Near Model Completeness and 0-1 Laws

John T. Baldwin *
Department of Mathematics, Statistics and Computer Science
University of Illinois at Chicago

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1 Introduction

We work throughout in a finite relational language L. Our aim is to analyze in as purely a model-theoretic context as possible some recent results of Shelah et al in which $0-1$-laws for random structures of various types are proved by a specific kind of quantifier elimination: near model completeness. In Section 2 we describe the major results of these methods ([11], [10] etc.) and some of their context. In Section 3 we describe the framework in which these arguments can be carried out and prove one form of the general quantification elimination argument. We conclude the section by sketching a general outline of the proof of a $0-1$ law. The hypotheses of this theorem have a 'back and forth' character. Establishing the 'forth' part depends heavily on probability computations and is not expounded here. The 'back' part is purely model theory. Section 4 carries out the 'back' portion of the proof in one context with some simplification from Shelah's original version.

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2 Survey of $0-1$-laws

Consider a pair of relational languages: $L \subseteq L^+$.

2.1 General Problem. An L-structure M_n (which in this paper will have cardinality n) is expanded to an L^+-structure N with probability $P_n(N)$. For L^+ sentences ϕ, what are the possible behaviors of the limit

$$\lim_{n\to\infty} P_n(\phi)?$$

If L is equality, this is the standard case of finding the limit probabilities of sentences.

2.2 Fact. The arguments described here work for any finite relational language L^+ such that for each relation symbol $R(\overline{x})$

1. For any permutation σ: $R(\overline{a}) \leftrightarrow R(\sigma(\overline{a}))$.
2. $R(\overline{a})$ implies the elements of $\overline{a}$ are distinct.

However, for notational ease we restrict to expanding an L-structure M_n by adding edges to form a random graph. Thus, $L^+ - L$ contains a single binary relation symbol: E. Shelah [10] considers iterating the adding of random relations. In this way, he extends to asymetric relations. E.g., a random directed graph can be thought of as first choosing a pair that will be connected and then choosing a direction.

2.3 Base Language. Here are the main examples:

1. M_n is $\langle n, =\rangle$.
2. M_n is $\langle n, S\rangle$ where $S(i,j)$ if $i < n-1$ and $j = i+1$.
3. M_n is $\langle n, S^*\rangle$ where $S^*(i,j)$ if $i < n-1$ and $j = i+1$ or $i = n-1$ and $j = 0$.
4. M_n is $\langle n, <\rangle$.

The reason that both examples 2 and 3 are considered is explained in Paragraph 3.24(5).

2.4 Probability. We consider 5 main examples of probability functions. M_n is a structure with universe n and $n \geq 2$. In each case the probability of an expansion N is determined from the 'edge probability' in the natural way. Let α be an irrational number in $(0, 1)$. In the following we write $p_{i,j}$ for the probability that there is an edge between i and j. In some cases this edge probability depends only on the size of the graph; Shelah writes p_n in those cases.

1. $p_{i,j} = 1/2$.
2. $p_{i,j} = \frac{1}{2^{|i-j|}}$
3. $$p_{i,j} = \begin{cases} \frac{1}{2^\alpha} & \text{if } |i-j| = 1 \\ \frac{1}{|i-j|^\alpha} & \text{if } |i-j| \geq 1 \end{cases}$$
4. $p_{i,j} = \frac{1}{n^\alpha}$,
5. $p_{i,j} = \frac{1}{2^{|i-j|}} + \frac{1}{n^\alpha}$.

2.5 Convergence Results.

	$1/2$	$\frac{1}{2^{\vert i-j\vert}}$	$\frac{1}{\vert i-j\vert^\alpha}$	$\frac{1}{n^\alpha}$	$\frac{1}{2^{\vert i-j\vert}} + \frac{1}{n^\alpha}$
$=$	0-1 law	0-1 law	0-1 law	0-1 law	0-1 law
S	conv	conv	conv	0-1 law	conv
S^*	0-1 law	0-1 law	0-1 law	0-1 law	0-1 law
$<$	v.w. 0-1 law	convergent	v.w.	v.w.	v.w.

In the first row, the first column is due to Fagin [6] (and Glebski et al). The is from [8], the fourth from [12] (see also [3]), the third and fifth from [11]. In the second and third rows, the first column is due to Lynch [9]. The second column is from [8], the fourth from [10], the third and fifth from [11]. For the first column of the fourth row see [5]; the second is from [8], for the very weak 0-1 law, see [13].

We use v.w. to abbreviate the *very weak 0-1 law*:

$$\lim_{n \to \infty} |(P_{n+1}(\phi) - P_n(\phi)| = 0.$$

2.6 Connections between the results. The family of results we are concerned with center on the Spencer-Shelah $0-1$-law for random graphs with edge probability n^α (α irrational). Shelah has generalized these results in two directions (usually in papers which contain both directions but whose proofs emphasize one). The probability measure can be of type 2,3,5 from Paragraph 2.4 (main point of [11]) or the base language can be 2 or 3 from Paragraph 2.3 (main point of [10]). In [10] a general framework for the iteration of adding a random relation is developed. Our focus in this paper is on understanding the general argument and expounding a key point of the extension to handle probabilites of type 3.

2.7 Classifying the limit theories. We briefly sketch a classification of the complexity of the limit theories. The notation that follows is rather standard for model theorists. See for example Section III.1 of [2]. We make no further use of this material in this paper; perhaps the model theoretic consequences of this classification will have a future application.

