Formalization, Primitive Concepts, and Purity

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August 21, 2011

Abstract

We emphasize the role of the choice of vocabulary in formalization of a mathematical area and remark that this is a particular preoccupation of logicians. We use this framework to discuss Kennedy's notion of 'formalism freeness' in the context of various schools in model theory. Then we clarify some of the mathematical issues in recent discussions of purity in the proof of the Desargues proposition. We note that the conclusion of 'spatial content' from the Desargues proposition involves arguments which are algebraic and even metamathematical. In particular, the converse to Desargues cannot be read as: the Desargues proposition implies there are non-coplanar points. Rather, Hilbert showed that Desargues proposition implies the coordinatizing ring is associative, which in turn implies the existence of a 3-dimensional geometry in which the given plane can be embedded. We (with W. Howard) give a new proof, removing Hilbert's 'detour' through algebra, of the 'geometric' embedding theorem and examine the issue of purity for this embedding theorem.

Mathematical logic formalizes normal mathematics. We analyze the meaning of 'formalize' in that sentence and use this analysis to address several recent questions. In the first section we establish some precise definitions to formulate our discussion and we illustrate these notions with some examples of David Pierce. This enables us to describe a variant on Kennedy's notion of formalism freeness and connect it with recent developments in model theory. In the second section we discuss the notion of purity in geometric reasoning based primarily on the papers of Hallet [17] and Arana-Mancosuo [3]. In an appendix written with William Howard we give a geometric proof (differing from Levi's in [29]) of Hilbert's theorem that a Desarguesian projective plane can be embedded in three-space.

Our general context is that there is some area of mathematics that we want to clarify. There are five components of a formalization. The first four 1) specification of primitive notions, 2) specifications of formulas and 3) their truth, and 4) proof provide the setting for studying a particular topic. 5) is a set of axioms that pick out the actual subject area. We will group these five notions in various ways through the paper to make certain distinctions.

Our general argument is: while formalization is the key tool for the general foundational analysis and has had significant impact as a mathematical tool [1] there are spe-

[1]Examples include the theory of computability, Hilbert's 10th problem, the Ax-Kochen theorem, o-minimality and Hardy fields, Hrushovski's proof the geometric Mordell-Lang theorem and current work on motivic integration.

cific problems in mathematical logic (Section 1.3) and philosophy (Section 2) where 'formalism-free' methods are essential.

I thank Paolo Mancosu for introducing me to this topic and acknowledge helpful comments from Andrew Arana, Phil Ehrlich, and Juliette Kennedy.

1 Formalization: Vocabulary, Logic and Proof

We describe our notion of a formalization of a mathematical topic. This involves not only the usual components of a formal system, specification of ground vocabulary, well-formed formulas, and proof but also a semantics. From a model theoretic standpoint the semantic aspect has priority over the proof aspect. The topic could be all mathematics via e.g. a set theoretic formalization. But our interest is more in the local foundations of, say, plane geometry.

Definition 1.0.1 *We see a* full formalization *as involving the following components.*

1. *Vocabulary: specification of primitive notions.*
2. *Logic*
   (a) *Specify a class*[2] *of well formed formulas.*
   (b) *Specify truth of a formula from this class in a structure.*
   (c) *Specify the notion of a formal deduction for these sentences.*
3. *Axioms: specify the basic properties of the situation in question by sentences of the logic.*

We will elaborate on this definition in Subsection 1.2. But first we want to emphasize the significance of the first item.

1.1 Vocabulary

We establish some specific notations which emphasize some distinctions between the mindsets of logicians, in particular, model theorists, and 'normal' mathematicians.

Definition 1.1.1
1. *A* vocabulary τ *is a list of function, constant and relation symbols.*
2. *A* τ-structure $\mathcal{A}$ *is a set with an interpretation of each symbol in* τ.

Specifying a vocabulary[3] is only one aspect of the notion of a formal system. But it is a crucial one and one that is often overlooked by non-logicians. From the standpoint of formalization, fixing the vocabulary is singling out the 'primitive concepts'. This

[2]For most logics there are only a set of formulas, but some infinitary languages have a proper class of formulas.

[3]Years ago Tarski used the phrase similarity type for essentially this notion; sometimes it is called the signature. Still another 'synonym' is language. We explain in Section 1.3 why we try to avoid this word.

choice is a first step in formalization. Considerable reflection from both mathematical and philosophical standpoints may be involved in the choice. The choice is by no means unique. For example, formulated in a vocabulary with only a binary function symbol, the theory of groups needs $\forall\exists$-axioms and groups are not closed under subalgebra. Adding a constant for the identity and a unary function for inverse, turns groups into a universally axiomatized class. Alternatively, groups can be formulated with one ternary relation as the only symbol in the vocabulary. The three resulting theories are pairwise bi-interpretable.

It is a commonplace in model theory that just specifying a vocabulary means little. For example in the vocabulary with a single binary relation, I can elect to formalize either linear order or successor (by axioms asserting the relation is the graph of a unary function). Thus, while I here focus on the choice of relation symbols – their names mean nothing; the older usage of signature or similarity type might be more neutral. The actual collection of structures under consideration is determined in a formal theory by sentences in the logic. In the formalism-free approaches discussed in Subsection 1.3 the specification is in normal mathematical language. Having fixed a vocabulary with one binary function (or alternatively one ternary relation), we say e.g. 'Let $\boldsymbol{K}$ be the class of well-orderings of type $< \lambda$ such that ....'

David Pierce [38] has pointed out the following example of mathematicians' lack of attention to vocabulary specification.

Example 1.1.2 (Pierce) Spivak's Calculus book [45] is one of the most highly regarded texts in late 20th-century United States. It is more rigorous than the usual Calculus I textbooks. Problems 9-11 on page 30 of [45] ask the students to prove the following are equivalent conditions on $\mathbb{N}$, the natural numbers. This assertion is made without specifying the vocabulary that is intended for $\mathbb{N}$. In fact, $\mathbb{N}$ is described as the counting numbers,

$$1, 2, 3, \ldots .$$

1 induction ($1 \in X$ and $k \in X$ implies $k+1 \in X$) implies $X = N$.

2 well-ordered Every non-empty subset has a least element.

3 strong induction ($1 \in X$ and for every $m < k$, $m \in X$ implies $k \in X$) implies $X = N$.

As Pierce points out, this doesn't make sense: 1 is a property of a unary algebra; 2 is a property of ordered sets (and doesn't imply the others even in the language of ordered algebras); 3 is a property of ordered algebras. e.g., 2) is satisfied by any well-ordered set while the intent is that the model should have order type ω.

It is instructive to consider what proof might be intended for 1) implies 3). Here is one possibility. Let X be a non-empty subset of N. Since every element of N is a successor (Look at the list!), the least element of X^c must be $k+1$ for some $k \in X$. But the existence of such a k contradicts property 1). There are two problems with this 'proof'. The first problem is that there is no linear order mentioned in the formulation of 1). The second is, "what does it mean to 'look at the list'?". These objections can be addressed. Assuming that $\mathbb{N}$ has a discrete linear order satisfying

$(\forall x)(\forall y)[x \leq y \vee y + 1 \leq x]$ and that the least element is the only element which is not a successor resolves the problem. This assertion follows informally (semantically) if one considers the natural numbers as a subset of the linearly ordered set of reals.

As Pierce notes [38] a fundamental difficulty in Spivak's treatment is the failure to distinguish the truth of each of these properties on the appropriate expansion of (N, S) and a purported equivalence of the properties– *which can make sense only if the properties are expressed in the same vocabulary*.

But in another sense the problem is the distinction between Hilbert's axiomatic approach and the more naturalistic approach of Frege. I'll call Pierce's characterization of Spivak's situation, Pierce's paradox. It will recur; Pierce writes:

> Considered as axioms in the sense of Hilbert, the properties are not meaningfully described as equivalent. But if the properties are to be understood just as properties of the numbers that we grew up counting, then it is also meaningless to say that the properties are equivalent: they are just properties of those numbers.

Note that this distinction about vocabulary is prior to distinctions between first and second order logic. We stated the difficulty in the purported equivalence of 1) and 2) in terms of second order logic. But the same anomaly would arise if PA were compared with 'every definable set is well-ordered'.

As this example illustrates, the specification of a vocabulary can be rather fluid. Much of this paper concerns the role of explicit definition.

Definition 1.1.3 *For any theory T in vocabulary τ and formula $\phi(\mathbf{x})$, if we expand τ to τ' by adding a new relation symbol, R_ϕ, and the axiom*

$$(\forall \mathbf{x})[\phi(\mathbf{x}) \leftrightarrow R_\phi(\mathbf{x})]$$

we say R_ϕ is explicitly defined *in T*

At first sight it seems extension by explicit definition is totally harmless. No new concept can be introduced[4]; it is just a kind of abbreviation. And if no new properties that are not provable in the base theory are introduced, this step should surely not infringe on the purity of an argument. It is a 'logical' not a 'mathematical' step. However, we will see that if arbitrary explicit definitions are allowed in a proof, the notion of such a proof being pure is almost meaningless.

It is natural when formalizing to try to find a minimal set of 'primitive concepts'. This was a frequent theme of Tarski[5]. And arguments for the naturality of various notions are part of the justification for a particular choice. However, there are other considerations.

Explicit definition can reduce complexity in a way that can be measured formally – by reducing quantifier complexity. In developing a first order theory to describe

[4] We think of the concepts formalizable in a theory as exactly the formulas. Making an explicit definition is focusing attention an existing concept, not adding a new one. See the discussion in Subsection 2.1.

[5] See [16] for a summary of Tarski's work on this area for geometry.

a given mathematical structure or class of structures, a model theorist would like to have the resulting theory T admit elimination of quantifiers (every formula $\phi(\mathbf{x})$ is equivalent in T to a formula $\psi(\mathbf{x})$ which has no quantifiers) and be formulated with concepts intrinsic to the subject. The quantifier elimination is easily established by fiat [33, 25]: add explicit definitions for each formula $\phi(\mathbf{x})$, a predicate $R_\phi(\mathbf{x})$ and an axiom $R_\phi(\mathbf{x}) \leftrightarrow \phi(\mathbf{x})$. But now the vocabulary contains many essentially incomprehensible relations. This defeats the goal of quantifier elimination in studying specific theories: every definable relation is a Boolean combination of *well-understood* relations. Sometimes, with judicious choice, a small family of well understood predicates can be added that suffice for the quantifier elimination. Two examples are Pressburger arithmetic where unary-predicates for divisibility by n are added and Macintyre's celebrated proof of quantifier elimination for the p-adic numbers; he adds predicates $P_n(x)$ meaning x is an nth power.

