigvee_{j<n} \psi_j(\mathbf{x}, \boldsymbol{a}_j)$ and each $\psi_j(\mathbf{x}, \boldsymbol{a}_j)$ divides over $A$.*

**Definition 3.1.5** *The formula $\phi(\mathbf{x}, \mathbf{y})$ has the tree property with respect to $k < \omega$ if there is a tree $(a_s : s \in \omega^{<\omega})$ (in some model of $T$) such that for all $\eta \in \omega^\omega$, the branch $\{\phi(\mathbf{x}, \boldsymbol{a}_{\eta\restriction n}) : n < \omega\}$ is consistent and for all $s \in \omega^{<\omega}$, the family of siblings $\{\phi(\mathbf{x}, \boldsymbol{a}_{s\frown i}) : i < \omega\}$ is $k$-inconsistent.*

**Definition 3.1.6** *$T$ is* simple *if there is no formula $\phi(\mathbf{x}, \mathbf{y})$ which has the tree property in $T$.*

There is a direct proof of the following result in [7].

**Theorem 3.1.7** *Let $T$ be a simple theory. A partial type $\pi(\mathbf{x}, \boldsymbol{a})$ divides over $A$ if and only it forks over $A$.*

None of the arguments given in [7] directly invoke replacement. But Lemma 1.1 of that paper, which is applied at a crucial point, is a variant of the standard:

**Theorem 3.1.8** *[Morley omitting types theorem] Let $T$ be a $\tau$-theory, $\Gamma$ a set of partial $\tau$-types (in finitely many variables) over $\emptyset$ and $\mu = (2^{|\tau|})^+$, Suppose $M_\alpha$ for $\alpha < \mu$ are a sequence of $\tau$-structures such that $|M_\alpha| > \beth_\alpha$.*

*Then, there is a countable sequence $I$ of order indiscernibles such that for every finite sequence $\boldsymbol{a} \in I$, $\mathrm{tp}(\boldsymbol{a}/\emptyset)$ is realized in each $M_\alpha$.*

The crucial message of Morley's theorem as opposed to the standard Ehrenfeucht Mostowski argument which takes place in ZC is the requirement that the finite types that will be realized in $J$ are specified in advance to come from the $M_\alpha$. I know of no proof of Theorem 3.1.7 that does not rely on replacement to invoke Erdos-Rado via Morley's omitting types theorem.

**Question 3.1.9** *Is Theorem 3.1.7 provable in ZC?*

**Remark 3.1.10** As we'll see later in other examples, restricting to stable theories often reduces the set theoretic strength needed for a result. In particular, Shelah proved the equivalence of forking and dividing for stable theories without any reliance on replacement. The strength of stationarity in stable theories allows the construction of the required indiscernibles using only Ramsey's theorem (as in Ehrenfreucht-Mostowski). See Section V.3, in particular Theorem V.3.9 of [2].

Recently, Chernikov and Kaplan [9] proved a slight weakening of the theorem for a much broader class of theories. They prove that for types over models[10] in NTP2 theories; these include all simple and all NIP theories. But they still rely on 'Kim's Lemma' which uses replacement.

[10]Examples of unstable theories where types over sets can fork without dividing are well known.

## 3.2 Bounded orbits of types

Newelski in [27] raised questions about how to calculate whether the action of a monster model of a first order theory of groups on its Stone space admitted a bounded orbit. This notion is related to definable amenability of the group. He reduced the problem to the calculation of the Hanf number for saturation and omission, which makes general model theoretic sense. In Section 3.3, we relate that this Hanf number is not calculable. In this we work only with the Hanf number for bounded orbits in an attempt to see if might be calculable (for arbitrary theories of groups).