Definition. A complete first order theory T has the tree property if there is a formula $\phi(x, y)$, an integer k, and tuples a_α for $\alpha \in \omega^{<\omega}$ such that for any $\alpha \in \omega^{<\omega}$, the set of formulas

$$\{\phi(x, a_{\alpha \frown n}) : n < \omega\}$$

is k-inconsistent but for any $\beta \in \omega^\omega$,

$$\{\phi(x, a_{\beta|n}) : n < \omega\}$$

is consistent.

Definition. A complete first order theory T is *complex* if it uniformly interprets every symmetric finite relation. A complete first order theory T is *simple* if it does not have the tree property.

The notion of a simple theory was introduced by Shelah in the early 70's; it has lately arisen in the study of finite fields. For our purposes, it is a measure of tractability. Every stable theory is simple but not vice versa. See [14] [7].

Definition. An incomplete theory is *simple* or *complex* if every completion of it is.

In the following table we describe the theory consisting of the almost sure sentences for the specified probability and set of base models.

	$1/2$	$\frac{1}{2^{\lvert i-j\rvert}}$	$\frac{1}{\lvert i-j\rvert^{\alpha}}$	$\frac{1}{n^{\alpha}}$	$\frac{1}{2^{\lvert i-j\rvert}}+\frac{1}{n^{\alpha}}$
$=$	simple	complex	complex	stable	complex
S	simple	complex	complex	stable	complex
$<$	complex	complex	complex	complex	complex

These results are observations about the theories.

3 Context and Strategy

3.1 Notation. Let K_∞ be a class of finite structures closed under substructure and isomorphism and containing the empty structure. Denote by K_n the collection of members of K_∞ that have cardinality n. Let $\overline{K}$ be the universal class determined by K_∞ and let $\mathbf{K}$ be an arbitrary subclass of $\overline{K}$.

Our main object of study is the pair $(K_\infty, \leq_i)$ where $A \leq_i B$ is a binary relation on pairs of structures $A \subseteq B$ from K_∞. Read $\leq_i$ as B is an *intrinsic extension* of A. Naturally, we insist that this relation be preserved by isomorphism. We will consider several ways to define the notion $\leq_i$, but dealing with $\leq_i$ axiomatically allows us to provide a unified proof of near model completeness for a number of different contexts.

3.2 Notation. Let $B \cap C = A$. The *free amalgam* of B and C over A, denoted $B \bigotimes_A C$, is the structure with universe BC but no relations not in B or C. If A, B, C are contained in N and N imposes the structure of $B \bigotimes_A C$ on BC (short for $B \cup C$), we say B and C are freely amalgamated over A in N.

We write $A \subseteq_\omega B$ to mean A is a finite subset of B. For any notion $A \leq_x B$ we will write $A <_x B$ for the same concept but with the additional assertion that $A \neq B$. A structure A is called discrete if there are no relations among the elements of A.

3.3 Definition. Let A be a finite substructure of $M \in \overline{K}$ with $A \subseteq B \in \overline{K}$.

1. $\chi_M(B/A)$ is the number of distinct copies of B over A in M.
2. $\chi^*_M(B/A)$ is the supremum of the cardinalities of maximal families of disjoint (over A) copies of B over A in M.

3.4 Examples. 1. For a fixed function f, from $N \times N$ to N, $A \leq_i B$ if for every $C \in K_\infty$, and every embedding g of A into C, $\chi_C(gA, B) < f(|A|, |B|)$.

2. $A \leq_i B$ defined in terms of a dimension function as in [4].

3. $A \leq_i B$ if for every $\epsilon > 0$, and for every sufficiently large $C \in K_\infty$, and every embedding g of A into C, $\chi_C(gA, B) < |C|^\epsilon$.

4. $A \leq_i B$ defined in terms of a wt_λ function as follows.

It is shown in [3] that the first two examples are actually the same and in [11] that the last two are. We stray from our general discussion of intrinsic extension to develop a few facts about weight to clarify the fourth example. This example will be explored in more detail in the last section.

3.5 Definition. Let $A \subseteq B \subseteq N \in \overline{K}$. Let λ be an equivalence relation on $B - A$ and let $\mathcal{E}(A, B)$ be the set of all such equivalence relations. B' with $A \subseteq B' \subseteq B$ is λ-closed if every λ-equivalence class which intersects B' is contained in B'. If $\lambda \in \mathcal{E}(A, B)$ and $A \subseteq B' \subseteq B$, we continue to denote by λ the equivalence relations $\lambda|B' \in \mathcal{E}(A, B')$ and, when B' is λ-closed, $\lambda|(B - B') \in \mathcal{E}(B', B)$.

1. $v_\lambda(B/A)$ is the number of equivalence classes of λ.

2. $e_\lambda(B/A) = |\{(c, d) \in B \times B - A \times A : R(c, d) \& \neg\lambda(c, d)\}|$.

3. $wt_\lambda(B/A) = v_\lambda(B/A) - \alpha e_\lambda(B/A)$.

3.6 Definition. For a class of models equipped with a weight function as in Definition 3.5, we define the notions of *intrinsic extension* $(\leq_i)$ and *strong extension* $(\leq_s)$ as follows.

1. $A \leq_i B$ if for every B' with $A \subseteq B' \subseteq B$, and every $\lambda \in \mathcal{E}(B', B)$, $wt_\lambda(B/B') < 0$.

2. $A \leq_s B$ if for every B' with $A \subseteq B' \subseteq B$, and some $\lambda \in \mathcal{E}(A, B')$, $wt_\lambda(B'/A) \geq 0$.

3. $A \leq_s^k B$ if for every B', $A \subseteq B' \subseteq B$ and $|B'| < k$, $A \leq_s B'$.

