Necessity of the VWGCH

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January 11, 2008

Abstract

The following statement is relatively consistent with ZFC. Suppose κ is a regular cardinal with $\kappa^{<\kappa} = \kappa$. Then, there is a complete sentence ψ of $L_{\omega_1,\omega}$ such that ψ is categorical in every $\lambda < \kappa$ but has 2^κ models of cardinality κ and no models of cardinality $> \kappa$.

A key distinction between first order and infinitary logic is the upwards Löwenheim-Skolem theorem. A first order sentence that has an infinite model or a sentence of $L_{\omega_1,\omega}$ that has a model of cardinality greater than $\beth_{\omega_1}$ has models of every infinite cardinal. But there are sentences [17, 11, 13, 15, 23] ϕ_κ of $L_{\omega_1,\omega}$ with no models of cardinality greater than κ for any cardinal κ below $\beth_{\omega_1}$. Shelah [18, 19] introduces the notion of an *excellent* class of models of a sentence of $L_{\omega_1,\omega}$ (or excellent sentence) and proves (in ZFC) that an excellent class has arbitrarily large models. Moreover, he recovers Morley's theorem for excellent classes: an excellent class that is categorical in one uncountable cardinal is categorical in all uncountable cardinals. Excellence is defined by the existence and uniqueness of k-ary amalgamations for certain systems of countable models.

Shelah further deduces excellence from conditions on the number of models of a sentence ϕ in small cardinalities. But this deduction requires an extension of ZFC. We call the assertion, $2^{\aleph_n} < 2^{\aleph_{n+1}}$ for $n < \omega$, the very weak generalized continuum hypothesis (VWGCH). It implies Devlin-Shelah weak diamond, which is the combinatorial tool for the next result. In [18, 19] Shelah proves using weak diamonds that if ϕ is $\aleph_n$-categorical) for all $n < \omega$ (or even has has at most $2^{\aleph_{n-1}}$ models in each $\aleph_n$), then ϕ is excellent. Hart and Shelah [10] (simplified by Baldwin-Kolesnikov [2]) prove that the assumption of categoricity up to $\aleph_\omega$ is necessary by finding sentences that are categorical up to $\aleph_n$ but no further. We show that the set-theoretic assumption is necessary as well. An immediate corollary of the main result of this note is that it is consistent that $2^{\aleph_0} = \aleph_{\omega+1}$ and there is a sentence ψ, categorical in $\aleph_n$ for $n \leq \omega$, that has $2^{\aleph_{\omega+1}}$ models of power $\aleph_{\omega+1}$ and no larger models. Thus the assertion: categority of a sentence of $L_{\omega_1,\omega}$ below $\aleph_\omega$ implies categoricity in all powers is independent of ZFC.

The Baldwin-Lachlan characterization of uncountable categoricity [3] yields that for any countable first order theory T, the property 'T is uncountably categorical' is absolute between transitive models of ZFC. No such absoluteness result holds when T is replaced by an arbitrary sentence of $L_{\omega_1,\omega}$ and we exploit that fact here. We describe a sentence which under VWGCH has the maximal number of models in every uncountable power below the continuum but which under Martin's axiom is categorical below the continuum. The complementary positive development of the theory of excellent classes is expounded in [1].

Shelah [21] had suggested examples subsumed in the discussion below to show that the method of approach in [18, 19] depended on VWGCH by showing there were sentences that failed both ω-stability and amalgamation in $\aleph_0$ but were $\aleph_1$-categorical under Martin's axioms. Carrying out this suggestion required that we introduce a new type of forcing conditions and we then saw that this forcing had the much stronger conclusions that we present here. We conclude the paper with some further discussion of the context and open problems.

