The Entanglement of Model Theory and Set Theory
Ohio State University

John T. Baldwin [1]
University of Illinois at Chicago

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[1] Thanks to J. Kennedy and A. Villeveces

Goal: Maddy

In Second Philosophy Maddy writes,

> *The Second Philosopher sees fit to adjudicate the methodological questions of mathematics – what makes for a good definition, an acceptable axiom, a dependable proof technique?– by assessing the effectiveness of the method at issue as means towards the goal of the particular stretch of mathematics involved.*

We discuss the choice of definitions of model theoretic concepts that reduce the set theoretic overhead:

Entanglement

Kennedy Parsons Väänänen

Such authors as Kennedy, Parsons,and Väänänen
have spoken of the entanglement of logic and set theory.

Theses

There is a deep entanglement between (first-order) model theory and cardinality.

There is No such entanglement between (first-order) model theory and cardinal arithmetic.

There is however such an entanglement between infinitary model theory and cardinal arithmetic and therefore with extensions of ZFC.

Equality as Congruence

Any text in logic posits that:
Equality '=' is an equivalence relation:

Further it satisfies the axioms schemes which define what universal algebraists call a congruence.

The indiscernibility of identicals

For any x and y, if x is identical to y, then x and y have all the same first order properties.
For any formula ϕ: $\forall \mathbf{x} \forall \mathbf{y}[\mathbf{x} = \mathbf{y} \rightarrow (\phi(\mathbf{x}) \leftrightarrow \phi(\mathbf{y}))]$

Equality as Identity

The original ‘sin’

The inductive definition of truth in a structure demands that the equality symbol be interpreted as identity:

$$M \models a = b \text{ iff } a^M = b^M$$

The entanglement of model theory with cardinality is now ordained! This is easy to see for finite cardinalities.

$$\phi_n : (\exists x_1 \ldots x_n) \bigwedge_{1 \leq i < j \leq n} x_i \neq x_j \wedge (\forall y) \bigvee_{1 \leq i \leq n} y = x_i$$

is true exactly for structures of cardinality n.

Entanglement with infinite Cardinality

Three examples of the entanglement with cardinality.

1. Downward Löwenheim Skolem –not so much
2. Upward Löwenheim Skolem
   Yes! Look at the proof.
3. Only finite structures are categorical.

Entanglement with Cardinal arithmetic and extensions of ZFC: Shelah

In 1970, model theory and axiomatic set theory seemed intrinsically linked. Shelah wrote

> *"... in 69 Morley and Keisler told me that model theory of first order logic is essentially done and the future is the development of model theory of infinitary logics (particularly fragments of $L_{\omega_1,\omega}$). By the eighties it was clearly not the case and attention was withdrawn from infinitary logic (and generalized quantifiers, etc.) back to first order logic."*

Shelah: Set theory and model theory

Shelah again:

> *During the 1960s, two cardinal theorems were popular among model theorists. . . . Later the subject becomes less popular; Jensen complained when I start to deal with gap n 2-cardinal theorems, they were the epitome of model theory and as I finished, it stopped to be of interest to model theorists. I sympathize, though model theorists has reasonable excuses: one is that they want ZFC-provable theorems or at least semi-ZFC ones the second is that it has not been clear if there were any more.*

Two Questions

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II. Why by the mid-70's had those apparent links evaporated for first order logic?

I. Apparent dependence on set theory

Löwenheim Skolem for 2 cardinals Vaught

Vaught: Can we vary the cardinality of a definable subset as we can vary the cardinality of the model?

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Two Cardinal Models

1. A two cardinal model is a structure M with a definable subset D with $\aleph_0 \leq |D| < |M|$.
2. We say a first order theory T in a vocabulary with a unary predicate P *admits* (κ, λ) if there is a model M of T with $|M| = \kappa$ and $|P^M| = \lambda$. And we write $(\kappa, \lambda) \to (\kappa', \lambda')$ if *every theory* that admits (κ, λ) also admits (κ', λ').