1.2 Formalization

We first expand a bit on what we mean by a full formalization. Definition 1.2.1 summarises the notion of a *model theoretic logic* $\mathcal{L}$ defined abstractly in [8]. This paper will only use some specific examples of such logics including, first order, second order, $L_{\omega_1,\omega}$ etc..

Definition 1.2.1 *A logic contains certain logical vocabulary: connectives, quantifiers and a set of variables. For each (non-logical) vocabulary τ, the collection of $\mathcal{L}(\tau)$-formulas is defined inductively in the natural way. An $\mathcal{L}(\tau)$-formula with no free variables is called an $\mathcal{L}(\tau)$-sentence.*

Thus, a logic *$\mathcal{L}$ is the pair $(\mathcal{L}, \models_{\mathcal{L}})$ such that $\mathcal{L}(\tau)$ is a collection of τ-sentences and for each $\phi \in \mathcal{L}(\tau)$ and each τ-structure $\mathcal{A}$, $\mathcal{A} \models_{\mathcal{L}} \phi$ is defined in the natural inductive way.*

Crucially, 'the natural inductive definition' implies that the truth of a τ-sentence in a τ-structure M depends only on the isomorphism type of M. Thus our entire framework is based on what Burgess[9] has called indifferentism: ignoring a specific set theoretic construction of the model. Ironically, Shelah has proved major results by deliberately ignoring this convention. Thus the Whitehead problem[6] is entirely about abelian groups. Nevertheless, Shelah[14] shows the answer is independent of the axioms of ZFC by representing a Whitehead group of cardinality $\aleph_1$ as a structure with universe $\aleph_1$ and identifies invariants depending on both the group operation and the ϵ-relation on $\aleph_1$. A further example of this use of the explicit construction of a model is Shelah's use of Ehrenfeuht-Mostowski models in any expanded language to construct many non-isomorphic models of first order theories under various instability hypotheses. Still another is the Ehrlich [13] use of Conway's surreal numbers to investigate real closed fields.

Definition 1.2.1 yields two natural notions of implication.

[6] J.H.C. Whitehead asked whether whenever B is an abelian group and $f : B \mapsto A$ is a surjective group homomorphism whose kernel is isomorphic to the group of integers $\mathbb{Z}$, then $B \approx \mathbb{Z} \otimes A$.

Definition 1.2.2 *For any $\phi, \psi \in \mathcal{L}(\tau)$, we say ϕ* logically implies *ψ and write $\phi \models_{\mathcal{L}} \psi$ if for every τ-structure M, $M \models_{\mathcal{L}} \phi$ then $M \models_{\mathcal{L}} \psi$.*

Alternatively, $\mathcal{L}$ may be assigned a collection of logical axioms and deduction rules giving rise to the natural notion of ψ can be deduced *from ϕ, $\phi \vdash_{\mathcal{L}} \psi$.*

Deduction is a syntactic notion. We divided the notion of formalization into five components: 1) specification of primitive notions, 2a) specifications of formulas and 2b) their truth, 2c) proof and 3) axioms. The standard account of a formal system (e.g. [43]) includes 2a), 2c) and 3) but not 2b). From our standpoint (and that of [8]), 2a), and 2b) are basic and 2c) may or may not exist. (There is no good proof system for second order logic and the proof systems for infinitary logic use inference rules with infinitely many premises.)

In Definition 1.0.1, the logic does not depend on the particular area of mathematics. In earlier days, from a more general logical perspective, the vocabulary might be universal, containing infinitely many n-ary relations for each n. In contrast, we seek primitive terms which pick out the most basic concepts of the field in question and axioms which in Hilbert's sense give us an implicit definition of the area. Thus, we can formalize concepts such as real closed fields (RCF) or algebraic geometry[7] or set theory without reference to the construction of specific models[8]. Our treatment of the primitive terms is analogous to the treatment of the element relation in set theory. But this analysis is relevant to either traditional (global) or local foundations. For any particular area of mathematics, one can lay out the primitive concepts involved and choose a logic appropriate for expressing the important concepts and results in the field. While in the last quarter century mathematical logic has primarily focused on first order logic as the tool, we discuss some alternatives in Subsection 1.3.

1.3 Formalism Freeness (Mathematical Properties)

In the opening paragraph of what might be viewed as the founding paper of model theory [47], Tarski writes,

> Every set Σ of sentences determines uniquely a class $\boldsymbol{K}$ of mathematical systems. ... Among questions which arise naturally in the study of these notions, the following may be mentioned: Knowing some structural (formal) properties of a set Σ of sentences, what conclusions can we draw concerning the mathematical properties of the correlated set of models? Conversely, ...

Tarski gives a number of examples of answers to questions of this sort; two are:

Tarski A class $\boldsymbol{K}$ of structures in a finite relational language is axiomatized by a set of universal sentences if and only if $\boldsymbol{K}$ is closed under isomorphism, substructure and if for every finite substructure B of a structure A, $B \in \boldsymbol{K}$ then $A \in \boldsymbol{K}$.

Birkhoff A class $\boldsymbol{K}$ of algebras is axiomatized by a set of equations if and only it it is closed under homomorphism, subalgebra, and direct product.

[7]See the discussion of Zariski geometries in Subsection 1.3.

[8]Geometry and analysis are presented in this way in e.g. [45, 20, 19].

This notion of 'mathematical property' is similar to that which Kennedy [26] traces as the notion of 'formalism freeness' in the works of Gödel. She writes, 'one can think of indifferentism[9] or formalism-freeness ... as the simple preference for semantic methods, that is methods *which do not involve or require the specification* of a logic– at least not *prima facie*.' We hope that the distinctions in Section1.2[10] can clarify this notion. It is tempting to speak of language here. We have avoided the word 'language' because of its several usages in the context at hand. One may speak of the 'language of rings' meaning $+, \times, 0, 1$. A different and more specific version is for this phrase to imply that these operations obey the axioms of ring theory; see Subsection 2.3. In a third version, one speaks of the language of first order logic, meaning the collection of formulas generated from a vocabulary by the finitary propositional connectives and existential and universal quantification. We interpret Kennedy to be making this distinction. That is, a formalism-free approach would take language in the first sense, not the second or third. An inquiry can be 'formalism-free' while being very careful about the vocabulary but eschewing a choice of logic and in particular any notion of formal proof. Thus it studies mathematical properties in the sense we quoted from Tarski above.

It is in this sense that certain recent work of Zilber and Shelah can be seen as developing a formalism-free approach to model theory. Both Zilber's notions of a quasi-minimal excellent class [49] and of a Zariski geometry [23], and Shelah's concept of an Abstract Elementary Class [41] give axiomatic but mathematical definitions of classes of structures in a vocabulary τ. That is, the axioms are not properties expressed in some formal language based on τ but are mathematical properties of the class of structures and some relations on it. In Shelah's case, the basic relation is a notion of 'strong submodel' relating the members of the class. Quasiminimal excellent classes require a combinatorial geometry on each model which has certain connections with the basic vocabulary. For example, we have:

Definition 1.3.1 *An abstract elementary class (AEC) $(\boldsymbol{K}, \prec_{\boldsymbol{K}})$ is a collection of structures[11] for a fixed vocabulary τ that satisfy the following, where $A \prec_{\boldsymbol{K}} B$ means in particular A is a substructure of B:*

1. *If $A, B, C \in \boldsymbol{K}$, $A \prec_{\boldsymbol{K}} C$, $B \prec_{\boldsymbol{K}} C$ and $A \subseteq B$ then $A \prec_{\boldsymbol{K}} B$;*
2. *Closure under direct limits of $\prec_{\boldsymbol{K}}$-embeddings;*
3. *Downward Löwenheim-Skolem. If $A \subset B$ and $B \in \boldsymbol{K}$ there is an A' with $A \subseteq A' \prec_{\boldsymbol{K}} B$ and $|A'| \leq |A| = \mathrm{LS}(\boldsymbol{K})$.*

In fact, these classes can be defined in a 'formalism-free' way using purely categorical terms; see Kirby: "Abstract Elementary Categories" [27] and Lieberman: "AECs as accessible categories" [30]. The connection with logic is at first only motivational.

[9] I distinguish Burgess's notion of indifferentism to identity page 9 of [9] from the issue studied here. Indifferentism seems to me to refer to working with structures up to isomorphism rather than caring about the set theoretic construction. Here we take that modus operandi for granted and consider how one is to describe the connection between various structures.

[10] Our articulation of them here was partially motivated by Kennedy's work.

[11] Naturally we require that both $\boldsymbol{K}$ and $\prec_{\boldsymbol{K}}$ are closed under isomorphism.

The AEC notion was developed to simplify the study of infinitary logics by generalizing some of the crucial properties and avoiding syntactical complications. The crucial Löwenheim Skolem property is derived from thinking of $\prec$ as a kind of elementary submodel. But there is no explicit syntax and no notion of a definable set.

In contrast, Zilber's notion of a *quasiminimal excellent class* [49] was developed to provide a smooth framework for proving the categoricity in all uncountable powers of Zilber's pseudo-exponential field. This example itself is developed in a standard model theoretic framework in $L_{\omega_1,\omega}(Q)$. The structure is patterned on (and conjecturally isomorphic to) to the complex exponential field $(\mathbb{C}, +, \times, e^x)$. For technical reasons having to do with the simplicity of dealing with relational languages, the vocabulary is taken to include all polynomially definable sets as basic predicates. The fundamental result that a quasiminimal excellent class is categorical in all uncountable powers can be presented in a formalism-free way. The key point is that there are no axioms in the object language of the general quasiminimal excellence theorem; there are only statements about the combinatorial geometry determined by what are in the application the ($L_{\omega_1,\omega}$)-definable sets.