From the notion of intrinsic substructure, we define a notion of intrinsic closure.

3.7 Definition. Let $A \subseteq M \in K$. The *intrinsic closure* of A in M is $icl_M(A) = \cup_{k<\omega} icl^k_M(A)$ where for any $M \in K$, any $k \in \omega$, and any $A \subseteq M$,

$$icl^k_M(A) = \cup\{B : A \cap B \leq_i B \subseteq M \& |B| < k\}.$$

That is, the k-intrinsic closure of A in M is the union of those substructures of cardinality less than k of M which are intrinsic extensions of their intersection with A. Since k-intrinsic closure is not transitive, we need a notion for iterating icl^k_M.

- $icl^{k,0}_M(A) = icl^k_M(A)$
- $icl^{k,m+1}_M(A) = icl^k_M(icl^{k,m}_M(A))$

3.8 Definability. Note that icl^k is first order definable in the following sense. For each finite n, k, there is a formula $\theta_{n,k}(\overline{x}, y)$ such that for any $M \in \overline{K}$, and any sequence $\overline{a}$ of length n from M, $M \models \theta_{n,k}(\overline{a}, b)$ if and only if $b \in icl^k_M(\overline{a})$. Thus, we can define in each M, (suppressing n and k) a set $A_m = A^m_M(\overline{a}) = icl^{k,m}_M(\overline{a})$.

3.9 Remark. To clarify later computations, we gave the definition of $\leq_s$ explicitly in terms of wt_λ. It is equivalent to give the general definition: for $A \subseteq B \in K_\infty$, $A \leq_s B$ if and only if there is no B' with $A \leq_i B' \subseteq B$. This definition is extended to possibly infinite A and B by $A \leq_s B$ if for every finite $X \subseteq A$, $icl_A(X) = icl_B(X)$.

The setting here differs from the similar one in [3] in one immaterial and several material ways. The immaterial difference is that we have chosen to axiomatize $\leq_i$ rather than $\leq_s$. A more significant difference is that there is no requirement that the dimension function is hereditarily nonnegative on members of K_∞. That is, $\emptyset \leq_s A$ for all $A \in K_\infty$ is not required (and is false in Example 3/4). If $\emptyset \leq_s A$ for all $A \in K_\infty$ then amalgamation for strong strong substructures entails the joint embedding property (for strong embeddings) in K_∞. Still more significantly, the bound on the number of allowable copies of an intrinsic extension of a structure is raised from a constant in Example 1/2 to a slow-growing function in Example 3/4.

3.10 Basic Axioms. **A1** If $A \leq_i B$ and $B \leq_i C$ then $A \leq_i C$.

A2 If $A \subseteq B$ and $A \leq_i C$ then $B \leq_i BC$.

A3 If $A \leq_i B$ and $f : B \mapsto B'$ is a $1-1$-homomorphism, then $fA \leq_i fB$.

It is easy to check that both Axioms A1 and A2 are verified in the examples in Paragraph 3.4. Axiom A3 is equally easy if the dimension/weight viewpoint (i.e. looking at Example 1 or 3) is taken.

The definition of intrinsic closure yields the following immediately.

3.11 Lemma.

1. *For every $A \subset_\omega M \in K$, every $k < \ell$, $icl_M^k(A) \subseteq icl_M^\ell(A)$*
2. *For every natural number k, if $A \subseteq B \subseteq C$ and $icl_C^k(A) \subseteq B$ then $icl_C^k(A) = icl_B^k(A)$.*

Axioms A1 and A2 respectively immediately yield the following properties of intrinsic closure. (See [3].)

3.12 Lemma. *For every k, m, ℓ, there exists t such that for every $M \in K$ and every $\overline{a} \in M$ of length ℓ,*

$$icl_M^{k,m}(\overline{a}) \subseteq icl_M^t(\overline{a}).$$

3.13 Lemma. *For any k and any $M \in K$, if $A \subseteq B \subseteq M$, then $icl_M^k(A) \subseteq icl_M^k(B)$.*

We make the following additional demand on $(K, \leq_s)$.

3.14 Axiom A4. For every $s, k \in \omega$, there are k^* and m such that for every $M \in K$, and every $\overline{a} \in M$ of length s and $b \in M$ the following conditions hold. Let $H = icl_M^k(\overline{a}b)$ and for each i, A_i denotes $icl_M^{k^*,i}(\overline{a})$:

$$A_m \cap H \leq_s H.$$

This 'back' condition is analogous to what Shelah [11] calls 'the universal demand' in defining such concepts as 'simply almost nice'. Our demand is stronger than Shelah's in that we have specified exactly how to construct A_m

rather than relying on a B with desirable properties and we require this A_m to be a strong submodel rather than something 'elementarily' equivalent to it. Thus, with this definition it is easier to prove the model completeness result but more difficult to verify the hypothesis (this condition). Nevertheless, we establish the hypothesis in Section 4, when the base language contains only equality. When successor or $<$ is allowed in the base language the situation becomes more complicated.

3.15 Definition. The model M is $(K_\infty, \leq_s)$-semigeneric, or just semigeneric, if

1. $M \in K$
2. If $A \leq_s B \in K_\infty$ and $g : A \mapsto M$, then for each finite m there exists an embedding $\hat{g}$ of B into M which extends g such that
   (a) $icl^m_M(\hat{g}B) = \hat{g}B \cup icl^m_M(gA)$
   (b) $M|icl^m_M(gA)\hat{g}B$ is the free join over gA of $icl^m_M(gA)$ and $\hat{g}B$.