Set Theory Intrudes Morley

Theorem: Vaught

$(\beth_\omega(\lambda), \lambda) \rightarrow (\mu_1, \mu_2)$ when $\mu_1 \geq \mu_2$.

Theorem: Morley's Method

Suppose the predicate is defined not by a single formula but by a type: $(\beth_{\omega_1}(\lambda), \lambda) \rightarrow (\mu_1, \mu_2)$ when $\mu_1 \geq \mu_2$.

Both of these results need replacement; the second depends of iterative use of Erdös-Rado to obtain countable sets of indiscernibles.

In the other direction, the notion of indiscernibles is imported into Set Theory by Jensen to define $O^{\#}$.

Set Theory Becomes Central

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Hypotheses included:

1. replacement: Erdos-Rado theorem below $\beth_{\omega_1}$.
2. GCH
3. $V = L$
4. Jensen's notion of a morass
5. Erdös cardinals,
6. Foreman [1982] showing the equivalence between such a two-cardinal theorem and 2-huge cardinals AND ON

1-5 Classical work in 60's and early 70's; continuing importance in set theory.

The links dissolve

Why did it stop? Lachlan  Bays 

Revised Theorem: solved in ZFC

Suppose

1. [Shelah, Lachlan $\approx$ 1972] T is stable
2. or [Bays 1998] T is *o*-minimal

then $\forall(\kappa > \lambda, \kappa' \geq \lambda')$
if T admits (κ, λ) then T also admits (κ', λ').

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Reversing the question

set theorist:
For which cardinals does $P(\kappa, \lambda, T)$ hold for all theories ?

model theorist:
For which theories does $P(\kappa, \lambda, T)$ hold for all cardinals ?

Really, Why did it stop?

Definition

[The Stability Hierarchy:] Fix a countable complete first order theory T.

1. T is stable in χ if $A \subset M \models T$ and $|A| = \chi$ then $|S(A)| = |A|$.
2. T is
   1. ω-stable[a] if T is stable in all χ;
   2. superstable if T is stable in all $\chi \geq 2^{\aleph_0}$;
      That is, for every A with $A \subset M \models T$, and $|A| \geq 2^{\aleph_0}$, $|S(A)| = |A|$
   3. stable if T is stable in all χ with $\chi^{\aleph_0} = \chi$;
   4. unstable if none of the above happen.

[a] This 'definition' hides a deep theorem of Morley that T is ω-stable if and only if it stable in every infinite cardinal.

So what? Sacks 

Sacks Dicta

"... the central notions of model theory are absolute and absoluteness, unlike cardinality, is a logical concept. That is why model theory does not founder on that rock of undecidability, the generalized continuum hypothesis, and why the Łos conjecture is decidable."

Gerald Sacks, 1972

General Program

1. Formalization of *specific mathematical areas* is a tool for studying issues in the philosophy of mathematics (methodology, axiomatization, purity, categoricity and completeness etc.);
2. The systematic comparison of local formalization of distinct areas is a useful tool for organizing and doing mathematics and the analysis of mathematical practice.

Stability is Syntactic

Definition

T is stable if no formula has the order property in any model of T.

ϕ is unstable in T just if for every n the sentence
$\exists x_1, \ldots x_n \exists y_1, \ldots y_n \bigwedge_{i<j} \phi(x_i, y_i) \wedge \bigwedge_{j \geq i} \neg\phi(x_i, y_i)$ is in T.

This formula changes from theory to theory.

1. dense linear order: $x < y$;
2. real closed field: $(\exists z)(x + z^2 = y)$,
3. $(\mathbb{Z}, +, 0, \times)$:$(\exists z_1, z_2, z_3, z_4)(x + (z_1^2 + z_2^2 + z_3^2 + z_4^2) = y)$.
4. infinite boolean algebras: $x \neq y$ & $(x \wedge y) = x$.

More precisely

While the stability spectrum function is another function about cardinality,
The notions defining the hierarchy are all absolute.