Theorem 1 (Martin's Axiom) *There is a sentence ψ in $L_{\omega_1,\omega}$ with the joint embedding property that is κ categorical for every $\kappa < 2^{\aleph_0}$. In ZFC one can prove ψ is $\aleph_0$-categorical but has neither the amalgamation property in $\aleph_0$ nor is ω-stable. (The example is an AEC with respect to $L_{\omega_1,\omega}$-elementary submodel.) Moreover, ψ has $2^{2^{\aleph_0}}$ models of power $2^{\aleph_0}$ and no larger models.*

For background on Martin's axiom, see e.g. [14, 12]. We use freely that fact that if κ is a regular cardinal with $\kappa^{<\kappa} = \kappa$ then $MA + 2^{\omega} = \kappa$ is relatively consistent with ZFC. We will use the following special case of the axiom that is tailored for our applications.

Definition 2 Martin's Axiom

1. *MA_κ is the assertion: If $\mathcal{F}$ is a collection of partial isomorphisms, partially ordered by extension, between two structures M and N of the same cardinality less than $2^{\aleph_0}$ that satisfies the countable chain condition then for any set of κ dense subsets of $\mathcal{F}$, C_a, then there is a filter G on $\mathcal{F}$ which intersects all the C_a.*
2. *$\mathcal{F}$ satisfies the countable chain condition if there is no uncountable subset of pairwise incompatible members of $\mathcal{F}$.*
3. *Martin's axiom is:* $(\forall\kappa < 2^{\aleph_0})(\mathrm{MA}_\kappa)$.

The basic idea behind the example studied here is to consider a two sorted universe. One side P is a countable set; the other is a filter of subsets of P. Clearly such structures can have cardinality at most $2^{\aleph_0}$. But with Martin's axiom one can prove that all models in a cardinality $\kappa < 2^{\aleph_0}$ are isomorphic. But we would like to study a 'nicely defined' class of models. Shelah [22, 21] suggests examples that are abstract elementary classes, axiomatizable in $L(Q)$ and finally in $L_{\omega_1,\omega}$. The first two cases are proved by variants of the forcing introduced by Baumgartner [4]; the situation here is a bit more complicated. The crux is to write a sentence in $L_{\omega_1,\omega}$ which enforces the countability of P while allowing enough flexibility for the forcing argument.

Example 3 (Shelah [21]) Let the vocabulary contain unary predicates P, Q, P_n for $n < \omega$ and a binary relation R. P will be a countable set contained in the algebraic closure of the empty set and the elements of Q will index a family of subsets of P.

We say a collection of distinct sets $X_0, \ldots X_{n-1}$ is *independent* if every intersection $\bigcap_{i<n} X_i^{\pm}$ is infinite. Similarly, we say a collection of distinct sets $X_0, \ldots X_{k-1}$ is *independent* on P_n if every intersection $\bigcap_{i<k} X_i^{\pm} \cap P_n$ has exactly 2^{n-k} elements. Finally, we say a collection $\mathcal{C}$ of subsets of P is *closed under finite difference* if $X \in \mathcal{C}$ and $Y \Delta X$ is finite implies $Y \in \mathcal{C}$.

We construct a model M: Take P as a disjoint union of sets P_n each with 2^n elements. Now choose subsets of P as follows:

1. Each $|X_i \cap P_n| = 2^{n-1}$.
2. Choose X_i for $i < \omega$ by induction so that if $k \leq n$, $X_1, \ldots X_k$ are independent on P_n.
3. Close the X_i under finite difference to form a collection $\mathcal{C}$.
4. Finally, form Q by adding elements q such that each $A_q = \{p \in P : R(q,p)\}$ is in $\mathcal{C}$ and each q names a unique subset of P via R.

It is easy to give a sentence ψ of $L_{\omega_1,\omega}$ that expresses the following properties of M. Our class is the models of ψ with respect to $L_{\infty,\omega}$ elementary submodel.