Hrushovski and Zilber [23] introduced Zariski geometries partly in an attempt to remedy a notorious gap in the model theoretic study of algebraic geometry. Algebraic geometry is concerned with the solution of systems of *equations*. But from a semantic standpoint, there is no way to distinguish among definable sets, 'all definable sets are equal'. In particular, the class of definable sets is closed under negation and equations and inequations have the same status. But from the perspective of algebraic geometry, 'some definable sets are more equal than others'. Systems of equations (varieties) are the objects of true interest. Hrushovski and Zilber remedy this situation by introducing a topology. The definition of a Zariski geometry [50] concerns the relations between a family of topologies (with a dimension) on the sets D^n for a fixed D. Generalizing the case of algebraic geometry the closed sets should be given by conjunction of equations. The main result of [23] is that every Zariski geometry satisfying sufficiently strong semantic conditions can in fact be realized as a finite cover of an algebraic curve: More precisely,

Theorem 1.3.2 (Hrushovski-Zilber) *If M is an ample Noetherian Zariski structure then there is an algebraically closed field K, a quasi-projective algebraic curve $C_M = C_M(K)$ and a surjective map*

$$p : M \mapsto C_M$$

of finite degree such that for every closed $S \subseteq M^n$, the image $p(S)$ is Zariski closed in C_M^n (in the sense of algebraic geometry); if $\hat{S} \subseteq C_M^n$ is Zariski closed, then $p^{-1}(\hat{S})$ is a closed subset of M^n (in the sense of the Zariski structure M).

The classes of Zilber and Shelah are presented 'mathematically': by properties of the class of models that are not connected to truth of formal sentences. But Zilber's quasiminimal excellent classes are definable in $L_{\omega_1,\omega}(Q)$ [28][12] and this axiomatization was the explicit goal of the project.

[12] Kirby's formulation makes the clear distinction between the 'mathematical' and 'logical' descriptions; Zilber blurs the distinction in the original paper [49].

To clarify the distinctions between 'formalism-free' and logical treatments, we provide some more examples of results in the study of AEC. Here is an example of a 'purely semantical' theorem. WGCH abbreviates the assertion, for all λ, $2^\lambda < 2^{\lambda^+}$.

Theorem 1.3.3 *[WGCH] Let $\boldsymbol{K}$ be an abstract elementary class (AEC). Suppose $\lambda \geq \mathrm{LS}(\boldsymbol{K})$ and $\boldsymbol{K}$ is λ-categorical. If amalgamation fails in λ there are 2^{λ^+} models in $\boldsymbol{K}$ of cardinality $\kappa = \lambda^+$.*

Shelah's celebrated 'presentation theorem' [40, 5] changes the role of logic from a motivation (AEC are supposed to abstract the properties of classes defined in various infinitary logics.) to a tool. The theorem asserts that an AEC with arbitrarily large models can be defined as the reducts of models of a first order theory which omit a family of types. But Morley [34] had calculated the Hanf number for such syntactically defined classes. Thus, passing through the syntax, Shelah obtains a purely semantic theorem: if an AEC with Löwenheim number $\aleph_0$ has a model of cardinality $\beth_{\omega_1}$ it has arbitrarily large models.

When spelled out, the syntactic condition in the presentation theorem is a set of sentences in roughly Tarski's sense. Thus, each of these three results provide examples of Tarski's consideration of a duality between description in a formal language and mathematical description. In the first two cases we gain a firmer grasp on a certain class of relations on the universe of model by seeing a specific logic in which they are definable. In the third, we are able to deduce purely semantical conclusions. Notably, the vocabulary arising in the presentation theorem arises naturally only in the context of the presentation theorem. There is no apparent connection of each symbol of the resulting vocabulary with any basic mathematical properties of the abstract elementary class in question.

2 Formalization and Purity in geometry

In this section, we use our earlier analysis of vocabulary as a tool for considering the question of whether/how formalization can be used to capture the notion of 'purity of method'. Our attention was drawn to this topic by the papers of [17, 3]. Our goal here is to use our perspective of the role of vocabulary to set out the mathematical issues involved in the study of Desargues theorem in a somewhat different way and to make some comment on how this affects purity concerns. This analysis is based rather directly on a first order axiomatization of geometry. Consideration in terms of Manders [31] 'diagram proofs' would raise other pertinent issues.

We discuss the relation between the choice of vocabulary and basic axioms and the 'content' of a subject in Subsection 2.1. In Subsection 2.2 we lay out some of the mathematical background in terms of formal axiomatic systems to clarify the relation between the Desargues proposition in affine and projective geometry. Subsection 2.3 analyzes several attempts to define the notion of purity and concludes that formalization can highlight impure methods but does not give a criteria for 'purity'. In Subsection 2.4, we come to grips with the essence of Hilbert's work on Desargues' theorem and ask about the purity of his argument that the 3-dimensional proof is impure.

The discussion of purity below will make clear that the context of a proposition is crucial for the issues we are discussing. But the notions of what geometry actually is have changed radically over the centuries. The distinction between the coordinate geometry of Descartes and synthetic geometry is crucial for our purposes. As [3] makes clear the study of geometry in the late 19th century involved multiple dimensions, surfaces of various curvature, coordinate and coordinate-free approaches. These distinctions remain today although perhaps fields of mathematics are laid out differently: algebraic geometry, real algebraic geometry, differential geometry, topology, etc. etc. Hilbert's work on Desargues theorem led directly to one subfield of geometry; the coordinatization theorems which are a main focus of this paper led to the study of projective planes over usually finite fields as a distinct area of mathematics. In such major sources for this area as [4, 24, 11] the proof of the Desargues theorem in 3-D geometry is not central, if it appears at all. Rather, the importance of Desargues theorem is an indicator of the algebraic properties of the coordinatizing field. The latter two books connect the projective geometries with combinatorics. There are important links with the statistical field of experimental design.

2.1 Content and Vocabulary

Given some area of mathematics, e.g. plane geometry, a formalization of the area has three essential components: the choice of primitive notions (vocabulary), the choice of logic[13], and the choice of basic assumptions (axioms). In this section, we restrict to first order logic.

The discussion of purity in geometry frequently [3, 17] draws on the following remark of Hilbert in his Lectures on Geometry of 1898/99 (page 316 of [21]), translation from [3]).

> This theorem gives us an opportunity now to discuss an important issue. *The content (Inhalt) of Desargues' theorem belongs completely to planar geometry; for its proof we needed to use space.* Therefore we are for the first time in a position to put into practice *a critique of means of proof*. In modern mathematics such criticism is raised very often, where the aim is to preserve *the purity of method*, i.e. to prove the theorem using means that are suggested by the content of the theorem.

We stress that the task is to 'critique a means of proof'. We are given both conclusion and some premises. Both the premises and the intermediary steps are at issue. And the issue is 'method'. It is not the mere existence of a valid implication; there can be pure and impure proofs of the same implication.

A key issue is whether concepts are introduced in the proof that are not implicit in the conclusion. But 'in the conclusion' makes no sense without some specification of the context. In Hilbert's approach such a specification is made by asserting axioms for the geometry which 'implicitly define' the primitive concepts. We follow that line here.

[13]The choice of logic includes three of the components we discussed in Section 1: formulas, truth, proof.

We somewhat refine below the 'standard account of definability' from [46] which argues that a proper definition must satisfy the eliminability and non-creative criteria[14]. As pointed out in [42], an extension by explicit definitions, is conservative over the base theory, i.e. non-creative. We will justify below the need to allow adding to the vocabulary of a given formalization (even in pure proofs) in order to prove a result. The next examples show that axioms for additional relations can change content; we provide directly relevant examples in Subsection 2.3. The crux is the word 'content'. What is the content of a proposition? Surely, the content changes if the models in the base vocabulary in which the proposition holds change when additional axioms are added.

Example 2.1.1 1. A vocabulary to study equivalence relations contains only $=$ and a binary relation symbol E_1. There we can assert by a theory T_1 that E_1 is an equivalence relation. But suppose the vocabulary is expanded by a binary relation symbol E_2 and T_1 to a theory T_2 asserting that E_2 is also an equivalence relation and each E_2 class intersects each E_1 class in a single element. Now all the reducts of models of the full theory are equivalence relations in which all equivalence classes have the same size.

   Example 2.1.1.1 makes the point very clearly. Surely the first theory tells us exactly what an equivalence relation is. And the concept of equivalence relation entails nothing about the relative size of the equivalence classes. Specifically, the sentence:

$$[(\exists x)(\forall y)(xE_1y \rightarrow x = y)] \rightarrow [(\forall x)(\forall y)(xE_1y \rightarrow x = y)]$$

   is a consequence of the added information about E_2.

2. Consider the proposition $x \cdot x \cdot x = x$ in the class of Abelian groups formulated in a vocabulary with a binary operation $\cdot$ and a constant symbol 1. Now expand the vocabulary to a vocabulary for fields by adding new symbols $+$ and 0; add the axioms for fields and the axiom $x + x + x = 0$.

   The meaning of the expression $x \cdot x \cdot x = x$ is very different in an arbitrary Abelian group (where there may be elements of arbitrary order) than it is in a field of characteristic 3 (where it is a law of the multiplicative group).

3. Consider the class of linear orders in a vocabulary with a single binary relation symbol $<$. Add an operation symbol $+$ and constant 0 and assert the structure is an ordered Abelian group. The original class contained $2^{\aleph_0}$ countable models; but only $\aleph_1$ can be expanded to a group. See page 207([22]).

Of course, our assertion that content changes if the class of models changes is based on the Hilbertian notion that the primitive concepts are implicitly defined by the axioms. One might think of content in a more Fregean way; the axioms are describing geometry. The danger of such a position is falling victim to Pierce's paradox. If we are speaking of a fixed geometry, the various properties are just true; one cannot be

[14]Suppes attribution of these criteria to Leŝnewieski is vigorously contested in [48].

meaningfully said to imply the other. I don't see how one can back off from describing a geometry to describing a family of geometries without embracing Hilbert; the analysis of our concepts of geometry is a crucial tool to formulate the axioms. But geometries we are able to study formally are whatever happens to satisfy the axioms.

2.2 Projective and Affine Geometry

Before turning to our specific analysis of the purity of proof of the Desargues proposition we set some notation and clear away some extraneous matters. In this paper we discuss first order axiomatizations of geometry. As in [20], we are trying to formalize the 'field of geometry (in the traditional sense)'. We make precise the distinction and connections between affine and projective geometry to clarify that while the two situations are distinct they behave the same with respect to purity of the Desargues propostion.