1. ω-stability (Morley rank defined: Π^1_1)
2. superstability (D-rank defined: Π^1_1)
3. stability (no formula has the order property: arithmetic)

The hierarchy is a partition

Theorem

[Stability spectrum theorem] Every complete first order theory falls into one of the 4 classes just defined.

Actually, studying a few more, simplicity and NIP (without the independence property), o-minimal theories etc. has extended the range to a much wider range of mathematically important topics.

The stability hierarchy: examples: Conant

http://homepages.math.uic.edu/~gconant/backupMap/

ω-stable

Algebraically closed fields (fixed characteristic), differentially closed fields (infinite rank), complex compact manifolds

strictly superstable

$(\mathbb{Z}, +)$, $(2^\omega, +) = (Z_2^\omega, H_i)_{i<\omega}$.

strictly stable

$(\mathbb{Z}, +)^\omega$, separably closed fields, the free group on 2 generators

Entanglement of model theory and the replacement axiom:Kim 

theorem

[Kim: ZFC] For a simple first order theory non-forking is equivalent to non-dividing.

The usual easily applicable descriptions of simple theories involve uncountable objects. But definitions of simple, non-forking, and non-dividing are equivalent in ZC to statements about countable sets of formulas.

Nevertheless, the argument for Kim's theorem employs Morley's technique for omitting types; that is: The standard argument uses the Erdos-Rado theorem on cardinals less than $\beth_{\omega_1}$.

Dis-Entanglement of model theory and the replacement axiom Vasey 

theorem

[Vasey: ZC^0] For a simple first order theory non-forking is equivalent to non-dividing.

ZC^0 is ZFC without replacement or power set but with the addition of a constant symbol Θ which is asserted to be infinite and the assertion that for X with $|X| \leq \Theta$, $\mathcal{P}(\mathcal{P}(X))$ exists.

Entanglement of Infinitary Logic and Axiomatic Set Theory

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     The elementary equivalence proved in the Ax-Kochen-Ershov theorem
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3. Entanglement:

Consistency yields Provability

Prove that a model theoretic property Φ holds in a model N of a weak set theory.
Extend the model N by ultralimits (to one or many) models N^* satisfying Φ and such that Φ is absolute between N and V.
Deduce Φ is provable in ZFC.

ABSTRACT ELEMENTARY CLASSES

A class of L-structures, $(\boldsymbol{K}, \prec_{\boldsymbol{K}})$, is said to be an *abstract elementary class: AEC* if both $\boldsymbol{K}$ and the binary relation $\prec_{\boldsymbol{K}}$ are closed under isomorphism plus:

1. If $A, B, C \in \boldsymbol{K}$, $A \prec_{\boldsymbol{K}} C$, $B \prec_{\boldsymbol{K}} C$ and $A \subseteq B$ then $A \prec_{\boldsymbol{K}} B$;

Examples

First order and $L_{\omega_1,\omega}$-classes
$L(Q)$ classes have Löwenheim-Skolem number $\aleph_1$.

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2. Closure under direct limits of $\prec_{\boldsymbol{K}}$-chains;
3. Downward Löwenheim-Skolem.

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First order and $L_{\omega_1,\omega}$-classes
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Shelah infinitary categoricity theorem

No Assumption of upwards Löwenheim-Skolem

Theorem [Shelah]

1. (For $n < \omega$, $2^{\aleph_n} < 2^{\aleph_{n+1}}$) A complete $L_{\omega_1,\omega}$-sentence which has very few models in $\aleph_n$ for each $n < \omega$ is excellent.
2. (ZFC) An excellent class has models in every cardinality.
3. (ZFC) Suppose that ϕ is an excellent $L_{\omega_1,\omega}$-sentence. If ϕ is categorical in one uncountable cardinal κ then it is categorical in all uncountable cardinals.