1. $\langle P_n^M : n < \omega \rangle$ is a partition of P^M.
2. P_n^M has exactly 2^n elements .
3. $(\forall x \in Q)(\forall u \in [P^M]^{<\aleph_0})(\exists y \in Q)[A_x^M \Delta A_y^M = u]$.
4. if $k < \omega$ and $y_0, \ldots, y_{k-1} \in Q$ satisfies $|A_{y_\ell} \Delta A_{y_m}| \geq \aleph_0$ for $\ell < m < k$ then for some m and all $n \geq m$, for any $\eta \in {}^k 2$ the set
$$\cap \{A_{y_\ell}^M : \eta(\ell) = 1\} \setminus \cup \{A_{y_\ell}^M : \eta(\ell) = 0\} \cap P_n^M$$
has exactly 2^{n-k} elements.
5. $Q(y) \wedge Q(z) \wedge (\forall x \in P)[xRy \leftrightarrow xRz] \rightarrow y = z$.
6. for every $k < \omega$ for some $y_0, \ldots, y_k \in G, \bigwedge_{\ell < m \leq k} |A_{y_\ell} \Delta A_{y_m}| \geq \aleph_0$

Remark 4 *1. Clause 3) means that the set of $\{A_x : x \in Q\}$ is closed under finite difference. In particular, it implies that for any subset u of any P_n there is an $x \in Q$ such that $A_x \cap P_n = u$. This is the clause that allows us to choose the extensions of the back and forth at the proper stage in the filtration.*

2. Clause 4) implies that if $k < \omega$ and $y_0, \ldots, y_{k-1} \in Q$ satisfies $|A_{y_\ell} \Delta A_{y_m}| \geq \aleph_0$ for $\ell < m < k$ then $\{A_{y_\ell}^M : \ell < k\}$ is an independent family of subsets of P^M.

Definition 5 *We write $x \sim y$ (or $A_x \sim A_y$) if $|A_x \Delta A_y|$ is finite.*

There are countably many $\sim$-inequivalent sets in M; if we add another (and close under finite difference) we have a proper $L_{\infty,\omega}$-extension of M.

All countable $M \models \psi$ are isomorphic. Indeed, if $M_0 \subseteq M_1$ it is easy to form a back and forth showing that $M_0 \prec_{\infty,\omega} M_1$. (The argument is really the same as the proof of the density conditions below.) Since there is an isomorphic $L_{\infty,\omega}$-extension of a model of ψ, ψ has an uncountable model.

There are essentially two kinds of pairs of elements in Q; those which are equivalent (i.e. denote two subsets with a finite difference) and those which are not. We have to strengthen the forcing conditions in order to distinguish these cases and describe the intersections of the A_{q_i} and their complements in P_n for finite tuples of elements q_i of Q. These relationships are described by formulas of the following form. Essentially we need the existential and universal types of sequences from Q.

Definition 6 *For any finite set $X \subset M$, an L-structure, the* description *of X, $\mathrm{Des}_M(\mathbf{x})$ (where $\mathbf{x}$ enumerates X) is the quantifier free diagram of X along the set of formulas (for each n and each $m \leq n$) of the following form (and their negations) that are true of X in M.*

$$(\exists v_0, \ldots \exists v_m) \bigwedge v_i \neq v_j \wedge A_{x_i}^{\pm}(v_j) \wedge \bigwedge_{i \leq m} P_n(v_i)$$

To show the isomorphism of models in κ under Martin's axiom, we use the following forcing conditions.

Definition 7 *Fix filtrations $\langle M_i : i < \kappa \rangle$ of M and $\langle N_i : i < \kappa \rangle$ of N by $L_{\infty,\omega}$-submodels such that each $M_{i+1} - M_i$, $(N_{i+1} - N_i)$ contains exactly one new $\sim$-equivalence class. $\mathcal{F}$ is the set of finite partial isomorphism f such that*
$$\mathrm{Des}_M(\mathrm{dom}\, f) = \mathrm{Des}_N(\mathrm{rg} f)$$
and such that for each $i < \kappa$, and each $x \in \mathrm{dom}\, f$, $x \in M_i$ if and only if $f(x) \in N_i$.

Note this definition is very different from saying $\operatorname{dom} f \subset M_i$ iff $\operatorname{rg} f \subset N_i$; this second version does not satisfy ccc. We have guaranteed that each element of M has only $\aleph_0$ possible images by maps in the set of forcing conditions. The notion of description implies that the forcing conditions must map two points that name inequivalent subsets of P to elements that name inequivalent subsets of P. The following remark is key. For any $\boldsymbol{a} \in M$, let $P_{\boldsymbol{a}}$ denote the union of the $P_n(M)$ such some component of $\boldsymbol{a}$ is in P_n (and similarly for N).