In this subsection we formulate some formal systems for geometry.We have slightly modified the statement of the axioms for projective planes from [19] and for affine geometry from [20]. A crucial distinction is that the axioms given here for a projective plane actually imply the structure is planar; any two lines intersect.

Definition 2.2.1 *A projective geometry is a structure for a vocabulary with one binary relation R. We interpret the first coordinate to range over points and the second to range over lines. The axioms for a projective* plane *assert:*

1. *Any two lines intersect in a unique point.*
2. *Dually, there is a unique line through two given points.*
3. *There are four points with no three lying on a line.*

These axioms are far from complete; analogous axioms for an affine plane assert:

Definition 2.2.2

1. *There is a unique line through two given points.*
2. *There are four points with no three lying on a line.*

We could extend Hilbert by passing from the informal axiomatization of the Grundlagen to fully formalized axiomatizations based on the conventions for first order logic not fully established until some thirty to forty years years after his work on geometry. While we don't write out the axioms symbolically, the translation is clear. We could introduce predicates for points and lines or we could simply insist that the domain and range of the relation R do not intersect (a 0th axiom) and then define them to be respectively the set of points and the set of lines. Note that the axioms for spatial geometry in [19] and [20] explicitly introduce a new primitive term: plane.

Following [10] by the 'high school parallel postulate' we mean the assertion: for any line ℓ and any point p not on the line, there *is a unique line* ℓ' through p and parallel to ℓ[15].

[15] Euclid proves the existence of a parallel line on the basis of his first four axioms; in this context, the 5th postulate asserts uniqueness. In the context of projective plane geometry existence fails. See [10] for an amusing and informative account of professional confusion over the difference between existence and uniqueness of parallel lines and the actual content of Euclid's fifth postulate.

Remark 2.2.3 There is an easy translation between projective and affine geometry.

Given a projective plane $\mathcal{P} = (\Pi, R)$, eliminate one line, ℓ, and all points that lie on it. Now two lines ℓ_1, ℓ_2 whose intersection point was on ℓ are parallel. It is easy to see that the affine plane satisfies the parallel postulate.

Similarly, suppose $\mathcal{A} = (\Pi, R)$ is an affine plane satisfying the high school parallel postulate. Add a new line ℓ_∞ and let all members of an equivalence class of parallel lines in Π intersect at a point on ℓ_∞; let these be the only points on ℓ_∞.

Definition 2.2.4

1. *The affine Desargues proposition asserts: if ABC and $A'B'C'$ are triangles with $AC \parallel A'C'$, $AB \parallel A'B'$ and $BC \parallel B'C'$ or intersect then AA', BB' and CC' are parallel.*

2. *The projective Desargues proposition asserts: if ABC and $A'B'C'$ are triangles such that the points of intersection of AC with $A'C'$, AB with $A'B'$ and BC with $B'C'$ are collinear then AA', BB' and CC' intersect in a point p.*

The version in [20] (Theorem 53) [16]) conflates the two by writing what I have given as the affine form with the conclusion that the lines are parallel or all three intersect in the same point.

It is easy to check:

Claim 2.2.5 *Under correspondence in Remark 2.2.3, the affine plane satisfies affine Desargues if and only if the projective plane satisfies projective Desargues.*

Hilbert introduced a ternary betweenness relation to fill what he regarded as gaps in Euclid; this extension is essentially irrelevant to our discussion here. While betweenness is appropriate for affine geometry, to consider both affine and projective geometry requires the more general quaternary separation predicate introduced by Pasch (See e.g., [3].) representing that the four points are cyclically ordered. Unlike betweenness this relation is projectively invariant. For the axioms of this relation see e.g. [19]. Notably, neither betweenness nor cyclic order appears in [19] until after the discussion of Desargues and coordinatization.

2.3 General schemes for characterizing purity

In this subsection, we discuss several suggestions for more clearly specifying the notion of a 'pure' proof, consider how they evaluate the purity of certain arguments, and then draw some conclusions about these specifications.

Detlefsen and Arana distinguish a notion of *topical purity*. Rephrasing their discussion in Section 3.4 of [12], there are certain resources which *determine* a problem (for a given investigator). In mathematics, the determinants include, definitions, axioms concerning primitive terms, inferences. These are referred to as the 'commitments of the problem' and specify what we call the context of the problem. The topic of a problem is a set of commitments. 'A purity constraint restricts the resources available to solve a problem to those which determine it. They then analyze the topical purity of a solution

[16] In the 1962 Open Court printing of Townsend's translation of the first edition, Theorem 32 is the converse and Hilbert writes this version in the text.

in terms of its stability under changes in the commitments. We want here to connect topical purity with several related notions considered by Arana [1, 2]. Our general conclusion (as Arana's) is that it is not possible to translate the problem of purity into a proposition about formal systems in the most traditional sense; it is essential to retain a notion of 'meaning' in the discussion. Our argument can be seen as a model theoretic analogue of the proof theoretic discussion in [2].

Arana introduces a notion of logical purity in [1].

Definition 2.3.1 (Logical Purity A) *1. The axiom set S is logically minimal for P if $S \vdash P$ but there is no proper subset of S proves P.*

2. The proof of P is pure if it is a proof from an S which is logically minimal for P.

He points out that there are some obvious difficulties with this definition, since we could conjoin a set of axioms and get something that is logically minimal. Here is a more robust formulation.

Definition 2.3.2 (Logical Purity B) *1. The axiom set S is fully logically minimal for P if $S \vdash P$ and there is no S' such that $S \vdash S'$, $S' \vdash P$ and $S' \not\vdash S$.*

2. The proof of P is pure if it is a proof from an S which is fully logically minimal for P.

The difficulty with the second formulation is that it turns out to be an even stronger version of the following notion of Arana. [1].

Definition 2.3.3 (Strong logical purity) *The proof of P from S over T is* strongly logically pure *over some basis theory T if also $T \vdash P \to S$.*

As, S can only be minimal in the sense of Definition 2.3.2 if S is logically equivalent to P; otherwise choose P as the S' to show S is not minimal. Thus the existence of logically pure proof of P from S in the sense of Definition 2.3.2 requires that S and P are logically equivalent.

Strong logical purity has a long logical history including Sierpinski's equivalents of the Continuum hypothesis in the 20's, Rubin and Rubin's 101 equivalents of the axiom of choice and Friedman's reverse mathematics. Pambuccian [36] pursues a similar 'reverse geometry', finding a minimal weak axiom system for four results in Euclidean geometry. These are searches for the weakest hypotheses in terms of proof-theoretic strength. They are *not* what Hilbert or Hallett claims for the Desargues property. And as Arana rightly points out 'reverse mathematics' is not an issue of purity. Each of these notions of logical purity, which I have just described, is about equivalence of statements not about proofs; none of them address the key issue: *'Which, if any, proof of a theorem is pure?'*.

Nevertheless, these *formal* notions are capable of detecting the non-existence of pure proofs.

Proposition 2.3.4 *No proof of Desargues proposition from the assumption of three dimensions is strongly logical pure.*

If there were such a proof, every geometry satisfying the Desargues proposition would actually be three dimensional. This is clearly false; we investigate the subtly different consequence (embedability) of the Desargues proposition in Subsection 2.4.

The notion of topical purity builds on an earlier formulation of Arana, 'a proof ... which draws only on what must be understood or accepted in order to understand that theorem.' (Page 38 of [1].)

Two issues arise. What does it mean to draw on? How can one determine 'must be understood or accepted to understand'. We follow Arana in leaving the second question to individual cases. But there is a more uniform way to understand 'draw on'. We say that a 'concept' is drawn on in a proof when it is given a name. We are going to discuss arguments below which could be formalized as derivations in a basic language. We first observe that introducing relations that are *not* definable in the base language is a definite sign of impurity. But we argue more strongly that an explicit definition may violate purity concerns. We will discuss what we claim are 'pure' and 'impure' proofs of the same fundamental result, explaining the reasons for this diagnosis. And then argue for the value of each proof.

In earlier sections we avoided the word language because of its multiple meanings in related contexts. We now introduce a specific meaning clarifying one of the three discussed in Subsection 1.3. We seek now a more mathematical formulation of the topical purity introduced in [12][17].

Definition 2.3.5 *The* language *for a mathematical topic A is a vocabulary for A with symbols for each of the primitive notions identified by the investigator and axioms for the relations which are sufficient to delimit these concepts in the specified context.*

Some examples are the language $(+, \cdot, 0, 1)$ for rings (where the ring axioms are specified) and the language $(\vee, \wedge, 0, 1)$ for bounded lattices (with appropriate axioms). The vocabularies differ only in notation; this difference is meaningless without the description of the properties.

Definition 2.3.6 (Topical purity) *Choose a first order formalization for the resources which determine the problems in the sense of the first paragraph of this section (Detlefsen-Arana). That is specify a language including a set of primitive concepts and axioms needed to describe the particular problem and its context A.*

More formally, fix a vocabulary τ and a theory T_0 that implicitly defines the concepts named by the symbols τ. Now a topically pure *proof of ϕ from ψ where ϕ and ψ are τ-sentence is a proof of ψ from ϕ in T_0 that* invokes *only concepts from the topic A.*

We will interpret 'invoke' in Definition 2.3.6 as 'introduce by explicit definition'.

The choice of first order formalization is a real choice. In [35, 37] Pambuccian provides examples distinct interpretations of the basic notions of a proposition which lead to distinct, indeed incompatible, systems; each system can be thought to provide a pure proof for the understanding of the concepts that has been formalized. [37] studies the Sylvester-Gallai theorem: If the points of a finite set are not all on one line, then

[17] Arana suggested the specification was close enough to topicality to not deserve a new name.

there is a line through exactly two of the points. One might conceive of a line in terms of betweenness or as ‘the shortest distance between two points’. These provide different contexts; Pambuccian explains three distinct proofs, one using the first concept and two the second. These are based on incompatible axioms systems. He remarks that still another proof holds for planes satisfying a certain Artin-Schreier condition.