**Assumption 3.2.1** *$T$ is a first order theory extending the theory of groups with countable vocabulary $\tau$. Each model of $T$ acts on itself and therefore on its Stone space by left translation. For $p \in S(M)$, $p \mapsto gp = \{\phi(x, g^{-1}\boldsymbol{a}) : \phi(x, \boldsymbol{a}) \in p\}$.*

**Notation 3.2.2** *Consider $M \prec M'$, which are groups. Let $p \in S(M')$, and let the $M'$-orbits of $p$ be represented by $p_\alpha$ ($\alpha < \mu$). Consider the restrictions $q_\alpha = p_\alpha \restriction M$.*

**P1:** *Each $q_\alpha$ extends uniquely to some type over $M'$ in the orbit of $p$.*

**P2:** *Every $p_\alpha$ restricts to some $q_\alpha$.*

The following definition is adapted from [27].

**Definition 3.2.3** *Let $\mathbb{C}$ be a $\kappa$-monster, $M \prec \mathbb{C}$, $q \in S(M)$, $\mathcal{O} = Mq$. That is, $\mathcal{O}$ is the orbit of $q$ under left translation by $M$.*

1. *$\mathcal{O}$ is* $\mathbb{C}$-bounded if there is $p \in S(\mathbb{C})$ with P1 and P2 and $q = p\restriction M$.
2. $\mathcal{O}$ is *$\infty$-bounded* if it is $\mathbb{C}'$-bounded for all $\kappa$-monsters $\mathbb{C}' \succ M$ (for any $\kappa > |M|$).
3. $T$ has an $\infty$-bounded orbit if for some $M \models T$, there is some $\infty$-bounded orbit in $S(M)$.
4. For a fixed countable theory $T$, let $\mu_T$ be the least cardinality of an $\infty$-bounded orbit (for some $\kappa$-monster modeling $T$) if such a model exists.
5. Let $\mu_{\rm bd} = \sup\{\mu_T : T$ has a model with an $\infty$-bounded orbit $\}$.

Now Newelski[27] asked[11].

**Question 3.2.4** *1. Find a bound on $\mu_{\rm bd}$ in terms of $\beth$-numbers.*

*2. Find an example where is an $\infty$-bounded orbit of size $> 2^{2^{\aleph_0}}$ in a countable theory.*

*3. Is there a bound on the size of an $\infty$-bounded orbit in a countable theory of a group.*

[11] Newelski's $\kappa$-monsters were not unique, complicating the issue even further.

By Remark 1.12 of [27] the Hanf number for $\mu_{\rm bd}$ is no more than $H(N_\lambda)$, (i.e. $H(P^\lambda_N)$ ) as discussed in Section 3.3. But we showed in [5] that this number is incalculable. However it is only an upper bound for the Hanf number for $\mu_{\rm bd}$.

We can't answer any of these questions; we try to place them in a broader context and address related questions. To begin with instead of trying to bound the size of the bounded orbit, one could bound the size of models which have a bounded orbit. If there was no such bound, one would like to think that the monster model had a bounded orbit in the sense of Definition 3.2.7.

Pillay and Conversano [10, 11] for o-minimal and even NIP theories and Newelski [28] for more general classes defined in terms of the action of groups have calculated the Hanf number. But our question here is:

**Question 3.2.5** *Is there a calculation of the Hanf number for bounded orbits that works for every first order theory of groups? Or are some kind of stability conditions necessary to get a bound.*

Note that the key aspect of $\infty$-bounded is the property of extending a $\kappa$-monster to a $\kappa'$-monster preserving the size of a small orbit. It was implicit in [27] that such an extension theorem holds in stable theories. For context, we include a proof.