Claim 8 *If* $\mathrm{Des}_M(\boldsymbol{a}) = \mathrm{Des}_N(\mathbf{b})$*, there is a bijection of* $P_{\boldsymbol{a}}$ *to* $P_{\mathbf{b}}$ *whose union with the map taking* $\boldsymbol{a}$ *to* $\mathbf{b}$ *is an isomorphism.*

Proof. Use the fact that the intersections of $A^{\pm}_{a_i}$ with $P(M)$ have the same cardinality as the intersections of $A^{\pm}_{b_i}$ with $P(N)$. $\square_8$

Let us prove that these forcing conditions have the ccc.

Lemma 9 $\mathcal{F}$ *satisfies the countable chain condition.*

Proof. Let $\langle f_\alpha : \alpha < \aleph_1 \rangle$ be a sequence of elements of $\mathcal{F}$. Without loss of generality, fix m and k so that the domain of each f_α contains m elements of P and k of Q. Applying the Δ-system lemma to the domain and the range, we can find Y (Y') contained in M (N) so that for an uncountable subset S, if $\alpha, \beta \in S$, $\operatorname{dom} f_\alpha \cap \operatorname{dom} f_\beta \cap M = Y$ ($\operatorname{rg} f_\alpha \cap \operatorname{rg} f_\beta \cap N = Y'$). Note that, in fact, all the f_α for $\alpha \in S_1$ intersect in a single bijection f. For, if there were some $b \in Y$ and some α with $f_\alpha(b) \notin Y'$, then (as Y' is the root for the range), the $\{f_\alpha(b) : \alpha \in S\}$ give uncountably many distinct images for b contrary to the choice of the filtration. (A similar argument for the domains and the fact that there are only finitely many maps from Y onto Y' yield the bijection f.) In the notation of Claim 8, let $P_{\operatorname{dom} f}$ be the union of the $P_n(M_0)$ such that $P_n(M_0) \cap \operatorname{dom} f \neq \emptyset$. There are only finitely many quantifier-free types of k-tuples from Q over $P_{\operatorname{dom} f}$. Thus we may assume each $\operatorname{dom} f_\alpha \restriction Q$ realizes the same quantifier-free type over $P_{\operatorname{dom} f}$ (and thus, using that $\operatorname{dom} f_\alpha$ and $\operatorname{rg} f_\alpha$ have the same description, each $\operatorname{rg} f_\alpha \restriction Q$ realizes the same quantifier-free type over $P_{\operatorname{rg} f}$).

The requirement that conditions preserve the filtration yields that for some $\langle i_j : j < \ell \rangle$ where $\ell = |Y|$, $Y \subset \bigcup_{j<\ell}(M_{i_j+1} - M_{i_j})$ and $Y' \subset \bigcup_{j<\ell}(N_{i_j+1} - N_{i_j})$. Applying the condition on filtrations (each element in the domain (range) has only countably many possible images (preimages)), we can demand that in restricting to S we guaranteed that no element of M_i (N_i) occurs in $\operatorname{dom} f_\alpha - Y$ ($\operatorname{rg} f_\alpha - Y'$) for more than one α. All the $\operatorname{dom} f_\alpha \restriction Q$ realize the same quantifier-free type over $P_{\operatorname{dom} f}$, so we have

$$\mathrm{tp}_{\mathrm{qf}}(\operatorname{dom} f_\alpha, \operatorname{dom} f_\beta) = \mathrm{tp}_{\mathrm{qf}}(\operatorname{rg} f_\alpha, \operatorname{rg} f_\beta)$$

for all $\alpha, \beta \in S_1$. To show ccc, we will find $\alpha \neq \beta \in S$ with $f_\alpha \cup f_\beta$ a condition. That is,

$$\mathrm{Des}_M(\operatorname{dom} f_\alpha, \operatorname{dom} f_\beta) = \mathrm{Des}_N(\operatorname{rg} f_\alpha, \operatorname{rg} f_\beta).$$