Boney/Vasey eventual categoricity theorems Boney

Theorem (Boney)

If κ is a strongly compact cardinal and $LS(\boldsymbol{K}) < \kappa$ then if $\boldsymbol{K}$ is categorical in some $\lambda^+ > \kappa$ then $\boldsymbol{K}$ is categorical in all $\mu \geq \lambda^+$.

Theorem (Vasey)

Assuming, κ is a strongly compact cardinal and $LS(\boldsymbol{K}) < \kappa$, VWGCH, and the result of a long preprint of Shelah,
if $\boldsymbol{K}$ is categorical in some $\lambda > \kappa$ then $\boldsymbol{K}$ is categorical in all $\mu \geq \lambda^+$.

The Dependence on cardinality

First order (Morley)

$\aleph_0$ is exceptional:

1. Categoricity is $\aleph_1$ implies categoricity in all uncountable cardinals.

Infinitary: Shelah, Boney/Vasey

Some small cardinals may be exceptional:

1. (VWGCH) Categoricity is all cardinals below $\aleph_\omega$ implies categoricity in all uncountable cardinals.
2. Categoricity beyond a strongly compact implies categoricity in all uncountable cardinals.

Which cardinals are exceptional?

Any $\aleph_n$. (Hart-Shelah; B-Kolesnikov)

The Paradigm Shift

Fundamental Distinctions

Logics

1. second order logic
2. infinitary logic (aec)
3. first order logic

The choice of logics presents a trade-off between greater ability to control the structure of models (via e.g. compactness) and lesser expressive power.

The Paradigm Shift

Model theory in the 1960's concentrated on the properties of logics.

This resulted in many problems being tied closely to axiomatic set theory.

The switch to classifying a theory T according to whether there were good recipes for decomposing models of T into simpler pieces resulted in

1. a divorce of model theory from axiomatic set theory
2. a fruitful interaction with many other areas of mathematics.

The study of infinitary logic offers more expressive power to study mathematics at a possible cost of set theoretic independence.

Axiomatization vrs Formalization

Bourbaki on Axiomatization:

Dieudonne  Bourbaki  Cartan 

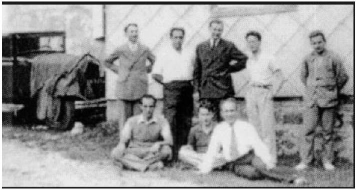
Bourbaki wrote:

> *Many of the latter (mathematicians) have been unwilling for a long time to see in axiomatics anything other else than a* futile logical hairsplitting not capable of fructifying any theory whatever*.*

More Bourbaki

This critical attitude can probably be accounted for by a purely historical accident.

The first axiomatic treatments and those which caused the greatest stir (those of arithmetic by Dedekind and Peano, those of Euclidean geometry by Hilbert) dealt with univalent theories, i.e. *theories which are entirely determined by their complete systems of axioms; for this reason they could not be applied to any theory except the one from which they had been abstracted (quite contrary to what we have seen, for instance, for the theory of groups).*

More Bourbaki: Bourbaki 

If the same had been true of all other structures, the reproach of sterility brought against the axiomatic method, would have been fully justified.

Bourbaki realizes but then forgets that the hypothesis of this last sentence is false.

They miss the distinctions between

1. axiomatization and theory
2. first and second order logic.

Bourbaki Again

Bourbaki distinguishes between 'logical formalism' and the 'axiomatic method'.

'We emphasize that it (logical formalism) is but one aspect of this (the axiomatic) method, indeed the least interesting one'.

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Bourbaki Again

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‘We emphasize that it (logical formalism) is but one aspect of this (the axiomatic) method, indeed the least interesting one’.

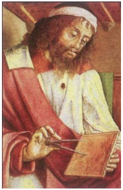
We reverse this aphorism:
The axiomatic method is but one aspect of logical formalism.

And the foundational aspect of the axiomatic method is the least important for mathematical practice.

Two roles of formalization

1. Building a piece or all of mathematics on a firm ground specifying the underlying assumptions
2. When mathematics is organized by studying first order (complete) theories, syntactic properties of the theory induce profound similarities in the structures of models. These are tools for mathematical investigation.