**Fact 3.2.6** *If $T$ is stable and $\mathbb{C}$ is a $\kappa$-monster of $T$ with a type $p \in S(\mathbb{C})$ with bounded orbit of cardinality $\lambda < \mu$ (with $\mu$ as in Definition 2.0.2.3), then in any $\kappa'$-monster $\mathbb{C}' \succ \mathbb{C}$, there is a type with orbit of cardinality $\lambda$.*

Proof. Let $p' \in S(\mathbb{C}')$ be a nonforking extension of $p = p_0 \in S(\mathbb{C})$. Fix $M_0 \prec \mathbb{C}$ that is $\kappa(T)^+$-saturated and $\langle q_\alpha \in S(M_0) : \alpha < \lambda\rangle$ which are the conjugates of $p\restriction M_0$. Each $q_\alpha$ is a nonforking extension of a stationary type over some $X_\alpha \subset M_0$ with $|X_\alpha| < \kappa(T)$. If every $\mathbb{C}'$-conjugate of $p'$ is a nonforking extension of some $q_\alpha$, we are finished. Now suppose some $m \in \mathbb{C}'$ conjugates $p'$ so that $mp'\restriction M_0 \neq q_\alpha$ for all $\alpha < \mu$. Then $mX_0$ is not contained in $M_0$. But there exists a subset $Y$ of $M_0$ with $|Y| = \kappa(T)$ such that $r = \mathrm{tp}(mX_0/M_0)$ is based on $Y$. Now by $|M_0|$-saturation of $\mathbb{C}$, construct in $\mathbb{C}$ a nonforking sequence $m_\beta Y_\beta$ for $\beta < |M_0|$ of realizations of $r$. Each $Y_\beta$ is the base of a conjugate of $p'\restriction M_0$ contrary to hypothesis. $\square_{3.2.6}$

Newelski's formulation focuses on whether a particular orbit remains bounded as the ambient monster changes. To connect with the true monster we consider a variant- which monsters have bounded (i.e. small) orbits?

**Definition 3.2.7 (A model has a Bounded orbit)**

1. *Let $\mathbb{M}$ be the monster model of $T$. We say $\mathbb{M}$ has a* bounded orbit *if there is a $p \in S(\mathbb{M})$ such that $\mathbb{M}p = \{ap : a \in \mathbb{M}\}$ is a set.*
2. *Let $\mathbb{C} = \mathbb{C}_\kappa$ be a $\kappa$-monster model of $T$. We say $\mathbb{C}_\kappa$ has a* bounded orbit *if there is a $p \in S(\mathbb{C})$ such that, with $\mathbb{C}p$ denoting $\{ap : a \in \mathbb{C}\}$, $|\mathbb{C}_p| < |\mathbb{C}|$.*

**Question 3.2.8** *When does $\mathbb{M}$ have a bounded orbit? That is, can we define a cardinal $\kappa$ and a* property of set models *such that $\mathbb{M}$ has a bounded orbit if and only if the set monster $\mathbb{C}_\kappa$ has the property.*

This is the formulation of Conjecture 2.0.4 in this context. We will see it is problematic.

**Notation 3.2.9** *We will write $\widehat{M}$ for a model which may be either a $\kappa$-monster $\mathbb{C}_\kappa$ or the true monster $\mathbb{M}$. The saturation hypothesis may not always be used.*

For the next few paragraphs we analyze the relation between the orbits of a type and its restrictions. Note that for any *set or class* model $M$ and $p \in S(M)$, the cardinality of the orbit of $p$ is the index of $\mathrm{stb}_M(p)$ in $M$. (The *stabilizer of* $p$, $\mathrm{stb}_M(p)$, is the subgroup of $a \in M$ such that $ap = p$.)

**Claim 3.2.10** *Let $M \prec N \prec \widehat{M}$ and $\hat{p} \in S(\widehat{M})$.*

1. *If $a \in \mathrm{stb}_{\widehat{M}}(\hat{p})$ and $a \in M$ then $a \in \mathrm{stb}_M(\hat{p}\restriction N)$. In particular, for any $M \prec \widehat{M}$, $(\mathrm{stb}_{\widehat{M}}(\hat{p}) \cap M) \subseteq \mathrm{stb}_M(\hat{p}\restriction M)$.*
2. *So if $M$ contains representative of all cosets of $\mathrm{stb}_{\widehat{M}}(\hat{p})$,*

$$|\widehat{M}/\mathrm{stb}_{\widehat{M}}(\hat{p}))| = |M/\mathrm{stb}_{\widehat{M}}(\hat{p}) \cap M| \geq |M/\mathrm{stb}_M(\hat{p}\restriction M)|.$$