To see the difficulty and establish notation, consider elements $x_\alpha \in \operatorname{dom} f_\alpha \cap Q(M)$ with image $y_\alpha \in \operatorname{rg} f_\alpha \cap Q(N)$. It may be that $\mathrm{Des}_M(x_\alpha, x_\beta) \neq \mathrm{Des}_N(y_\alpha, y_\beta)$. x_α, x_β are in different $\sim$ classes so they are eventually independent and the same is true for y_α, y_β. But for some finite number of i, the pattern of $Ax_\alpha \cap Ax_\beta \cap P_i(M)$ may differ from the pattern of $Ay_\alpha \cap Ay_\beta \cap P_i(M)$. However there exist y'_α such that $y'_\alpha \sim y_\alpha$ and $\mathrm{Des}_M(x_\alpha, x_\beta) = \mathrm{Des}_N(y'_\alpha, y_\beta)$. This means that there are disjoint finite sets $(A_{\alpha,\beta}, B_{\alpha,\beta})$ of $P(N)$ so that $y'_\alpha = (y_\alpha \cup A_{\alpha,\beta}) - B_{\alpha,\beta}$. We can choose such $(A_{\alpha,\gamma}, B_{\alpha,\gamma})$ for each $\gamma \in S_1$. But there are only countably many choices for such pairs so for some γ, γ', $(A_{\alpha,\gamma}, B_{\alpha,\gamma}) = (A_{\alpha,\gamma'}, B_{\alpha,\gamma'})$. But then,

$$\mathrm{Des}_M(x_\gamma, x_{\gamma'}) = \mathrm{Des}_N(y_\gamma, y_{\gamma'}).$$

To actually complete the proof we consider for each α a sequence $\mathbf{x}_\alpha = \langle x^1_\alpha \dots x^k_\alpha \rangle$ where the x^i_α enumerate $\mathrm{dom}\, f_\alpha \cap Q(M)$ and also an image $\mathbf{y}_\alpha$. Similarly we define vectors $(\overline{A}_{\alpha,\gamma}, \overline{B}_{\alpha,\gamma})$ for each $\gamma \in S_1$. There are only countably many choices for such pairs so for some γ, γ', $(\overline{A}_{\alpha,\gamma}, \overline{B}_{\alpha,\gamma}) = (\overline{A}_{\alpha,\gamma'}, \overline{B}_{\alpha,\gamma'})$. But then

$$\mathrm{Des}_M(\mathrm{dom}\, f_\gamma, \mathrm{dom}\, f_{\gamma'}) = \mathrm{Des}_N(\mathrm{rg} f_\gamma, \mathrm{rg} f_{\gamma'})$$

and we finish. $\square_9$

We need to show that ψ implies the obvious density conditions to show the function constructed by MA is a bijection. Let D_a be the conditions with a in the domain and R_a be the conditions with a in the range. We show the density condition for the R_a; a similar argument works for the domain.

Lemma 10 *If $M, N \models \psi$ and $\boldsymbol{a} \in M, \mathbf{b} \in N$ with $\mathrm{Des}_M(\boldsymbol{a}) = \mathrm{Des}_N(\mathbf{b})$ then for every $c \in M$, there is a $d \in N$ such that $\mathrm{Des}_M(\boldsymbol{a}c) = \mathrm{Des}(\mathbf{b}d)$. Moreover, if M and N have cardinality κ with filtrations $\langle M_i : i < \kappa \rangle$ and $\langle N_i : i < \kappa \rangle$ by members of $\boldsymbol{K}$, then $c \in M_{i+1} - M_i$ implies $d \in N_{i+1} - N_i$ (and vice versa).*