We insist that the base language should reflect both the context of the topic and the particular problem. If not, formalizations of [20, 19] would be impure for studying Desargues theorem because they take‘plane’ as a primitive concept. And Desargues’s theorem is about lines, points and incidence. But ‘plane’ is part of the context and it is not important whether it is taken as a primitive or introduced by explicit definition.

A natural question is whether this notion is different from the notion of logical purity described above. To show it differs from strong logical purity, we need only exhibit a proposition which has both topically pure and impure proofs from the same base theory. We will note this in Corollary 2.4.7.

It is tempting to insist that the analysis of the context and conclusion should elicit all relevant concepts and thus the set of concepts used in the proof should be fixed in the choice of language.

Sobociǹski [44] discusses this criteria on a formal system, primarily in the context of propositional logics. Givant and Tarski [16] argue that including defined concepts in an axiom system leads to a misleading appearance of simplicity of the axioms. They discuss simplicity in terms of both the ‘length’ of the axiom system and it complexity in terms of the number of quantifier alternations. While the second has important structural consequences, as we discussed in Subsection 1.2, either measure of simplicity appears to irrelevant to the notions of purity considered here. Both relate to technicalities of the formalization.

We reject this criteria primarily because it is not true to mathematical practice. Mathematical proofs are not carried out as derivations in a fixed formal language. In particular, new concepts are introduced by definition for the purpose of particular proofs. We explore examples of this type of extension in detail in Subsection 2.4. But a simple example is to consider a formulation of projective geometry [39] (see the Appendix) that contains only points, lines, and incidence as primitive terms. In order to carry out the proof that Desargues theorem holds in three-dimensional space, one needs the notion of plane. And it is straightforward (Definition 4.3.13) to introduce the notion of plane as an explicit definition in this system.

If these new definitions are mere abbreviations it seems they should be harmless. Certainly if axioms are added about the new relations, this is no longer harmless (See Example 2.1.1.). In fact, we will argue that, even without additional axioms, explicit definitions can violate purity. That is why we add the requirement that the new definitions remain within context of the original topic. We will illustrate the meaning of this phrase in Subsection 2.4.

The first use of our characterization of topical purity is to determine cases where there is no topically pure proof of a proposition. Here is an example. We generalize Proposition 2.3.4 from ‘strongly logically pure’ to the much broader notion of topical purity.

Proposition 2.3.7 *There is no topically pure proof of the Desargues proposition in the* *plane.*

Clearly the Desargues proposition is stated in terms of points, lines, and incidence. Thus for projective or affine geometry, the existence of a topically pure proof would entail that every model of the axioms in Definition 2.2.1 is a Desarguesian projective plane. Many counterexamples to this assertion have been exhibited in the last 100-odd years.

This claim is controversial. In particular, it has been argued[18].

Remark 2.3.8 *Counterclaim: Affine Desargues can be proved from planar axioms so there is a pure proof in the plane*

To evaluate this claim, we first clarify the mathematical situation. In [20], Hilbert proves two mathematical results:

Fact 2.3.9

1. *In three dimensional (affine or projective) geometry, Desargues theorem holds. This depends only on the incidence and order*[19] *axioms.*
2. *In two dimensions, the affine Desarguesian theorem can be proved from the incidence axioms, the parallel axiom, and the congruence axiom.*

We showed in Example 2.1.1 that adding additional relations and structure can change the interpretations of the basic structure. Here, we note that the problems arise in the specific geometric context. It is true that affine Desargues can be proved from the parallel postulate and congruence axioms (basically side-angle-side) and these are surely planar concepts. But while parallelism is necessary to understand affine Desargues, congruence is not. The proof of Fact 2.3.9.2 requires extensions of the basic geometric axioms in two distinct ways. First there are additional axioms in the same vocabulary, the parallel postulate. But secondly a new concept of congruence must be introduced; in fact several of them. A priori, one needs relations for segment congruence (4-ary), triangle congruence (6-ary) and angle congruence (perhaps formulated as a four-ary predicate on lines). And a congruence axiom such as SAS must also be posited. The fact that additional axioms are introduced is immediate evidence of impurity. In fact congruence is definitely foreign to the situation as the theorem of Desargues holds in an affine plane over any algebraically closed field. There is no notion of congruence definable in the geometry over such fields (consider the Riemann mapping theorem). To define congruence one must introduce further relations, e.g., regard the complexes as a two dimensional real vector space). Thus, in the spirit of [12], there can be no topically pure proof of the Desargues theorem in the plane, even for affine geometry.

The basic point here is that two distinct notions of geometry are being considered. Metric geometry (or in the Euclidian formulation, geometry with parallels and congruence) is a different subject than projective geometry which encodes only the properties of lines and incidence. But in fact there is no pure proof in the context of affine metric

[18] See sections 4.1 and 4.7 of [3]

[19] In fact, the order axioms are a red herring. They are used only to guarantee that the coordinatizing field is ordered. See Bernays Supplement IV in [20].

geometry, because the congruence axioms require 'flatness'; Desargues theorem fails in various non-Euclidean geometries. But this illustrates an important attribute of the search for pure proofs. It forces the clarification of hypotheses.

The next section explains that although the definition of topical purity allows the introduction of new terms by definition, this introduction is restrained by the informal context A. We will see that without this restriction proofs using manifestly impure notions would meet the requirement for topical purity.

2.4 Purity and the Desargues proposition

As reported by Hallett in (page 227 of [17]), Hilbert argues that although the content of the Desargues proposition is manifestly two-dimensional; three dimensional methods are necessary for its proof. We explore the role of vocabulary versus axioms in understanding this claim. We first want to formalize Hilbert's results on the strength of planar Desargues.

Notation 2.4.1 *1. PG is the theory of projective geometry (asserting the existence of at least 3 dimensions) and PP is the theory of projective planes as in Definition 2.2.1.*

2. Let Σ be the collection of sentences σ about projective planes (i.e. satisfied in some projective plane) such that $PG \vdash \sigma$. I.e., $\sigma \in \Sigma$ just if σ is true in at least one projective plane and in every projective geometry of dimension at least 3.

Hilbert(page 220 of [17] or page 240 of [21]) conjectured and later proved two results which establish the pivotal role of the Desargues theorem from a geometrical standpoint.

> Is Desargues Theorem also a sufficient condition for this? i.e. can a system of things (planes) be added in such a way that all Axioms I, II are satisfied, and the system before can be interpreted as a sub-system of the whole system? Then the Desargues Theorem would be the very condition which guarantees that the plane is distinguished in space, and we could say that everything which is provable in space is already provable in the plane from Desargues.

Using Notation 2.4.1, we formulate the two assertions of this quote in modern terms.

Theorem 2.4.2 (Hilbert) *1. If Π is a Desarguesian projective plane, Π can be embedded in three-space.*

2. If $\psi \in \Sigma$ then (PP + Desargues) $\vdash \psi$.

We place the situation in a more general framework. Let T_1 and T_2 be two extensions in the same vocabulary of a theory T. For a formula $\theta(\mathbf{y}, x)$ a tuple $\boldsymbol{a}$ and a sentence ψ, $\psi^{\theta(\boldsymbol{a},x)}$ denotes the relativization of ψ to $\theta(\boldsymbol{a}, x)$.

Definition 2.4.3 (Context) *Suppose there is a formula $\theta(\mathbf{y}, x)$ such that*

1. *For every $\psi \in T_2$, $T_1 \models \forall \mathbf{y} \psi^{\theta(\mathbf{y},x)}$*
2. *For every ψ, $T_1 \models \forall \mathbf{y}\mathbf{y}'[\psi^{\theta(\mathbf{y},x)} \leftrightarrow \psi^{\theta(\mathbf{y}',x)}]$*
3. *If $M \models T_2$ and $M' \models T_1$ and $M \subseteq M'$, there is an $\boldsymbol{a} \in M'$ such that $M = \phi(M', \boldsymbol{a})$.*
4. *If $T_2 \cup \{\psi\}$ is consistent and $T_1 \models \psi$ then $T_1 \models \forall \mathbf{y} \psi^{\theta(\mathbf{y},x)}$*

Lemma 2.4.4 *Let Σ be the collection of sentences that are consequences of T_1 and consistent with T_2. Then Σ is contained in the consequences of T_2.*

Proof. Fix $\sigma \in \Sigma$. Let $M \models T_2$, then M extends to a model M' of T_1, so $M' \models \sigma$. By 4) each instance $\phi(\boldsymbol{a}, x)$ satisfies σ; in particular, by 3) $M \models \sigma$. Now by the extended completeness theorem $T_1 \vdash \sigma$. $\square_{2.4.4}$

Proof of 2.4.2: Taking T_2 as $PP + Desargues$ and T_1 as PG, 2) is immediate from 1) by Lemma 2.4.4. We discuss 1) at length and prove it in the appendix. $\square_{2.4.2}$

Thus if θ is a sentence about projective planes that we show in a formalism-free way to be true in every plane that can be embedded in three-space, then θ can be formally derived from the Desargesuian property. The Pappus theorem is an example of a statement concerning projective planes, which is *false* in some planes that can be embedded in three-space.

Hilbert's analysis of the quality of a proof extends beyond topical purity. He wrote, (in unpublished notes of Hilbert that are quoted in [17]).

> Nevertheless, drawing on differently constituted means has frequently a *deeper and justified* ground, and this has uncovered beautiful and *fruitful relations*; e.g. the prime number problem and the $\zeta(x)$ function, potential theory and analytic functions, etc. In any case one should never leave such an occurrence of the mutual interaction of different domains unattended.

The role of 'spatial assumptions' is better seen by a more careful examination of Hilbert's proof of Fact 2.3.9 and Theorem 4.0.9. He begins [20] by noting that the three-dimensional proof of Desargues theorem (Fact 2.3.9.1) from the axioms of connection, order and parallels is well-known. The structure of his proof of each of Fact 2.3.9.2. and Theorem 4.0.9 follows that pointed out for the proof of embeddability from Desargues on page 228 of [17]. A ternary field is a structure with a single ternary operation; roughly $t(a, x, b)$ corresponds to $ax + b$, which satisfies a set of axioms as specified in [11, 24]. But for this correspondence to be literally true the plane coordinatized by the ternary field must satisfy the Desargues property [20].