Proof. For any $\phi(x, \mathbf{c}) \in \hat{p}\restriction N$, $\phi(x, a^{-1}\mathbf{c}) \in \mathrm{stb}_{\widehat{M}}(\hat{p})$ and $a^{-1}\mathbf{c} \in N$ so $\phi(x, a^{-1}\mathbf{c}) \in \mathrm{stb}_N(\hat{p}\restriction N)$. For the equality in the second assertion, note that for any $b \in M$ there is an $a_i$ in the set of representatives such that $b^{-1}a_i \in \mathrm{stb}_{\widehat{M}}(\hat{b})$ but also in $M$. $\square_{3.2.10}$

We establish downward monotonicity for having a bounded orbit.

**Claim 3.2.11** *Let $M \prec N \prec \widehat{M}$ and $\hat{p} \in \widehat{M}$. If $\widehat{M}$ has a bounded orbit of size $\mu$ then for every $\kappa$ with $\mu < \kappa < |\widehat{M}|$ and every model $M$ of size $\kappa$, there is a model $M'$ of size $\kappa$ with $M \prec M'$ such that $M'$ has an orbit of size $\leq \mu$.*

Proof. Fix a model $M$ of size $\kappa$. Let $\hat{p} \in S(\widehat{M})$ have a bounded orbit of cardinality $\mu$, i.e. $\mathrm{stb}_{\widehat{M}}(\hat{p})$ has $\mu$ cosets with representatives $\langle a_i : i < \mu\rangle$. Let $N$ be a submodel containing the $a_i$. Choose $M'$ as an elementary extension of $N$ and $M$ of cardinality $\kappa$. By Claim 3.2.10, the orbit of $\hat{p}\restriction M'$ has size at most $\mu$. $\square_{3.2.11}$

We show there are enough models which reflect the orbits of $\hat{p}$.

**Claim 3.2.12** *Fix $\hat{p} \in S(\widehat{M})$. For any $M$, there is an $M'$ with $M \prec M'$ and $|M| = |M'|$ such that*

$$\mathrm{stb}_{\widehat{M}}(\hat{p}) \cap M' = \mathrm{stb}_{M'}(\hat{p}\restriction M').$$

Proof. By Claim 3.2.10 $\mathrm{stb}_{\widehat{M}}(\hat{p}) \cap M' \subseteq \mathrm{stb}_M(\hat{p}\restriction M)$. Now let $M_1 = M^*$ be a model of size $|M|$ such that for every $a \in \mathrm{stb}_M(\hat{p}\restriction M) - \mathrm{stb}_{\widehat{M}}(\hat{p}) \cap M'$, there is $\mathbf{c} \in M^*$ with $\phi(x, \mathbf{c}) \in \hat{p}$, $\phi(x, a^{-1}\mathbf{c}) \notin \hat{p}$. (Thus, every such $a$ is not in $\mathrm{stb}_{M^*}(\hat{p}\restriction M')$.) Now if $M_{i+1} = M_i^*$, then $M' = \bigcup_{i<\omega} M_i$ satisfies the desired condition, as by Claim 3.2.10 $\mathrm{stb}_{\widehat{M}}(\hat{p}) \cap M' \subseteq \mathrm{stb}_{M'}(\hat{p}\restriction M')$.

With these technical lemmas we can relate the size of the orbit a fixed type over the (class or set) monster to the size of the orbits of its restrictions.