Proof. Suppose $Q(c)$ and $c \sim a_i$ for some $a_i \in \boldsymbol{a}$. A first approximation at the image of c is b_i. It is correct on P_n for the infinitely many n such that $A_c \cap P_n = A_{a_i} \cap P_n$. As $c \sim a_i$, we must have $d \sim b_i$. To also satisfy the finitely many exceptional P_n, modify b_i to d using clause 3) so that $[Des(\boldsymbol{a}c) \leftrightarrow Des(\mathbf{b}d)]$. (More precisely, let u enumerate the finitely many P^M_n on which the a_i and c are not equal. Then map u to an enumeration of the corresponding P^N_n extending the given correspondence. Use the fact that the $a_i \in Q^M$ and the $a'_i \in Q^N$ realize the same existential type to make sure this mapping preserves the cardinality of intersections of the A_{a_i} (and complements). Finally choose the intersection of $A_{d'}$ with u' to mimic the intersection of c with u.) By the definition of the forcing conditions b_i and a_i are at the same level in the filtration and since c differs from b_i and d from a_i by a finite set, so are c and d.

Suppose $Q(c)$ and $c \not\sim a_s$ for each $a_s \in \boldsymbol{a}$. By Remark 4.2, there is an m_c for each $s < k = \mathrm{lg}(\boldsymbol{a})$ such that for all $r \geq m_c$, the $A_{a_s} \cap P_r$ and $A_c \cap P_r$ are independent. Suppose i is least such that $c \in Q(M_i)$; then i is 0 or a successor $j+1$. Choose $d' \in N_{j+1} - N_j$ (or N_0 if $i = 0$) so that $d' \not\sim b_s$ for $s < k$. Now, as in the first paragraph, modify d' to d by altering the values of A_d on the P_r for $r < m_c$ so that $[Des(\boldsymbol{a}c) \leftrightarrow Des(\mathbf{b}d)]$. Since d and d' differ on only finitely many values, $d \in N_{j+1} - N_j$.

Suppose $c \in P$, then $c \in P_n$ for some n. Now let $B \subset P_n(M)$ be the intersection of the $A^{\pm}_{a_i} \cap P_n$ for $a_i \in Q$ that are satisfied by c. Then $[\mathrm{Des}_M(\boldsymbol{a}) = \mathrm{Des}_N(\mathbf{b})]$ implies $|B| = |B'|$ where B' is obtained by replacing a_i by b_i in the formula defining B. So an appropriate d can be chosen in same $P_n(N)$. There is no problem with the filtration since all elements of P are in the first model. $\square_{10}$

Now by Martin's axiom there is a generic filter G such that $\bigcup G$ is an isomorphism between M and N. This concludes the proof of categoricity. Clearly every model of ψ has power at most the continuum.

Lemma 11 *For every $\kappa \leq 2^{\aleph_0}$ there is a model M_κ of ψ with $|M_\kappa| = \kappa$. Indeed, there are $2^{2^{\aleph_0}}$ models of cardinality $2^{\aleph_0}$. And under VWGCH for $\aleph_0 < \kappa < \aleph_\omega$, ψ has 2^κ models of power κ.*

Proof. There are $2^{\aleph_0}$ independent subsets of P. (In the language of [5], there is a family Υ with $|\Upsilon| = 2^{\aleph_0}$ that has ω-large isolation. Thus, as in 7.3 of [5], there are $2^{2^{\aleph_0}}$ filters $\mathcal{F}_i$ on P such that if $i \neq j$, there is an $A \in \mathcal{F}_i$ with $\omega - A \in \mathcal{F}_j$. Now, for each $\kappa \leq 2^{\aleph_0}$ we can restrict the $\mathcal{F}_i$ to $\mathcal{F}^\kappa_i$, such that $|\mathcal{F}^\kappa_i| = \kappa$, while preserving the disjointess of the filters. Each filter gives rise to a model M^κ_i by interpreting $\mathcal{F}^\kappa_i$ as Q. Note that two of these models, M, N are isomorphic if and only if there is an isomorphism α of $P(M)$ with $P(N)$ (mapping $P_n(M)$ to $P_n(N)$ that extends to an isomorphism of M and N by mapping $x \in Q(M)$ to $\{\alpha(p) : p \in x\}$. For $\kappa < 2^{\aleph_0}$, under MA, we have shown the all the M_i^κ are isomorphic. Since each equivalence class under this relation has only $2^{\aleph_0}$ elements, there are still $2^{2^{\aleph_0}}$ pairwise non-isomorphic models of ψ with cardinality $2^{\aleph_0}$. And, under VWGCH, since $2^{\aleph_0} < 2^\kappa$ if $\aleph_0 < \kappa < \aleph_\omega$, ψ has 2^κ models of power κ.

