CAYLEY'S THEOREM FOR ORDERED GROUPS: O-MINIMALITY
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It has long been known [4] that any group could be represented in a strongly minimal theory by just writing down the relations of the group as unary functions. We show the same process works for ordered groups and yields an o-minimal group. Baizanhov showed:

Fact 1. *If M is an o-minimal structure and $p \in S(A)$ is non-algebraic then there is an ordered group of unary definable functions defined on the realizations of p.*

He asked,

Question 2. *What ordered groups can be realized in this manner?*

We show

Theorem 3. *If G is a linearly ordered group, then G can be represented as the collection of definable unary functions acting on a complete type in an 0-minimal structure.*

Proof of Fact 1. This follows from a short analysis of $p-$stable formulas [1]. A 2-ary formula, $\phi(x, y)$, over A is $p-$stable if for some (equivalently for every) realization $\alpha \in p(M)$ there exist $\gamma_1, \gamma_2 \in p(M)$ such that $\gamma_1 < \phi(M, \alpha) < \gamma_2$. (In particular, $x = y$ is such a 2-formula.)

By o-minimality, $\phi(M, \alpha)$, can be written as a finite number of $L(A\alpha)$-definable intervals and points. Thus each p-stable formula determines a finite number of functions: f_i maps α to the ith point in a linear ordering of the end points of these intervals. Let f be such function. The image $f(\alpha)$ and pre-image $f^{-1}(\alpha)$ of an arbitrary element $\alpha \in p(M)$ belong to $p(M)$. We can define f^{-1}, because

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models of an o-minimal theory satisfy the algebraic exchange principle [2]. So f is a bijection on $p(M)$.

Classical work in o-minimality ([2], [3]) shows every function on an o-minimal model has only a finite number of singular points, where the function changes its monotonicity (e.g. from increasing to decreasing), and all such points are A-definable. Consequently, these points do not satisfy the non-algebraic one-type $p \in S_1(A)$. So, f is strictly monotone on some interval I_f, $p(M) \subseteq I_f$. It can not be constant, because $p \in S_1(A)$ is non-algebraic.

In fact, we will show that any such f is strictly increasing. Note that for any f either $\alpha < f(\alpha)$ or $\alpha < f^{-1}(\alpha)$. Suppose $\alpha < f(\alpha)$. Let $\beta = f(\alpha)$, $\gamma = f(\beta)$. We have $\alpha < \beta < \gamma$, because $\models \forall x(x \in I_f \rightarrow x < f(x))$. So we have f is strictly increasing, and because $\alpha = f^{-1}(\beta) < f^{-1}(\gamma) = \beta$, f^{-1} is strictly increasing too. *Mutatis Mutandis*, we handle the case that $\alpha < f^{-1}(\alpha)$.

Let $G_{p,A}$ be set of all $L(A)-$definable bijections on $p(M)$; this is a group of actions under composition. Define the order $(<^1)$ on this group: for every pair $f, g \in G_{p,A}$, $f <^1 g$, if for some $\alpha \in p(M)$, $f(\alpha) < g(\alpha)$. It is straightforward to show this notion is well-defined and the group is ordered in the sense described in Definition 4. $\square_1$

The proof of Theorem 3 consists of two steps.

(1) Observe that any ordered group can be represented as $|G|$ unary functions acting on itself.
(2) Show that the theory of $(G, <, t_g)_{g \in G}$ just defined admits quantifier elimination and is therefore 0-minimal

Recall:

Definition 4. $(G, \cdot, <)$ *is an (linearly) ordered group if*

(1) $(G, \cdot)$ *is a group;*
(2) $<$ *linearly orders* G*;*
(3) *(left order) if* $g_1 < g_2$ *then* $hg_1 < hg_2$*;*
(4) *(right order) if* $g_1 < g_2$ *then* $g_1 h < g_2 h$*;*
(5) *if* $g < 1$ *then for any* x*,* $gx < x$*;*
(6) *if* $g > 1$ *then for any* x*,* $gx > x$*.*

The following fact is evident from a little reflection on the definition.

Fact 5. *Let* $(G, \cdot, <)$ *be an ordered group.*

(1) *The order* $(G, <)$ *is* 1*-transitive.*
(2) *Consequently* $(G, <)$
    (a) *is discrete and isomorphic to copies of* $(Z, <)$ *or*

(b) *is dense.*

Form a language L^* with $<$ and unary function symbols t_g for g in G.