**Lemma 3.2.13** *Fix a type $\hat{p} \in S(\mathbb{M})$. The following are equivalent.*

1. *The orbit of $\hat{p}$ is bounded.*

2. *Let $\mu = |\mathbb{M}\hat{p}|$. For every $M$ with $|M| \geq \mu$, there exists an $M'$ containing $M$ with $|M'| = |M|$ such that $\hat{p}\restriction M'$ has cardinality $\mu$.*

3. *Let $\mu = |\mathbb{M}\hat{p}|$. For every $M$ with $|M| \geq \mu$, there is a $\kappa$-monster $\mathbb{C}_\kappa$ with $M \prec \mathbb{C}_\kappa$ and the restriction of $\hat{p}$ to the $\kappa$-monster $\mathbb{C}_\kappa$ has an orbit of cardinality less than $\kappa$.*

Proof. 1) implies 2) is Claim 3.2.11 with $\widehat{M}$ taken as $\mathbb{M}$. For 2) implies 1), we prove the converse. If $\hat{p} \in S(\mathbb{M})$ is unbounded then for every $\kappa$, we can choose $M$ of cardinality $\kappa$ so that $\kappa$ elements of $M$ are in distinct cosets of $\mathrm{stb}_{\mathbb{M}}(\hat{p})$. By Lemma 3.2.12, we can extend $M$ to $M'$ with $\mathrm{stb}_{\mathbb{M}}(\hat{p}) \cap M' = \mathrm{stb}_{M'}(\hat{p}\restriction M')$ and the index of this subgroup is at least $\kappa$.

3) To prove 3) from 1) make the same construction as in proving 3) but continue for $\kappa^+$ steps with the $\alpha + 1$'st structure saturated over its predecessor.

3) implies 1). Build a chain $\mathbb{C}_\kappa$ for arbitrarily large $\kappa$ satisfying the condition in iii). Since $\hat{p}$ is fixed for the entire construction $\hat{p}$ is unbounded.

$\square_{3.2.13}$

**Remark 3.2.14** Note that condition 3) depends on the particular embedding of $M_\kappa$ in the monster. That is, if we take a different copy of $M_\kappa$, there will be a conjugate of $\hat{p}$ under the automorphism group of $\mathbb{M}$ whose restriction to this $M_\kappa$ has a bounded orbit.

Thus 3) implies

4): For arbitrarily large $\kappa$, there is a $\hat{p}' \in S(\mathbb{M})$ whose restriction to the $\kappa$-monster $M_\kappa$ has an orbit of cardinality less than $\kappa$.

But 4) is ostensibly weaker; the extension keeps changing.

**Question 3.2.15** *Is it possible that there is a type $p \in S(M)$ that is bounded, say with orbit size $\mu$ and for arbitrarily large $\kappa$ there are extensions $p_\kappa$ over set models $M_\kappa$ which all have orbit size $\kappa$. But there is no increasing chain of such types?*

We can rephrase this analysis with the following implication.

**Corollary 3.2.16** *From Lemma 3.2.13 and Remark 3.2.14, we know 1) implies 2):*

1. *$\mathbb{M}$ has a bounded orbit.*

2. *For arbitrarily large $\kappa$, the $\kappa$-monster $M_\kappa$ has an orbit of cardinality less than $\kappa$.*

**Question 3.2.17** *But does the converse hold?*

If the converse held, then we would know that the true monster does not have a bounded orbit, exactly if there is an upper bound on the cardinalities of set monsters with bounded orbits. And then we could reduce the existence of a true monster with bounded orbits to the following ZFC question.

**Question 3.2.18** Find an *explicit description* of a function $f : \mathrm{On} \to \mathrm{On}$ such that if $T$ has cardinality $\sigma$ and the $f(\sigma)$-monster has a bounded orbit then $\mathbb{M}$ has a bounded orbit.

More precisely, is $f$ *calculable*?

## 3.3 Saturation and omission

In this section, we state what seems to be a natural model theoretic property. It arose from Newelski's consideration of the issues in Section 3.2. But the Hanf number for the property is not calculable.