The following 'forth' condition corresponds to the existential demand in Shelah's definition. There exist semigenerics; indeed with probability one each structure is semigeneric. That is, each of the $\phi^m_{A,B,C}$ defined in [3], which together axiomatize the semigeneric structures (see next lemma), has limit probability 1.

3.16 Lemma. *There is a collection of sentences $\phi^m_{A,B,C}$ indexed by appropriate triples of finite structures such that a structure $N \in K$ is semigeneric, if and only if for each appropriate $\langle A, B, C\rangle$, $N \models \phi^m_{A,B,C}$*

3.17 Definition. A theory T is said to be *nearly model complete* if every formula is equivalent in T to a Boolean combination of Σ_1-formulas.

Thus, T is nearly model complete if the type of any finite sequence is determined by exactly the family of Σ_1-formulas it satisfies. Near model completeness lies strictly in strength between model completeness and 1-model completeness (every formula is equivalent to a Σ_2-formula). Note that, in contrast to the random graph with edge probability $1/2$, the axioms for a nearly model complete theory will be Π_3 not Π_2.

Now, we want to prove that under these conditions, the class of semigeneric structures is nearly model complete. The proof is practically identical to that in [3] and is included only for convenience. However, the hypotheses have been weakened to give a more general result applying to Example 3/4 as well as 1/2. In particular, the hypotheses are formulated entirely in terms of $\leq_i$ and $\leq_s$. Thus a weight function only enters the quantifier elimination argument to establish A1-A4.

3.18 Theorem. *If $(K, \leq_i)$ satisfies Axioms A1 through A4, then for every formula $\phi(\overline{x})$ there is a Boolean combination of existential formulas $\psi_\phi(\overline{x})$ such that if M is $(K, \leq_i)$-semigeneric then $\psi_\phi(\overline{x})$ is equivalent to $\phi(\overline{x})$ on M.*

Proof. We actually show:

3.19 Lemma. *For any formula $\phi(x_1 \ldots x_r)$ there is an integer $\ell = \ell_\phi$, such that for any pair of semigenerics $M, M' \in K$ and any r-tuples $\overline{a} \in M$ and $\overline{a}' \in M'$ if $icl_M^{\ell_\phi}(\overline{a}) \approx icl_{M'}^{\ell_\phi}(\overline{a}')$ then $M \models \phi(\overline{a})$ if and only if $M' \models \phi(\overline{a}')$.*

To deduce the elimination of quantifiers result from this formulation note that it implies that the type of any sequence (in any semigeneric structure $M \in K$) is determined by the Σ_1 and Π_1 formulas it satisfies.

Proof. The proof is by induction on formula complexity; the Boolean connectives are easy. So suppose $\phi(\overline{x}) = (\exists y)\psi(\overline{x}, y)$. Suppose $M \models \phi(\overline{a})$ so there is a b such that $M \models \psi(\overline{a}, b)$. Let $H = icl_M^k(\overline{a}b)$ and for each i, $A_i = icl_M^{k,i}(\overline{a})$:

Apply Axiom A4, to $\overline{a}, b$ with $k = \ell_\psi$ to obtain k^* and m such that $A_m \cap H \leq_s A_m H$. Now applying Lemma 3.12, choose ℓ_ϕ so that for every $\overline{a}$ of length r, and every semigeneric N, $A_{m+1}^N(\overline{a}) \subseteq icl_N^{\ell_\phi}(\overline{a})$.

We want to show that for any semigenerics M and M', for any $\overline{a} \in M^r$, $\overline{a}' \in M'^r$, and $b \in M$ if $icl_M^{\ell_\phi}(\overline{a}) \approx icl_{M'}^{\ell_\phi}(\overline{a}')$ then there is a $b' \in M'$ with $icl_M^{\ell_\psi}(\overline{a}, b) \approx icl_{M'}^{\ell_\psi}(\overline{a}', b')$.

Fix g which maps $\overline{a}$ to $\overline{a}'$ and $icl_M^{\ell_\phi}(\overline{a})$ isomorphically onto $icl_M^{\ell_\phi}(\overline{a}')$. By the choice of ℓ_ϕ, for each $i \leq m+1$, g maps $A_i^M(\overline{a})$ isomorphically onto $A_i^{M'}(\overline{a}')$. (Use Lemma 3.11 and induct.) To avoid superscripts, for each i, let A_i' denote the image of $A_i = A_i^M$ under g.

Since M' is semigeneric, $M' \models \phi_{A_m, H, A_{m+1}}$. Thus, there is an isomorphism $\hat{g}$ extending g and mapping H into M' with $icl_{M'}^{\ell_\phi}(A_m' \hat{g} H) = icl_{M'}^{\ell_\phi}(A_m') \cup \hat{g} H$

and so that $M'|(icl^{\ell_\phi}_{M'}(A'_m)\hat{g}H)$ is a free join of $icl^{\ell_\phi}_{M'}(A'_m)$ and $\hat{g}H$ over A'_m. Let $H'_1 = \hat{g}H$ and $b' = \hat{g}(b)$. We need to show $icl^{\ell_\psi}_M(\overline{a}, b) \approx icl^{\ell_\psi}_{M'}(\overline{a}', b')$.