Definition 6. *The structure $\langle X, <, t_g \rangle_{g \in G}$ is a unary representation of G on X if:*

(1) $t_g(t_h) = t_{gh}$;
(2) *If for some x, $t_g(x) = t_h(x)$ then $g = h$;*
(3) *if $g_1 < g_2$ then for all g, $t_{g1}(x) < t_{g2}(x)$;*
(4) *each t_g is increasing;*
(5) *if $g < 1$ then for any x, $t_g(x) < x$;*
(6) *if $g > 1$ then for any x, $t_g(x) > x$;*
(7) *each t_g is onto.*

Just translating from Definition 4 to Definition 6, we see:

Lemma 7. *If G is any linearly ordered group, the structure $\langle G, <, t_g \rangle_{g \in G}$, with for any $h \in G$, $t_g(h) = gh$, gives a unary representation of G on itself.*

Theorem 8. *For any linearly ordered group G the theory T_G of $\langle G, <, t_g \rangle_{g \in G}$ (defined above) admits quantifier elimination and is therefore o-minimal.*

Proof. The theory T_G is axiomatized by the properties in Definition 6 and from identities of the form $\prod_{i \leq k} g_i = h$ that hold in G, axioms $t_{g_0} \circ t_{g_2} \ldots \circ t_{g_k} = t_h$. From these axioms we deduce the following Conditions which are in the theory T_G. For any g_1, g_2, x, y:

(1) $t_{g_1}(x) < t_{g_2}(y) \equiv x < t_{g_1^{-1} g_2}(y)$
(2) $\neg[t_{g_1}(x) < t_{g_2}(y)] \equiv [t_{g_2}(y) < t_{g_1}(x)] \vee [t_{g_1}(x) = t_{g_2}(y)]$
(3) $\neg[t_{g_1}(x) = t_{g_2}(y)] \equiv [t_{g_2}(y) < t_{g_1}(x)] \vee [t_{g_1}(x) < t_{g_2}(y)]$

Now consider an arbitrary existential formula

$$\psi(\mathbf{y}) = (\exists x)\phi(x, \mathbf{y}) = (\exists x) \bigvee \bigwedge \theta_{ij}(x, y_1, \ldots, y_k)$$

where each θ_{ij} is atomic or neg-atomic. Using Conditions 2 and 3, each θ_{ij} which is neg-atomic can be replaced by a disjunction of atomic formulas. Since T_G is unary, the representation of each term by a single t_h, and using Condition 1), each atomic $\theta_{ij}(x, y_1, \ldots, y_k)$ has the form $x < t_g(y_i)$ or $t_{g_s}(y_i) < t_{g_r}(y_j)$. Let $x, y_1, \ldots y_k, t_{g_j}(y_i)$ (for $i < k$ and appropriate j) be a complete list of all terms occurring in $\phi(x, \mathbf{y})$. Now each $\bigwedge_j \theta_{ij}$ is equivalent to a finite disjunction (caused by the removal of the neg-atomic formulas) of formulas of the form: $\bigvee_s \bigwedge_t \mu_{st}$ where $\bigwedge_t \mu_{st}$ has the form:

$$t_{\sigma(1)}(y_{\tau(1)}) < \ldots t_{\sigma(i)}(y_{\tau(i)}) < x < t_{\sigma(i+1)}(y_{\tau(i+1)}) < \ldots t_{\sigma(k)}(y_{\tau(k)})$$

where by varying σ and τ we get all linear orders of the terms occurring in θ_{ij}.

Now there are two cases depending on whether the ordering on G is dense or discrete.

If $(G, <)$ is dense, we just drop x from the matrix to have the equivalent formula in T_G.

If $(G, <)$ is discrete, suppose that $t_g(y_1) < x < t_h(y_2)$ are the termx immediately surrounding x in $\bigwedge_t \mu_{st}$. Suppose f is the least element of G greater that 1. Then $(\exists x) \bigwedge_t \mu_{st}(x, \mathbf{y})$ is equivalent to replacing the occurrence of $t_g(y_1) < x < t_h(y_2)$ in μ_{st} by $t_f(t_g(y_1)) < t_h(y_2)$.

This establishes the quantifier elimination and o-minimality follows immediately.

$\Box_8$

Let us remark that a discrete G can also be represented as acting on a dense linear order by applying Theorem 8 to $G \times Q$ and then taking the reduct to the unary functions naming elements of G.

This note reflects simplifications by the three authors of an earlier argument by Baizhanov.

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