[20] In [6] I constructed a non-Desarguesian projective plane which is $\aleph_1$-categorical. In [7], I prove that despite its well-behaved nature from a model theoretic standpoint, this plane admits little 'algebraic' structure; in particular the ternary operation can not decomposed into two well-behaved binary operations and no group is interpretable in the structure. I also proved this projective plane is in the definable closure of any line (with no parameters) That is, the plane admits no perspectivities. The task of giving a geometric proof of this result remains open.

Remark 2.4.5 *The structure of the argument:*

1. Any *geometry can be coordinatized by a ternary field.*
2. *If the geometry satisfies*
   (a) *the Desargues proposition or*
   (b) *the parallel postulate and SAS (the congruence axiom in Hilbert's parlance)*

   then the coordinatizing ring is associative (and in fact a skew field[21]).
3. *An n-dimensional affine (projective) geometry can be constructed as a set of n $(n+1)$-tuples from a Skew field and the plane can be embedded in the three-space.*

This proof from [20] introduces a different set of purity concerns. In Hilbert's argument, a field is defined whose elements are equivalence classes of segments. These are not geometric notions and the objects are not in the model but are what model theorist now call 'imaginary [32] elements'[22]. This objection is somewhat reduced by Heyting's proof. Heyting still defines a field, but its elements are points of the given plane. Even if we have the fields as the points on a line, the construction of the three dimensional model goes far afield from geometry. These new objects do not have 'geometric interpretations'. The modern geometry of homogenous quadruples is employed in the construction. This is essentially a metamathematical argument constructing a three-space out of whole cloth and embedding the original plane in it. This seems to be a really new method introduced by Hilbert[23]. It is very different from Hilbert's geometric construction of counterexamples to Desargues or the geometric argument for Theorem 4.0.9 given by Levi [29] or in the appendix. At the least it is a precursor of the modern notion of the interpretation of one theory in another. Moreover, Hilbert's proof of Desargues in an affine plane (Proposition 2.3.9.2) with congruence also goes through this metamathematical trick of embedding in three-space and deducing the result from the known proof of Desargues in 3-space. In fact, as pointed out in [3], Desargues gave a geometric proof of his theorem (in three dimensions) in the affine case using the theorem of Menelaus.

Consideration of some of the standard texts in projective geometry of the last half century [4, 11, 24] reveals an interesting phenomena. The proof of Desargues proposition is at best barely mentioned[24]. The crux is the understanding of the Desargues proposition in terms of the properties of the group of collineations and in terms of the properties of the coordinatizing ternary ring.

[21] A skew field or division ring is a structure for the vocabulary $(+, \times, 0, 1)$ which satisfies all the axioms for a field except commutativity of multiplication.

[22] This is a deliberate decision of Hilbert so as to study the geometry of segments. Already in his 1893-1894 lectures he had established a correspondence between the points on a line and numbers. See pages 68-69 of [21].

[23] This was remarked by Hallett[17].

[24] Hartshorne, [18], is an exception.

Our use of ‘metamathematical’ in the previous paragraph has two senses. Metaphorically, Hilbert is constructing a model and so this is a precursor of model theory. But he has also given a ‘formalism-free’ proof of Theorem 4.0.9.2. (I.e. with the conclusion expressed as in the quotation before Theorem 4.0.9.) But as we noted in proving Theorem 4.0.9.2, this formalism-free proof translates to the existence of a formal proof by the extended completeness theorem. (Of course, this translation was not available to Hilbert in 1900.)

As we noted in Remark 2.4.5 and Proposition 2.3.4, the Desargues proposition does not imply there are non-coplanar points. Thus, it is not true that the Desargues proposition implies there is a third dimension. Rather, Hilbert showed, by a *fundamentally non-geometric construction*, one can embed the given plane in three-space. But there is a ‘geometric’ construction of this embedding, which we present in the appendix. Thus we have an example where an impure proof provides very significant information. Indeed the very impurity of Hilbert’s argument is crucial for the 20th century development of the theory of plane projective geometry. In fact, this may be mathematical impact of a proof of impurity. It focusses attention on the proposition in question as an axiom for selecting a new field of study. For example, the fact that there is no pure proof of the Desargues proposition in the plane calls attention to the importance of studying Desarguesian planes. In fact, the crucial property, as Hilbert saw, is not the geometric configuration itself but the associated algebraic structure; it was later codified in terms of transitivity properties of the automorphism group (Lenz-Barlotti classification).

In contrast, I claim that the argument for Theorem 4.0.9.1 in the appendix is topically pure. The crucial point is that Hilbert’s argument introduces the notions of coordinatization and field which are foreign to synthetic geometry. In the Appendix, we reinterpret the words, point, line, and plane in terms of certain planar configurations to interpret a 3-space containing π in a Desarguesian plane π but don’t introduce significantly new concepts.

Since Hilbert’s proof is impure, we conclude.

Fact 2.4.6 *The assertion that every Desarguesian plane is embedded in three-space has both topically pure and topically impure proofs from the axioms PP of projective planes.*

Corollary 2.4.7 *The notions of* strong logical purity *and* topical logical purity *differ.*

The Desarguesian proposition is a dividing line in the sense of Shelah (e.g. introduction to [41]). Its truth implies strong coordinatization properties; its failure implies planarity (in an axiom system that is agnostic on dimension). Specifically, the associativity of the coordinatizing field is used to prove that the relation of tuples from the field: $\mathbf{x} \sim \mathbf{y}$ if there is a ‘number’ c with $\mathbf{y} = c\mathbf{x}$ by coordinate-wise multiplication (used to introduce homogenous coordinates) is transitive and thus an equivalence relation. Thus we have:

Fact 2.4.8 *The following are equivalent: A projective plane is*

1. *coordinatized by an* associative *skew field.*

2. *satisfies the Desargues property;*
3. *can be embedded in three-space.*

There is another connection between spatial axioms and associativity. In three-dimensional Euclidean geometry the volume of a cube can be computed. Interpreting XI.32 of [15] in modern language yields the formula $V = \ell wh$. (He proves that the volume of a parallelpiped is determined by the area of the base and the height.) The fact that the geometric notion is independent of which side is chosen as the base of the parallepiped implies the associative law for the coordinatizing field [25].

Finding the associative field is, in modern terms, an interpretation of the 'field' into the geometry. It proceeds by a sequence of explicit definitions. The proof of the algebraic axioms follows from the geometry. And then the plane is interpreted back into the 3-space over the field. Thus if there is any distinction between algebra and geometry this fails to be a topically pure proof. But this conclusion cannot be established by a characterization of purity such as strong logical purity which concerns the mere existence of proof. The failure of topical purity is seen by consideration of the meaning of concepts introduced in the proof: the introduction of the notion of a skew field which is not a geometric notion is decisive.

This illustrates Tait's maxim: The notion of formal proof was invented to study the the existence of proofs, not methods of proof. Or as Burgess [9] puts it, 'For formal provability to be a good model of informal provability it is not necessary that formal proof should be a good model of informal proof.'

We should not ignore the virtues of a demonstration that there is no pure proof. It shows that additional resources are needed for a particular claim. The Desargues proposition is particulary instructive in showing the value of the searching for the content of those additional resources. Hilbert isolated the ability to coordinatize in terms of the Desargues configuration and its connections with the interpretability of division rings. A significant part of 20th-century mathematics, the further development of projective planes, particularly finite projective planes, relied both on the algebraization and on the discovery of the underlying properties of the group of perspectivities of the plane.

Hilbert succeeded in showing a deep connection between algebraic and geometric conceptions by identifying both the algebraic (associativity) and geometric (Desargues proposition) conditions necessary and sufficient for Descartes coordinatization to succeed.

3 Distinguishing Algebraic and Geometric proof

This section is a commentary on the Appendix, arguing that it provides a pure geometric proof of the embedability theorem while Hilbert's proof is manifestly not pure. For this we need some distinction between algebra and geometry. Algebra deals with numbers (of various sorts); geometry deals with magnitudes. Geometric arguments

[25] Serendipitously, this argument was given by Ken Gross in a professional development program for elementary school teachers while I was working on this paper.

admit and (as the writing of the appendix demonstrated) often demand pictures. The distinction is clearly made in the quotation from Newton in [12]; there should be no arithmetical computations except of 'Quantities truly geometrical'. The essence of coordinatization, fundamental to Hilbert's proof of the embedding theorem, is to reject this notion.

Thus in the appendix the crucial vocabulary remains points, lines, and planes. There is no introduction of multiplication and addition and no reliance on the development of coordinate geometry. Crucially, however, lines are introduced as a set of triangles and certain equivalence relations play a significant role. This is a more complex argument than models of non-Desarguesian planes that interpret (pieces) of curves as lines. This level of complexity is implicit in Hilbert.

The number of special cases that appear in the proof below are characteristic of geometric arguments. Algebraic methods (as is clear in the developments of 20th century algebraic geometry) can clarify the notion of a 'generic configuration'. Thus, the coordinatization of a Desargesuian plane requires a not-quite arbitrary choice of coordinate points. In contrast Claim 4.3.12 requires a delicate argument to replace 'arbitrary' points by ones that are in general position.

Our argument makes clear the geometric picture that motivates the coding of points in three-space by triples in the plane. Levi does not bring this out and it is unclear if he had the same picture in mind. In particular, he gave the proof in the affine case and then extended to projective planes on general grounds as in Remark 2.2.3.

4 Appendix: A geometric proof that Desargues implies embeddability with William Howard

Theorem 4.0.9 *A Desarguesian projective plane π can be embedded in a three dimensional geometry.*

Levi's first 'geometric' proof of Theorem 4.0.9 appeared in [29] in 1939. The second author remembers reading a version of the proof but not where [26]. The argument here stems mostly from the second author's reanalysis and differs from Levi by presenting the motivation more clearly and by a quite different application of Desargues. It differs from both Hilbert and Levi in being purely projective with no reference to affine geometry. The crux is that either this argument or Levi's differ from Hilbert's as there is no mention of coordinatization by a skew field. For concreteness, we view projective geometry as axiomatized in [39] and recite the exact axioms later.

4.1 Some plane projective geometry

The following key property of a Desarguesian projective plane drives the main work.

[26] Victor Pambuccian pointed us to an exact reference, [29].