By the choice of $\hat{g}$ and H'_1, $A'_m H'_1 \cong A_m H$ which contains $icl^{\ell_\psi}_M(\overline{a}, b)$, so it suffices (by Lemma 3.11) to show $A'_m H'_1$ contains $icl^{\ell_\psi}_{M'}(\overline{a}', b')$. Note by Lemma 3.13, $icl^{\ell_\psi}_{M'}(\overline{a}', b') \subseteq icl^{\ell_\phi}_{M'}(\overline{a}', b') \subseteq icl^{\ell_\phi}_{M'}(A'_m\hat{g}H) = A'_{m+1}H'_1$. By Lemma 3.11, $icl^{\ell_\psi}_{M'}(\overline{a}', b') = icl^{\ell_\psi}_{A'_{m+1}H'_1}(\overline{a}', b')$. Since A'_{m+1} and H'_1 are freely joined over A'_m, $\hat{g}^{-1} \cup g^{-1}$ is a $1-1$ homomorphism from $A'_{m+1}H'_1$ onto $A_{m+1}H$. Applying Axiom A3 and since $H = icl^{\ell_\psi}_M(\overline{a}, b)$ we see $icl^{\ell_\psi}_{A'_{m+1}H'_1}(\overline{a}', b') \subseteq A'_m H'_1$ whence $icl^{\ell_\psi}_{A'_{m+1}H'_1}(\overline{a}', b') = icl^{\ell_\psi}_{A'_m H'_1}(\overline{a}', b')$.

The next corollary follows exactly as in [3].

3.20 Corollary. *Suppose there is a $(K, \leq_i)$-semigeneric L-structure. The theory of the class of $(K, \leq_i)$-semigeneric L-structures is nearly model complete.*

The next result follows from the definability of the intrinsic closure (Paragraph 3.8).

3.21 Lemma. *There is a collection of first order sentences Φ such that if $M \models \Phi$, for each m, $icl^m_M(\emptyset) = \emptyset$.*

An immediate application of Lemma 3.19 yields:

3.22 Corollary. *Any consistent theory T which contains both Φ, the set of sentences expressing that $icl_M(\emptyset)$ is empty, and Σ, the sentences axiomatizing the semigeneric models, is complete.*

3.23 A strategy for proving $0-1$-laws.

1. Define a notion of $\leq_i$ satisfying the axioms in this section.
2. Show by a model theoretic argument that this notion of $\leq_i$ satisfies the 'back' condition.
3. Establish the 'forth' condition by proving that the sentences defining semigeneric structures have probability one.
4. Apply Corollary 3.20 to conclude that the class of semigenerics is nearly model complete.

5. If for each semigeneric M, $icl_M(\emptyset) = \emptyset$, completeness follows by Corollary 3.22.

3.24 Remarks. 1. We establish the 'back' condition when icl is defined in terms of λ-weight in Section 4.

2. Two major extensions of [11] are to allow extension of successor and to allow edge probability $\frac{1}{2^{|i-j|}}$. The definition of semigeneric given here is appropriate for the more general probability situation but only working over equality. The language extension problem is treated in more detail in [10] and [1].

3. We have not dealt with the exact relationships among the probability, λ-weight, and $\leq_i$. See [11].

4. The proof of steps 2 and 3 requires further direct use of the weight function.

5. The problem of nonempty closure is illustrated by expanding (n, S). The first element (and much more) is in the closure of the empty set. Transferring to the circle (S^*), is one way to dodge this bullet.

6. More generally, when $icl_M(\emptyset) \neq \emptyset$, we have reduced the theory of a semigeneric M to the sequence of structures $\langle icl^m_M(\emptyset) : m < \omega\rangle$. So completeness follows if this sequence does not depend on the choice of M. This situation arises when considering expansions of successor and edge probability $n^{-\alpha}$.

7. Strictly speaking, it is not near model completeness but the more technical Lemma 3.19 which is applied to obtain completeness.

8. Suppose $icl^m_M(\emptyset)$ depends on M. For any sentence ϕ, Lemma 3.19 reduces the truth of ϕ in M to the isomorphism type of $icl^m_M(\emptyset)$, for appropriate m. If (e.g. expanding successor with edge probability $\frac{1}{2^{|i-j|}}$) a probability can be assigned to the isomorphism type of $icl^m_M(\emptyset)$, convergence is obtained even though the $0-1$-law is not.

4 The 'back' argument

We want to establish the following principle, Axiom A4, when *icl* is defined in terms of λ-weight. This is a key model theoretic step in generalizing the $0-1$-law from a random graph with edge probability $n^{-\alpha}$ to one with edge probability $\frac{1}{|i-j|^{\alpha}}$. These arguments reformulate the results in Section 6 of [**?**]. We do not deal here with the difficulties of showing the sentences expressing semigenericity have probability one.

4.1 Axiom A4. For every $s, k \in \omega$, there are k^* and m such that for every $M \in K$, and every $\overline{a} \in M$ of length s and $b \in M$ the following condition holds. Let $H = icl^k_M(\overline{a}b)$ and for each i, if A_i denotes $icl^{k^*,i}_M(\overline{a})$:

$$A_m \cap H \leq_s H.$$

Shelah [**?**] has introduced the following terminology.

4.2 Definition. $(K, \leq_i)$ *smooth* if whenever B and C are freely amalgamated over A inside N, $B \leq_i BC$ if and only if $A \leq_i C$.

While this condition is related to A3 and plays a similar role in Shelah's proof of the quantifier elimination result to that played by A3 here (in the sense that the other hypotheses are the same), the conditions are quite different.

Note that by smoothness, if A_m and H are free over $A_m \cap H$, the conclusion of **A4** is equivalent to

$$A_m \leq_s A_m H.$$

4.3 Fact.
1. There exists an $\epsilon_n > 0$ such that if $|B - A| < n$ and $wt_\lambda(B/A) < 0$ then $wt_\lambda(B/A) < -\epsilon_n$.
2. If $\leq_s$ is defined from λ-weight then $(K, \leq_s)$ is smooth (Definition 4.2).