$\square_{11}$

Remark 12 1. Note that if X is independent from any of the $\sim$ equivalence classes of M, it is consistent to extend M by adding to Q names for either X or $P(M) - X$. Thus, amalgamation fails in $\aleph_0$. Considering infinitely many such inequivalent subsets, there are $2^{\aleph_0}$ elements in Q that realize distinct types over the prime model so ω-stability fails as well.

2. Since under Martin's axiom for any $\kappa < 2^{\aleph_0}$, $2^\kappa = 2^{\aleph_0}$, it is immediate that under Martin's axiom ψ has only $2^{\aleph_0}$ models in each cardinal below the continuum; but the proof of categoricity uses the more subtle forcing conditions.

The role of MA and VWGCH are very different. Via the weak diamond VWGCH catalyses the development of a structure theory. We are using MA to prove a particular example is categorical in many cardinals; this example is otherwise ill-behaved. Although categoricity is absolute for first order theories, these examples show it is not for $L_{\omega_1,\omega}$. A natural problem is to find a further requirement that strengthen categoricity to an absolute notion.

The works of Grossberg-VanDieren [6, 7] and Lessmann [16] establish categoricity transfer in ZFC under the additional requirements of tameness, amalgamation and arbitrarily large models. Thus, tameness is an obvious candidate for such a strengthening. Unfortunately, the example at hand is $(\aleph_0, \infty)$-tame. We sketch the argument for tameness. Note that it requires the notion of Galois types of triples: $\mathrm{gatp}(a/M, N)$ (the Galois type of b over M in N) because we do not have amalgamation. (See [1, 8, 20]. Consider elements $b, c \in Q$. It is easy to check that for any model M of ψ, $\mathrm{gatp}(b/M, N_1) = \mathrm{gatp}(c/M, N_2)$ if and only if b and c name the same subset of P.

Shelah's more dramatic aim is to prove that if an $L_{\omega_1,\omega}$-sentence ψ has only a set of (non-isomorphic) models then for some κ there are 2^κ models of power κ. The slightly weaker result that if ϕ has at most $2^{\aleph_n}$ models in $\aleph_{n+1}$ for each n then ϕ is excellent, is proved under VWCGH in [18, 19, 1]. It remains open both whether either of these statements is provable in ZFC and whether $2^{\aleph_n}$ can be improved to $2^{\aleph_{n+1}}$ under VWCGH.

Shelah [18, 19] and Grossberg and Hart [9] have shown the eventual behavior of excellent classes is determined by the models of small cardinality. Thus, the general 'main gap thesis' that classes of models (definable say in $L_{\omega_1,\omega}$) can be partitioned into those that admit a structure theory and *creative* classes (such as the dense linear orders) that introduce essentially new structures of arbitrarily cardinality holds for excellent classes. One would like, as in the first order case to make a direct translation between the behavior of an initial segment of the spectrum function (counting the number of models) and its eventual behavior. In one sense, this dichotomy continues to hold. The abberrant example has a bounded number of models and so might be considered to have a structure theory, by listing. This raises the question of whether it is consistent to have an $L_{\omega_1,\omega}$-sentence with arbitrarily large models that is categorical up to $\aleph_\omega$ but not eventually categorical.

I thank Alf Dolich, Alexei Kolesnikov, Chris Laskowski, Dave Marker, Ahuva Shkop, and Andres Villaveces for useful conversations in sorting this example out and some comments of Mike Benedikt in thinking through the context.

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