Newelski reduced the existence of bounded orbits to the question. When is a there a triple of a theory in vocabulary $\tau$, a sub-vocabulary $\tau_1$ and a $\tau$-type $p$ such that there is a model $M$ with $M\restriction\tau_1$-saturated but $M$ omits $p$.

Let us state the property more formally.

**Definition 3.3.1** *We say $M_1 \models \boldsymbol{t}$ where $\boldsymbol{t} = (T, T_1, p)$ is a triple of two theories in vocabularies $\tau \subset \tau_1$ with $|\tau_1| \leq \lambda$, $T \subseteq T_1$ and $p$ is a $\tau_1$-type over the empty set if $M_1$ is a model of $T_1$ which omits $p$, but $M_1\restriction\tau$ is saturated.*

*Let $\boldsymbol{N}_\lambda$ denote[12] the set of $\boldsymbol{t}$ with $\tau_1 = \lambda$. Then $H(\boldsymbol{N}_\lambda)$ denotes the Hanf number of $\boldsymbol{N}_\lambda$, $H(\boldsymbol{N}_\lambda)$ is least so that if some $\boldsymbol{t} \in \boldsymbol{N}_\lambda$ has a model of cardinality $H(\boldsymbol{N}_\lambda)$ it has arbitrarily large models.*

To avoid difficulties about finding saturated models, we assume in this subsection:

**Assumption 3.3.2** *Assume the collection of $\lambda$ with $\lambda^{<\lambda} = \lambda$ is a proper class.*

This assumption follows from the generalized continuum hypothesis or from 'There are a proper class of strongly inaccessible cardinals'.

Under the mild set theoretic hypotheses of 3.3.2, we showed in [5] that $H(\boldsymbol{N}_\lambda)$ equals the Löwenheim number of second order logic, which is incalculable[36]. In [4] we restrict the question by requiring that the theory $T$ satisfy stability conditions. The distinction between superstable and stable is immense. For the superstable case the number is easily calculable in terms of Beth numbers. For stable theories, it again depends on the choice of set theory, although it is lower than the general case.

**Theorem 3.3.3** *Let $|\tau| = \lambda$.*

1. *If $T$ is superstable*

$$H(\boldsymbol{N}_\lambda^{ss}) < H(L_{(2^\lambda)^+,\omega}) \leq \beth_{(2^{(2^\lambda)^+})^+}.$$

2. *If $T$ is stable*

$$H(L_{\lambda^+,\kappa(T)}) \leq H(\boldsymbol{N}_\lambda^{str}) \leq H(L_{(2^\lambda)^+,\kappa(T)}).$$

[12] Thus, 'there is an $M \in \boldsymbol{N}_\lambda$ with cardinality $\kappa$' replaces the more cumbersome notation in [5], '$P_N^\lambda(\boldsymbol{K}_{\boldsymbol{t}}, \kappa)$ holds'.

*3. In general,*

$$H(\boldsymbol{N}_\lambda) \geq \ell^2(L^{II})$$

*where $\ell^2(L^{II}L^{II}$ denotes Löwenheim number of second order logic. Thus $H(\boldsymbol{N}_\lambda)$ is a large cardinal if such exist.*

For the superstable case, we have stated a calculable bound. For the stable case, we can restrict to countable languages to give a simpler account. Then the lower bound is $H(L_{\omega_1,\omega_1})$ and the upper bound is $H(L_{(2^\omega)^+,\omega_1})$. As Vaananen observes in [35], this makes the lower bound at least the first inaccessible, first Mahlo, etc. if they exist. So it is not calculable in the sense of this paper.

On the other hand, $H(L_{\omega_1,\omega_1})$ is below the first weakly compact cardinal (if there is a proper class of weakly compacts) by [35]. Thus, it is smaller than $H(\boldsymbol{N}_{\aleph_0})$, which is bigger than the first weakly compact, since the Löwenheim number of second order logic is bigger the first weakly compact cardinal, Indeed it is bigger than the first (second, third, etc) fixed point of any normal function on cardinals that itself can be described in second order logic. So if there is e.g. a Ramsey cardinal this Löwenheim number is bigger than the first one etc. However it is less than the first supercompact [21]. Thus, it is not calculable in the sense we have defined.