We will be dealing with sequences $\langle C_i : i < \alpha\rangle$ of structures containing a fixed set B and of bounded size. For every t, a long enough such sequence ($> g(t)$) contains an extremely homogeneous subsequence of size t:

4.4 Lemma. *There is a function g from natural numbers to natural numbers with the following property for each t. Suppose for $i < g(t)$, $A \subseteq B \subseteq C_i \subseteq N \in K$ where $|C_i| \leq r$. Let $C^* = \bigcup\{C_i : i < g(t)\}$ and let λ be an equivalence relation on $C^* - A$. Denote $\lambda|C_i$ by λ_i. There exists $u \subseteq g(t)$ and X with $B \subseteq X \subseteq \cap_{i\in u} C_i$ such that*

1. $|u| \geq t$.

2. $|B| \leq |X| < r$.

3. If $i, j \in u$ *then*

   (a) $C_i \cap C_j = X$

   (b) $\lambda_i|X = \lambda_j|X$ *and there is an* L*-isomorphism* $\psi_{i,j}$ *between* C_i *and* C_j *over* X *that also maps* λ_i *to* λ_j.

   (c) Each λ*-equivalence class intersects either a unique* $C_i - X$ *or intersects all* $C_i - X$.

Proof. The Δ-system lemma establishes 2 and 3a) for some $u' \subseteq g(t)$. Then selecting a fixed quantifier-free type in the language $L \cup \{\lambda\}$, we determine the $\psi_{i,j}$ for 3b). Applying Ramsey's theorem yields 3c). The partition is defined by $(i, j) \in P_s$ (for $s \in 2^k$) if and only if for each $a_r \in C_i, r < k$, $\lambda(a_r, \psi_{i,j}(a_r))^{s(r)}$). (For any ϕ, $\phi^{s(r)}$ denotes ϕ if $s(r) = 1$ and $\neg\phi$ if $s(r) = 0$.) The function $g(t)$ can be computed from the bounds for the Δ-system lemma and Ramsey's theorem, and the number of quantifer free k-types in $L \cup \{\lambda\}$.

4.5 Remark. The function g depends uniformly on $|A|$, $|B|$, and r. These will be parameters of the main result (where r is derived from the k mentioned in the main theorem).

Given such a homogeneous sequence, we establish some further nomenclature.

4.6 Notation. 1. We say $c \in C_i$ is *large* if $c \in C_i - X$. (Actually, it is the orbit of c over X that is large.)

2. We say c/λ is *isolated* if $c/\lambda \subseteq C_i - X$.

3. Finally, c/λ is *dense* if $c/\lambda \cap C_i \neq \emptyset$ for all i.

Note that if $b \in B$, then b/λ is dense. The following lemma will be exploited in the proof of Theorem 4.9 and Lemma 4.15.

4.7 Lemma. *Suppose* $\langle C_i : i < t \rangle$ *is a homogeneous sequence as in the conclusion of Lemma 4.4 and let* C^* *denote the union of the* C_i. *Thus,* $|C_i| < r$*; choose* ϵ_r *as in Fact 4.3. Suppose* $\lambda \in \mathcal{E}(A, C^*)$ *and* λ_i *denotes* $\lambda|C_i$.

Suppose further that there is c in C_i with c large and c/λ is not dense. If $t > r/\epsilon_r$ and for all i, $B \leq_i C_i$ then $wt_\lambda(C^*/A) < 0$. Thus if $wt_\lambda(C^*/A) > 0$, then for any $c \in C^*$, if c is large, c/λ is dense.

Proof. Let C' denote the union of the dense λ-equivalence classes. Note that $X \subseteq C'$ and C' is λ-closed. Further, the number of dense classes is bounded by the cardinality of C_i so $wt_\lambda(C'/A) \leq r$. Thus, for each i:
$wt_\lambda(C_i/(C' \cap C_i)) < -\epsilon_r$.

$$\begin{array}{rcc} wt_\lambda(C^*/A) & \leq & wt_\lambda(C'/A) + wt_\lambda((C^* - C')/C') \\ & \leq & wt_\lambda(C'/A) + \sum_{i<t} wt_\lambda(C_i/(C' \cap C_i) \\ & \leq & r + (t \times -\epsilon_r) < 0. \end{array}$$

4.8 Definition. We say A is m-strong in B and write $A \leq_s^m B$ if for every B' with $A \subseteq B' \subseteq B$ and $|B| < m$, $A \leq_s B'$.

Now we can establish the following local principle. This principle is immediate with $k^* = \ell$ if there is an ℓ with $|icl_M^k(B)| < \ell$ whenever $|B| < n$ (as when the edge probability is $n^{-\alpha}$). But when the size of the intrinsic closure is unbounded (as when the edge probability is $\frac{1}{|i-j|^\alpha}$) a serious argument is needed.

4.9 Theorem. *For every m, n, k there exists a k^* such that for every A, B with $|A| < m, |B| < n$ and every $M \in K$, if $A \leq_s^{k^*} icl_M^k(B)$ then $A \leq_s icl_M^k(B)$.*

Proof. Let H denote $icl_M^k(B)$. Let $r = 2k$ and fix t with $t > r/\epsilon_r$. Let $k_0 = r$ and $k_{i+1} = r \times 2^{{k_i}^2}$. Let $p = g(2t)$ and let $k^* = k_p$. For $i < p$ and each appropriate $\mu \in \mathcal{E}(A, E_i)$ we will define a structure $D_{i,\mu}$ with $|D_{i\mu}| < 2k$. Then E_{i+1} denotes $\bigcup\{D_{i,\mu} : \mu \in \mathcal{E}(A, E_i)\}$. Note that for each i, $|E_i| < k^*$ so $A \leq_s E_i$.