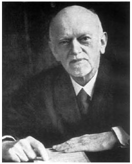
Euclid-Hilbert formalization 1900:

Euclid  Hilbert 

The Euclid-Hilbert (the Hilbert of the Grundlagen) framework has the notions of axioms, definitions, proofs and, with Hilbert, models.

But the arguments and statements take place in natural language. For Euclid-Hilbert logic is a means of proof.

Hilbert-Gödel-Tarski-Vaught formalization 1917-1956:

Hilbert  Gödel  Tarski  Vaught 

In the Hilbert (the founder of proof theory)-Gödel-Tarski framework, logic is a mathematical subject.

There are explicit rules for defining a formal language and proof. Semantics is defined set-theoretically.

First order logic is complete. The theory of the real numbers is complete and easily axiomatized. The first order Peano axioms are not complete.

Formalization

Anachronistically, *full formalization* involves the following components.

1. Vocabulary: specification of primitive notions.
2. Logic
   - a Specify a class of well formed formulas.
   - b Specify truth of a formula from this class in a structure.
   - c Specify the notion of a formal deduction for these sentences.
3. Axioms: specify the basic properties of the situation in question by sentences of the logic.

Item 2c) is the least important from our standpoint.

The success of the hierarchy

A crucial consequence of stability is the ability to define family of dimensions and classify structures.

The stability classification of T gives detailed information about the fine structure of definable sets in each model of T.
This information is encoded by stability ranks that are in many cases (e.g. algebraic geometry) the same as those arising in other content areas.

A sophisticated theory for studying the interactions of these various dimensions has had applications in many fields.

Mathematically relevant areas of mathematics can be axiomatized by complete first order theories of various stability classes.

Model theory entangles with Algebra

Theorem (Hrushovski 1989) Let T be a stable theory. Let $\tilde{p} \not\perp \tilde{q}$ be stationary, regular types and let n be maximal such that $\tilde{p}^n \perp^a \tilde{q}^\omega$. Then there exist p almost bidominant to $\tilde{p}$ and q dominated by $\tilde{q}$ such that:

- $n = 1$ q is the generic type of a type definable group that has the regular action on the realizations for p.
- $n = 2$ q is the generic type of a type definable algebraically closed field that acts on the realizations for p as an affine line.
- $n = 3$ q is the generic type of a type definable algebraically closed field that acts on the realizations for p as a projective line.
- $n \geq 4$ is impossible.

The Entanglement with group and field theory: Importance

The hypotheses are purely model theoretic.

There is no assumption that a group or ring is even interpretable in the theory.

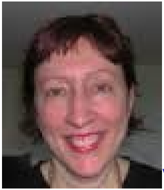
The conclusion gives precise kinds of group and field actions that are *definable* in the given structures.

There are important consequences in model theory, diophantine geometry, differential fields, . . .

Summation: Hrushovski 

Hrushovski ICM talk 1998

Instead of defining the abstract context for the [stability] theory, I will present a number of its results in a number of special and hopefully more familiar, guises: compact complex manifolds, ordinary differential equations, difference equations, highly homogeneous finite structures. Each of these has features of its own and the transcription of results is not routine; they are nonetheless readily recognizable as instances of a single theory.

Thanks: Kennedy  Villaveces 

Related Work

Completeness and Categoricity (in power): Formalization without Foundationalism
The Bulletin of Symbolic Logic 2014

Formalization, Primitive Concepts and Purity
Review of Symbolic Logic vol 6, 2013

Axiomatizing Changing Conceptions of the geometric continuum I and II
First order justification of $C = 2\pi r$
submitted

`http://homepages.math.uic.edu/~jbaldwin/model11.html`

Relevant Model/Set Theory Papers

replacement: http://homepages.math.uic.edu/~jbaldwin/pub/monster4.pdf
Vasey paper: http://math.cmu.edu/~svasey/papers/morley-seq/vasey-morley-seq_v4.pdf
consistency yields provability:
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