# 4 What kind of conditions imply only weak set theory is needed to prove a proposition?

The usual answers to this question are in terms of syntactic properties of the properties under consideration. Properties of infinitary logic are inherently more complicated that properties of first order logic. Or as Macintyre stressed, various properties are phrased with second order quantification.

None of the three examples in Section 3 seem *a priori* to have any real connection with stability. Even in the first case, forking and dividing can be defined for arbitrary theories and one can ask, 'does forking equal dividing'. But we have three propostions where there is apparent (in the first case), definite (in the third) (and (open in the second) dependence on set theory to obtain the proposition for an arbitrary theory[13]. But in each case a stability hypothesis gives a positive solution to the problem. In stable theories forking is the same as dividing. In stable theories, there are bounded orbits. In superstable theories, the Hanf number of $N_\lambda$ is calculable.

Note that in these examples the dependence of the result on the length of the set theoretic universe varies depending on the stability of the underlying theory. In contrast, calculating the spectrum function of a first order theory has no such dependence (although the proof uses the stability hierarchy).

Can we find a way to characterize such conditions as stability or o-minimality or the geometric conditions such as genus mentioned by Macintyre so as to account for

[13]Strictly speaking to apply this comment to the case of NTP2 theories we should only consider types over models, since we know there are non-simple theories for which forking (over arbitrary (countable) sets) does not equal dividing.

lesser set theoretic strength necessary to study 'tame' mathematics? We see here that the cardinalities of the objects studied are not the defining characteristic of tame.

# References


[1] John T. Baldwin. *Categoricity*. Number 51 in University Lecture Notes. American Mathematical Society, 2009. www.math.uic.edu/˜ jbaldwin.

[2] J.T. Baldwin. *Fundamentals of Stability Theory*. Springer-Verlag, 1988.

[3] J.T. Baldwin. Amalgamation, absoluteness, and categoricity. In Arai, Feng, Kim, Wu, and Yang, editors, *Proceedings of 11th Asian Logic Conference, 2009*. World Scientific, 2012. Available at www.math.uic.edu/ jbaldwin.

[4] J.T. Baldwin and S. Shelah. A Hanf number for saturation and omission II. preprint:`http://www.math.uic.edu/\~jbaldwin/pub/shnew22`, to appear Fund. Math.

[5] J.T. Baldwin and S. Shelah. A Hanf number for saturation and omission. *Fund. Math.*, 213:255–270, 2011. preprint:`http://www.math.uic.edu/\~jbaldwin/pub/shnew22`.

[6] Steven Buechler. *Essential Stability Theory*. Springer-Verlag, 1991.

[7] Enrique Casanovas. The number of types in simple theories. *Annals of Pure and Applied Logic*, 98:69–86, 1999.

[8] C.C. Chang and H.J Keisler. *Model theory*. North-Holland, 1973. 3rd edition 1990.

[9] A. Chernikov and I. Kaplan. Forking and dividing in ntp2 theories. *Journal of Symbolic Logic*, 77:1–20, 2012.

[10] A. Conversano and A. Pillay. Connected components of definable groups and o-minimality i. preprint.

[11] A. Conversano and A. Pillay. Connected components of definable groups and o-minimality II. preprint.

[12] Džamonja+Shelah. On $\triangleleft^*$-maximality. *Annals Pure and Applied Logic*, 125:119–158, 2004. item 692 in Shelah archive.

[13] Harvey Friedman. Higher set theory and mathematical practice. *Ann. Math.Log.*, pages 325–357, 1971.

[14] Sy-David Friedman and Martin Koerwien. On absoluteness of categoricity in aecs. preprint, 200x.