4.10 Definition. Suppose $B \subseteq E \subseteq H$ and $A \leq_s E$.

1. For $c \in H - E$ and $e \in E$, $R^*(c, e)$ holds if there are a C_c with $B \leq_i C_c$, $|C_c| < k$ and a path from c to e with all intermediate points contained in $C_c - E$.

2. For $\mu \in \mathcal{E}(A, E)$, we say (E, μ) is *secure* in H if μ witnesses $A \leq_s E$ and

(a) for every $c \in H - E$ there is at most one μ-class of an element $e \in E$ with $R^*(c, e)$.

(b) There do not exist $c, d \in H$ and $a, b \in E$ with $R(c, d)$, $R^*(c, a)$, $R^*(d, b)$ and $\neg\mu(a, b)$.

3. We say E is *secure* if for some $\mu \in \mathcal{E}(A, E)$, (E, μ) is secure in H.

We will eventually deduce that for some $i < p$, E_i is secure in H.

4.11 Claim. If $A \leq_s E \subseteq H$ and E is secure in H then $A \leq_s H$.
Proof. Fix $\lambda \in \mathcal{E}(A, E)$ such that for every B' with $A \subseteq B' \subseteq E$, $wt_\lambda(B'/A) \geq 0$ and (E, λ) is secure in H. Now extend λ to $\lambda^* \in \mathcal{E}(A, H)$ as follows. For $c \in H - E$ and $e \in E$, $\lambda'(c, e)$ holds just if $R^*(c, e)$ and for $c, d \in H - E$, $\lambda'(c, d)$ holds if neither $R^*(c, e)$ nor $R^*(d, e)$ holds for any $e \in E$. λ^* is the transitive closure of $\lambda \cup \lambda'$. Since E is secure in H, λ^* is well-defined. Now by the first clause in the definition of secure, $v_{\lambda^*}(H/A) = v_\lambda(B/A) + 1$ and by the second clause $e_{\lambda^*}(H/A) = e_\lambda(B/A)$. This completes the proof of the claim.

Now we continue the main construction. If at step i, E_i is secure; stop. If not, for each $\mu \in \mathcal{E}(A, E_i)$ choose $D_{i,\mu}$ as follows. One of the two clauses in the definition of secure is violated. In the following we deal with the second clause; the first is a simpler version obtained by identifying c_i and d_i. First, find a_i, b_i, c_i, d_i with $a_i, b_i \in E_i$ and $c_i, d_i \in H - E_i$ so that $R^*(c_i, a_i)$, $R^*(d_i, b_i)$, $R(c_i, d_i)$ and $\neg\mu(a_i, b_i)$. Since $H = icl^k_N(B)$, for $j < 2$, choose $C^j_{i,\mu}$ with $B \leq_i C^j_{i,\mu}$, $|C^j_{i,\mu}| < k$, with $C^0_{i,\mu}$ and $C^1_{i,\mu}$ witnessing $R^*(c_i, a_i)$ and $R^*(d_i, b_i)$ respectively. Let $D_{i,\mu}$ be the union of the $C^0_{i,\mu}$ and $C^1_{i,\mu}$ and $E_{i+1} = E_i \cup \cup_\mu D_{i,\mu}$.

Suppose for contradiction, no E_i for $i < p$ is secure in H so p structures E_i are chosen. Let λ witness $A \leq_s E_p$ and apply Lemma 4.4 (for $r = 2k$) to the $\langle D_{i,\lambda|E_i}\rangle$ with $i < p$ to get a homogeneous sequence which we renumber as $\langle D_i : i < t\rangle$. Since $p = g(2t)$, we may assume all D_i violate the same clause of the definition of secure. We analyze the slightly more complicated second clause. Without loss of generality we may assume that the isomorphisms among the D_i also respect a_i, b_i, c_i, d_i. Fix a particular D_i for analysis. Since $c_i, d_i \in H - E_i$, they are both large. As remarked in Lemma 4.7, since $A \leq_s E_p$, this implies that every element c of C_i, c/λ is dense. getting (possibly by renaming) that If c_i/λ is not dense, Lemma 4.7 contrary to hypothesis, as Since there is a path from a_i to b_i in D_i, we can find a pair of elements $x, y \in D_i$ such that $R(x,y)$, $\neg\lambda(x,y)$, and both are dense. Now $wt_\lambda(\{[x/\lambda],[y/\lambda]\}/A) = wt_\lambda([x/\lambda]/[y/\lambda]A) + wt_\lambda([x/\lambda]/A) \leq 2 - t < 0$. This contradiction completes the proof.

4.12 Theorem. *Axiom A4 holds.*

Proof. Recall that H denotes $icl^k_M(\overline{a}b)$ and $A_i = icl^{k^*,i}_M(\overline{a})$. It suffices to find k^* and m so that $A_{m+1} \cap H \subseteq A_m$, as this inclusion implies $A_m \cap H \leq^{k^*}_s H$; the result then follows by Theorem 4.9.

We complete the proof by first establishing a dichotomy and then showing that the undesirable alternative is impossible.