Definition 4.1.1 *Fix a point P and let $\mathcal{S}_P$ denote the set of pairs $\hat{a} = \langle a_1, a_2 \rangle$ such that P lies on the line $a_1 a_2$. For any $\ell \in \pi$, and $\hat{a}, \hat{b} \in \mathcal{S}_P$ such that $\hat{a}$ and $\hat{b}$ do not lie on the same line through P, define R_ℓ by*

$\hat{a} R_\ell \hat{b}$ if and only a_1, a_2 and b_1, b_2 are centrally perspective by a point on ℓ.

Lemma 4.1.2 *For any $\ell \in \pi$,*

1. *If $\hat{a}, \hat{b}, \hat{c} \in \mathcal{S}_P$ lie on distinct lines through P then $\hat{a} R_\ell \hat{b}$ and $\hat{c} R_\ell \hat{b}$ imply $\hat{a} R_\ell \hat{c}$*
2. *R_ℓ is an equivalence relation on any set of pairs which determine distinct lines through P.*

Proof. Suppose $\hat{a}, \hat{b}, \hat{c}$ determine distinct lines through P. Suppose $\hat{a} R_\ell \hat{b}$ and $\hat{c} R_\ell \hat{b}$. Note that the result holds trivially if either $a_1 b_1 c_1$ or $a_2 b_2 c_2$ are collinear. So we may assume the triangles $a_1 b_1 c_1$ and $a_2 b_2 c_2$ are proper; by definition of $\mathcal{S}_P$ they are centrally perspective through P. Note that the points of central perspectivity $O_{\hat{a},\hat{b}}, O_{\hat{a},\hat{c}}, O_{\hat{c},\hat{b}}$ are the intersections of the sides of $a_1 b_1 c_1$ and $a_2 b_2 c_2$. By Desargues' theorem $O_{\hat{a},\hat{b}}, O_{\hat{a},\hat{c}}, O_{\hat{c},\hat{b}}$ are collinear. By the definition of R_ℓ the first and third are on ℓ so all three are. Applying the first part of the lemma multiple times yields the result. $\square_{4.1.2}$

4.2 Motivation

The goal here is a direct construction: Given a Desarguesian plane π, we construct a three dimensional projective space containing π. We will describe incidence in π in English; incidence in the new structure will be given by 'element of'. We use capital Roman letters for the points in a general projective plane, lower case Roman letters for points in the new space (i.e. 'triangles' in π^3) and list the points on the triangle a as a_1, a_2, a_3 in π. Points in π may be labeled by capital Roman letter or subscripted lower-case Roman letters depending on the context. Planes are labeled by Greek letters and lines by lower case Roman letters.

The basic idea is to reverse the following operation. Fix a plane π in three-space. Then map the three-space onto π by fixing a triangle $M_1 M_2 M_3$ in a plane α not equal to π and mapping each P to the triangle, say $\langle a_1, a_2, a_3 \rangle$ that P projects $M_1 M_2 M_3$ onto in π. A little thought shows there are certain subtleties in this idea. Let ω denote the line where α and π meet and let H denote $M_1 M_2 \wedge \omega$, V denote $M_1 M_3 \wedge \omega$ and D denote $M_2 M_3 \wedge \omega$. Any point not in $\alpha \cup \pi$ will project to a proper triangle in π. Points on α have various special forms which are precisely defined as the sets Ω_i in Definition 4.3.1. If P is on α but not colinear with one of the edges of $M_1 M_2 M_3$ it projects to three distinct points on ω (in Ω_0). If P is on $M_1 M_2$ but not M_1 or M_2 it projects to $\langle H, H, s \rangle$ for some $s \in \omega - \{H, V, D\}$ (in Ω_1). The cases of M_1, M_2, M_3 are even more special. Each of them projects to a pair from $\{H, V, D\}$ (into Ω_2). A line ℓ in three-space not in α will project to a 'line' of triangles in π which are in perspective from $\ell \wedge \pi$.

4.3 The Construction

With a little care for special cases and appropriate use of the Desargues theorem we can reverse the map in Section 4.2 and code the three-space. While, especially in footnotes, we return to the picture of the previous paragraph for motivation, formally, we are now defining a three-space by defining in π a set of points (certain triangles) and the notions of line, incidence and eventually plane on those points.

Definition 4.3.1 (The Model: I) *Fix a line ω in the Desarguesian plane π and distinct points H, V, D, called the coordinate points, on ω. $\mathcal{S}$ is a collection of ordered triples from π.*

1. *The set of regular points $\mathcal{S}_r$ in the new three space consists of the proper triangles $a = a_1a_2a_3$ in π such that H is on a_1a_2, D is on a_2a_3, and V is on a_1a_3. We call them* regular triangles*.*

   $\mathcal{S}$ also contains several exceptional cases.

   (a) *degenerate triangles: $a_1 = a_2 = a_3$. That is, $a \in \pi$.*

   (b) *'triangles' on ω: [27]:*

      i. *Ω_0 denotes the set of special triangles, ordered triples $\langle a_1, a_2, a_3\rangle$, of distinct points on ω [28].*

      ii. *Ω_1 is the set of very special triangles, triples of points from ω of the form $\langle H, H, x\rangle, \langle V, x, V\rangle, \langle x, D, D\rangle$ with $x \in \pi - \{H, V, D\}$[29].*

      iii. *Ω_2 denotes the three 'triples' $\langle *, H, V\rangle, \langle H, *, D\rangle, \langle V, D, *\rangle$ where $*$ indicates that one coordinate is undefined[30].*

   $\Omega = \Omega_0 \cup \Omega_1 \cup \Omega_2 \cup \omega$ is called the plane at infinity.

2. *A* regular line *m in the three-space is a set of $a \in \mathcal{S}$ such that for some fixed $0_m \in \pi$ any $a, b \in m$ are centrally perspective[31] with center O_m. Note that O_m is a degenerate point in $\mathcal{S}$ of type 1a) and $O_m \in m$.*

   $\mathcal{L}$ denotes the set of all lines including those entirely contained in Ω and $\mathcal{L}_r$ denotes the set of regular lines (those which contain a regular triangle).

   The lines on Ω are described in Definition 4.4.5.

We will say a special triangle b is centrally perspective with a regular triangle a if the lines a_ib_i (for $i \leq 3$) meet in a point. We write $O_{a,b}$ for the point on π of central perspectivity of a and b. We say two or more points in $\mathcal{S}$ are *regularly collinear* if there is a regular line through them. Suppose $a, b, c \in \mathcal{S}$ are three points that are not

[27] These points corresponding to the images of elements of the plane α.

[28] These points are in central perspective with a regular triangle and are discussed in Remark 4.3.3.

[29] These are also in perspective with a regular triangle; see Remark 4.3.2.

[30] In our prototype, $M_1 \mapsto \langle *, H, V\rangle$, $M_2 \mapsto \langle H, *, D\rangle$, $M_3 \mapsto \langle V, D, *\rangle$. These points lie on the type 3 lines in Remark 4.3.2.

[31] More precisely, $a_1a_2a_3$ and $b_1b_2b_3$ are centrally perspective at $a_1b_1 \wedge a_2b_2 \wedge a_3b_3 = O_m$. There are three special cases, e.g. when a_1a_2 and b_1b_2 are the same line through H in π. In this case, letting s denote $a_3b_3 \wedge \omega$, the very special point $\langle H, H, s\rangle$ is on the line.

regularly collinear and that a is a regular triangle. Then ab and ac are two distinct regular lines.

We begin by observing a natural classification of regular lines in $\mathcal{S}$.

Remark 4.3.2 (Classifying regular lines) Each regular line $r \in \mathcal{L}_r$ (i.e. not on Ω) is given by a triple (r_1, r_2, r_3) of lines in π. (In case 3a below one of the r_i is not defined; all triangles on the line share a point.) r_i goes through the x_i coordinate of the triangles in r. The intersection of the r_i is O_r, the vertex.

type 1 *None of the coordinates H, V, D is on any r_i; the r_i are distinct.*

type 2 *Two of the r_i agree and go through a coordinate point, H, V, or D*[32].

		$r_1 \wedge \omega$	$r_2 \wedge \omega$	$r_3 \wedge \omega$
$2a)$	$r_1 = r_2$	H	H	$\omega - \{V, D\}$
$2b)$	$r_1 = r_3$	V	$\omega - \{H, D\}$	V
$2c)$	$r_2 = r_3$	$\omega - \{H, V\}$	D	D

type 3 *One of the r_i is undefined and so has no intersection with ω. O_r is a vertex of every regular triangle in r.*

	$r_1 \wedge \omega$	$r_2 \wedge \omega$	$r_3 \wedge \omega$
$3a)$	$undefined$	H	V
$3b)$	H	$undefined$	D
$3c)$	V	D	$undefined$

Remark 4.3.3 (Special triangles) Specifying two points on a regular triangle b determines the third and the labeling is determined by e.g. b_1 is the vertex that is the intersection of lines through Hb_2 and Vb_3. We establish the analogous claim for special triangles in Lemma 4.3.4.

Lemma 4.3.4 *Let s_1, s_2 be two points in $\omega - \{H, V, D\}$. Then there is a unique point s_3 on ω so that $s = s_1 s_2 s_3$ is in central perspectivity with a regular triangle $a_1 a_2 a_3$. Thus, $s \in \Omega_0$.*

Proof. Choose any regular triangle $a_1 a_2 a_3$. Let P on π be the intersection of $s_1 a_1$ and $s_2 a_2$. Since the s_i are not in $\{H, V, D\}$, P is not on a line extending a side of $a_1 a_2 a_3$. Choose s_3 as Pa_3 intersect ω. We show that s_3 does not depend on the choice of $a_1 a_2 a_3$. Let $b_1 b_2 b_3$ be another regular triangle and by the same procedure choose a point of perspectivity Q and an $s'_3 = Qb_3 \wedge \omega$ so that $s_1 s_2 s'_3$ and $b_1 b_2 b_3$ are in perspective from Q. By the converse of Desargues, the regular triangles $a_1 a_2 a_3$ and $b_1 b_2 b_3$ are centrally perspective through some point R. Further,

1. $a_1 b_1 s_1$ and $a_2 b_2 s_2$ are centrally perspective by H and

[32] If one thinks of the affine part of the plane the lines in this class are horizontal, vertical, and diagonal in that order. Note the special case when, e.g. in 2a), the intersection point is $\langle H, H, H\rangle$; this is the image of a triangle whose plane contains $M_1 M_2$. Lines of this form contain no special triangles.