[15] A Grothendieck. Elements de géometrie algébriques (rédigés avec la collaboration de J. Dieudonné), IV Etudes Locale des Schémas et des Morphismes de Schémas. *Publ Math I.H.E.S.*, 28:5–255, 1966.

[16] William Hanf. Models of languages with infinitely long expressions. In *Abstracts of Contributed papers from the First Logic, Methodology and Philosopy of Science Congress, Vol.1*, page 24. Stanford University, 1960.

[17] W. Hodges. *Model Theory*. Cambridge University Press, 1993.

[18] Byunghan Kim. Forking in simple theories. *Journal of the London Mathematical Society*, 57:257–267, 1998.

[19] Angus J. Macintyre. Model theory: Geometrical and set-theoretic aspects and prospects. *Bulletin of Symbolic Logic*, 9:197 – 212, 2003.

[20] Angus J. Macintyre. The impact of Gödel's incompleteness theorems on mathematics. In Baaz et al, editor, *Kurt Gödel and the Foundations of Mathematics*, pages 3–26. Cambridge University Press, 2011.

[21] M. Magidor. On the role of supercompact and extendible cardinals in logic. *Israel Journal of Mathematics*, 10:147–157, 1971.

[22] D. Marker. *Model Theory: An introduction*. Springer-Verlag, 2002.

[23] C. Mclarty. What does it take to prove Fermat's last theorem? Grothendieck and the logic of number theory. *Bulletin of Symbolic Logic*, 16:359–377, 2010.

[24] C. McLarty. Coherent čech and zariski cohomology in second order arithmetic. preprint:`http://arxiv.org/pdf/1207.0276.pdf`, 2012.

[25] C. McLarty. A finite order arithmetic foundation for cohomology. preprint:`http://arxiv.org/pdf/1102.1773.pdf`, 2012.

[26] M. Morley. Omitting classes of elements. In Addison, Henkin, and Tarski, editors, *The Theory of Models*, pages 265–273. North-Holland, Amsterdam, 1965.

[27] Ludomir Newelski. Bounded orbits and measures on a group. *Israel Journal of Mathematics*, 2012. to appear 2012.

[28] Ludomir Newelski. Bounded orbits and strongly generic sets. preprint, 2012.

[29] B. Osserman. The weil conjectures. In T. Gowers, editor, *The Princeton Companion to Mathematics*, pages 729–732. Princeton University Press, 2008.

[30] S. Shelah. *Classification Theory and the Number of Nonisomorphic Models*. North-Holland, 1978.

[31] S. Shelah. Simple unstable theories. *Annals of Mathematical Logic*, 19:177–203, 1980.

[32] S. Shelah. Categoricity for abstract classes with amalgamation. *Annals of Pure and Applied Logic*, 98:261–294, 1999. paper 394. Consult Shelah for post-publication revisions.

[33] S. Shelah. Model theory without choice: categoricity. *Journal of Symbolic Logic*, 74:361–401, 2009. paper 840.

[34] Katrin Tent and Martin Ziegler. *A course in Model Theory*. Lecture Notes in Logic. Cambridge University Press, 2012.

[35] Jouko Vaananen. The Hanf number of $L_{\omega_1,\omega}$. *Proceedings of American Mathematical Society*, 79:294–297, 1980.

[36] Jouko Vaananen. Abstract logic and set theory II: Large cardinals. *Journal of Symbolic Logic*, 47:335–346, 1982.

[37] various contributors. Inaccessible cardinals and andrew wiless proof. mathoverflow:`http://mathoverflow.net/questions/35746/inaccessible-cardinals-and-andrew-wiless-proof`, 2012.

[38] M. Ziegler. Introduction to stability theory and Morley rank. In E. Bouscaren, editor, *Model Theory and Algebraic Geometry : An Introduction to E. Hrushovski's Proof of the Geometric Mordell-Lang Conjecture*, pages 19–44. Springer-Verlag, 1999.