4.13 Notation. For any $A, B \subseteq N \in K$, $\mathcal{R}(A,B)$ denotes the set of edges between A and B.

4.14 Lemma. *For every $s, k \in \omega$, for every $M \in K$, and every $\overline{a} \in M$ of length s and $b \in M$, for any $k^* \geq k$, for every m^*, there is an $m < m^*$ such that either*

1. *$A_{m+1} \cap H \subseteq A_m$ or*
2. *$\mathcal{R}(H - A_m, A_m) > m^*/2^{2k} - k$.*

Proof. If the first alternative fails, for each $j < m^*$, there exist C_j, d_j with $d_j \in C_j$, $|C_j| < k$, $\overline{a}b \cap C_j \leq_i C_j$, and $d_j \in (A_{j+1} - A_j) \cap H$. By the pigeon-hole principle and the finite Δ-system lemma we can choose $X \supseteq \overline{a}b$ and $u \subseteq m^*$ with $|u| \geq m^*/2^{2k}$ such that for $i, j \in u$, $C_i \cap C_j = X$ and $C_i \approx_X C_j$. Since $X \subseteq \bigcap_{i \in u} C_i$, possibly decreasing u by less than k elements, we can assume that if $i \in u$, $(A_{i+1} - A_i) \cap X = \emptyset$. Let α be the minimal element of u and β the maximal. Fix $\ell \in u$, (at least third in increasing order on the elements of u.)

Let $B_1 = C_\ell \cap A_\alpha$ and $B_2 = C_\ell \cap A_\beta$. By our last restriction on u, $B_2 \cap X \subseteq B_1$.

Claim. For each $i \in u$, $R(C_i - B_2, B_2 - B_1) = \emptyset$ or alternative 2 holds with $m = \beta$.

Proof of claim. Fix i and suppose (c,d) is an edge of M with $c \in C_i - (B_2 - B_1)$ and $d \in B_2 - B_1$. Then d is not in X so the images of d in the various C_i give $|u|$ distinct edges between $icl^k_M(\overline{a},b) - icl^{k^*,\beta}_M(\overline{a})$ and $icl^{k^*,\beta}_M(\overline{a})$.

So if both alternatives of Lemma 4.14 fail, we have $C_\ell - (B_2 - B_1)$ and B_2 are freely joined over B_1. But $C_\ell - (B_2 - B_1) \leq_i C_\ell$. So by smoothness, $B_1 \leq_i B_2$. But $|B_2| \leq |C_i| < k$ so $B_2 \subseteq icl_M^{k^*,\alpha+1}(\overline{a})$. This contradicts the diagonalizing definition of B_2 and yields Lemma 4.14.

Now we want to show that the second alternative of Lemma 4.14 is impossible.

4.15 Lemma. *For every $s, k \in \omega$, for every $M \in K$, and every $\overline{a} \in M$ of length s, there exists t such that for every m if $b \in M - A_m$, then for sufficiently large $k^* \geq k$, letting H denote $icl_M^k(\overline{a}b)$ and $A_i = icl_M^{k^*,i}(\overline{a})$:*

$$|\mathcal{R}(H - A_m, A_m)| < t.$$

Proof. Apply Fact 4.3 to choose $\epsilon \in \Re$ $0 < \epsilon < 1$ such that for all λ, if $|C_1| < k$, $wt_\lambda(C_0, C_1) \notin (-\epsilon, \epsilon)$. Choose $|u|$ so that $|u| > k/\epsilon$ and $\alpha \times |u| > 1$. Then choose $t \geq k^2 g(|u|)$. If the Lemma fails, let $\mathcal{R}(H - A_m, A_m) = \{(d_i, c_i) : i < t\}$, where the edges are distinct. For each $i < t$, choose C_i such that $\overline{a}b \cap C_i \leq_i C_i$, $|C_i| < k$, and $d_i \in C_i$. We will first show there exists D with $A_m \subseteq D \subseteq H$ with $|D| > t/k^2$ and $\ell < k$ such that $A_{m+\ell} \cap D = D' \leq_s D$.

Define

$$f(i) = \mu q(A_{m+q+1} \cap C_i \subseteq A_{m+q}).$$

Then f is a function from t to k. There exists a set $u \subseteq t$ with $|u| \geq t/k^2$ such that

1. $i, j \in u$ implies $|C_i| = |C_j| =_{df} p$.

2. i in u implies $f(i) =_{df} \ell$.

Since $k^* \geq k \cdot t$ and $t \geq k^2 g(|u|)$, $k^* \geq |u| \cdot k$. Let $D = \bigcup_{i \in u} C_i \cup A_m$. Let $D' = icl_M^{k^*,\ell}(\overline{a}) \cap D$. By the choice of ℓ, $D' \leq_s D$. Let $\lambda \in \mathcal{E}(D', D)$. Apply Lemma 4.4 to obtain a homogeneous sequence $\langle < C_i, \lambda | C_i >: i \in u \rangle$. By Lemma 4.7, $Y = d_i/\lambda$ is dense. so

$$wt_\lambda(D', D' \cup Y) = v_\lambda(D', Y) - \alpha \times |\mathcal{R}(D', D \cup Y)| \leq 1 - \alpha \times |u| < 0.$$

Since, $D' \leq_s D$, this is impossible. This contradiction completes the proof of Lemma 4.15 and thus establishes Axiom A4.

The following corollary does not seem to be necessary for the argument presented in Theorem 3.19. It appears in [11] as part of Conclusion 6.11 and may be necessary for the expansion of nontrivial languages.

4.16 Corollary. *In the situation of Axiom A4 there is a set $B \subseteq A_m$, with $|B| < t$, with t computed as in Lemma 4.15 such that A_m and HB are freely amalgamated over B.*

Proof. Let B be the elements of A_m that are connected to elements of $H - A_m$.

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