2. $a_2b_2s_2$ and $a_3b_3s_3$ are centrally perspective by $a_2a_3 \wedge \omega = D$.

Applying Desargues, 1) yields $a_1b_1 \wedge a_2b_2 = R, b_1s_1 \wedge b_2s_2 = Q$, and $a_1s_1 \wedge a_2s_2 = P$ are collinear. And 2) implies $a_2b_2 \wedge a_3b_3 = R, a_2s_2 \wedge a_3s_3 = P$ are collinear with $b_2s_2 \wedge b_3s_3$. That is $b_2s_2 \wedge b_3s_3$ lies on PR and so must equal Q. But then $Qb_3 \wedge \omega = b_3s_3 \wedge \omega$ and $s_3 = s_3'$. $\square_{4.3.4}$

We make two crucial observations.

Fact 4.3.5 *If a and b are in $\mathcal{S}$ and at most one of them is on Ω, $O_{a,b}$ is well-defined. Every regular line meets Ω.*

Proof. The first sentence is clear from Lemma 4.3.4 when $a \in \mathcal{S}_r$ and $b \in \Omega_0$ has two coordinates from $\omega - \{H, V, D\}$. But even in case $b \in \Omega_1 \cup \Omega_2$, there are two lines a_ib_i and a_jb_j (for some i, j among $1, 2, 3$) which are distinct; their intersection is $O_{a,b}$. If $a \in \pi$ then for any b, $O_{a,b} = a$. The second sentence is easy to check. $\square_{4.3.5}$

We now give an external definition using the tools of our construction of the plane generated by three points, where one is regular. We will show that in $(\mathcal{S}, \mathcal{L}, \epsilon)$ it describes the same notion as the intrinsic Definition 4.3.13.

Definition 4.3.6 (The Model: II) *If $a, b, c \in \mathcal{S}$ are three points that are not regularly collinear and a is a regular triangle, the* plane $\alpha = \langle a, b, c\rangle$ generated by abc *is the union of all lines $\ell_x \in \mathcal{S}$ that pass through a and some point x on the line $\ell_\alpha = O_{a,b}O_{a,c}$ (called the trace of ℓ_α) in π. Formally*

$$\alpha = \bigcup_{x \in \ell_\alpha} \ell_x.$$

Remark 4.3.7 *Since the 7 element Fano plane is the only projective plane with exactly three points on a line and it is coordinatized by the 2 element field, we can assume that each line in π has at least 4 elements. Consequently each regular line in $\mathcal{S}$ contains at least two regular points.*

We want to apply Desargues's theorem to study planes containing regular points. The difficulty is that applications of Desargues theorem to establish the transitivity of the relations R_ℓ from Definition 4.1.1 require that certain triangles not have sides on a common line. From now though Lemma 4.3.12 we see how to adjust situations to avoid this problem.

Definition 4.3.8 *By a* direction *of a regular triangle b we mean any of the lines that extend a side of b to one of $\{H, V, D\}$ (E.g. the line b_1b_2H; to specify the direction we write X-direction for some $X \in \{H, V, D\}$.).*

Definition 4.3.9 *Suppose $a, b, c \in \mathcal{S}$. We say a* pair *of regular triangles a, b are* aligned *if they share a direction.*

No regular triangle is aligned with a special triangle. Any pair of elements of Ω is aligned.

We say three triangles a, b, c are askew *if there is a direction in which no pair of them is aligned.*

We will need some stronger variants on Lemma 4.1.2. First we extend the relation R_ℓ from pairs to triangles.

Definition 4.3.10 *For any $\ell \in \pi$, and $a, b \in \mathcal{S}$ such that $O_{a,b}$ exists define R_ℓ by $aR_\ell b$ if and only $O_{a,b} \in \ell$.*

Lemma 4.3.11 *Suppose $a, b, c \in S$, and at most one is in Ω, then R_ℓ is transitive on $\{a, b, c\}$. R_ℓ is transitive on any finite set of elements in $\mathcal{S}$ with at most one on Ω and the rest in S_r.*

Proof. Suppose a, b, c are all aligned in a direction, say D for simplicity. Then $a_1, a_3, b_1, b_3, c_1, c_3$ lie on a line ℓ and each of $O_{a,b}, O_{a,c}, O_{b,c}$ must lie on that line. Assuming this case does not occur, we prove.

Claim 4.3.12 *We can choose d such that*

1. *for two of the triangles (say a, c) $O_{a,d} = O_{c,d} = O_{a,c}$;*
2. *a, d, b and b, c, d are askew.*

Proof of Claim: If one of the points is in Ω, a, b, c are automatically askew. So we consider the case where all are regular. There are two cases.

First suppose that two of the triangles are aligned in two directions. Fix these as b, c and suppose they both D-aligned and V-aligned. (The choice of names of the triangles in the statement of Claim 4.3.12 depend on the choice made here.) Then $c_3 = b_3$ and since the three are not X-aligned for any X, if there is any alignment with a, it is along H. Suppose a and c are H-aligned. Choose d_3 on c_3a_3 not equal to either of them. Then let $d_1 = d_3D \wedge a_1c_1$ and $d_2 = d_3V \wedge a_2c_2$. Now neither d, b, a nor d, b, c is D-aligned.

The second case is when no pair is aligned in two directions but perhaps each pair is aligned. Suppose for example, that a_1, a_2 and b_1, b_2 are collinear. By our case analysis, we may suppose that a, c are not both D and V aligned. For concreteness, say they are D aligned. And we may also assume that $a_2 \neq c_2$ as equality would imply that a and c are H-aligned. If they were, c_1 would be on both a_1a_3D and a_1b_1H so $a_1 = c_1$ and we would have all three H-aligned contrary to the first line of the proof of Lemma 4.3.11. Thus, $a_3 \neq c_3$ and $a_2 \neq c_2$. Choose regular d_1 on $a_1c_1O_{a,c}$ but not among a_1, c_1 or $O_{a,c}$. Now let d_2 be $Hd_1 \wedge a_2c_2$ and d_3 be $Dd_2 \wedge a_3c_3$. Both of these points are defined

The construction guarantees that c, b, d are not H-aligned and b, a, d are not aligned in the D (i.e. a_2a_3) direction. $\square_{4.3.12}$

This implies that R_ℓ is transitive on any triple of elements in $\mathcal{S}$ with at most one on Ω and the rest in S_r. For $a, b, c \in S_r$, the result follows from Lemma 4.1.2 applied to the two triples in 2) along the non-aligned direction. That is, suppose $aR_\ell c$ and $bR_\ell c$. We have $O_{c,d} = O_{a,d}$ and $O_{b,c}$ are both on ℓ so applying Lemma 4.1.2 in the H -direction to c, b, d $O_{b,d} \in \ell$, i.e $bR_\ell d$. We have $bR_\ell d$ and by construction $aR_\ell d$, so applying the Lemma 4.1.2 in the D-direction on a, b, d, we deduce $aR_\ell b$. If one of the points is on Ω, the result is easier as we don't have to do so much adjustment to apply Lemma 4.1.2.

For the second sentence, suppose we are given a set A of n regular triangles $a_1, \dots a_n \in S_r$ and $a_{n+1} \in \Omega$. It suffices to show R_ℓ is transitive on each three element subset of A. We can establish this by applying part 1) independently to each such triple. $\square_{4.3.11}$

Robinson [39], whose axiomatization we are using, unlike Hilbert, does not take *plane* as a primitive but makes the following intrinsic definition.

Definition 4.3.13 *The plane generated by a, b, c is the collection of all points on lines that contain a and intersect bc.*

We show in Lemma 4.3.18 that our external notion of plane, Definition 4.3.6, coincides with the intrinsic definition in the language of the geometry that we give now. For this, we need the consequences of Desargues we have just developed and work extensively with the extrinsic definition until proving the equivalence in Lemma 4.3.18.

Lemma 4.3.14 *Suppose a, b, c are distinct points in $\mathcal{S}$ and a is a regular triangle. If d is a regular triangle on the* plane *α generated by abc (ℓ_α and a) in the sense of Definition 4.3.6, then α is also generated by dbc (by ℓ_α and d) in the same sense.*

Proof. Suppose the trace of α on π is $\ell = \ell_\alpha$. If $b, c \in \Omega$, each of the triples a, b, d and a, c, d is askew. In particular there is a direction X such that the three lines through X extending sides of a, b, d are distinct. If at most one point is in Ω we apply Lemma 4.3.11 to guarantee the triangles are askew. For any of these points u in $\mathcal{S}$, let $\hat{u}$ denote the restriction to the relevant pair in $\mathcal{S}_X$. We have $\hat{d}R_\ell\hat{a}$ and $\hat{b}R_\ell\hat{a}$. Hence we can apply the Lemma 4.1.2.1 for $\mathcal{S}_X$, to conclude $\hat{d}R_\ell\hat{b}$. Similarly, $\hat{d}R_\ell\hat{a}$ and $\hat{c}R_\ell\hat{a}$ imply by Lemma 4.1.2.1, $\hat{d}R_\ell\hat{c}$. So $O_{d,b}O_{d,c} = \ell_\alpha$ so $\alpha = \langle d, b, c\rangle$ and α is generated by ℓ_α and d. $\square_{4.3.14}$

Lemma 4.3.15 *If $\alpha = \langle a, b, c\rangle$ is generated by abc, where a is regular, then there is a $k \in \{1, 2, 3\}$ such that*

1. *For every regular point $d = \langle d_1, d_2, d_3\rangle$ in α, d_k is not on ℓ_α.*
2. *$s_k \wedge \omega$ is defined for the line through bc, which is given by $s = \langle s_1, s_2, s_3\rangle$.*
3. *For every regular line $s = \langle s_1, s_2, s_3\rangle$ in α, s_k is defined.*

Proof. Note first that some coordinate a_i of a, must not be on ℓ_α since a is proper triangle in π. We will show that in this case no regular triangle d in α has d_i on α. For simplicity, suppose $i = 1$. Suppose for contradiction some $d \in \alpha$ has d_1 on ℓ_α. Then $d_1 = O_{a,d}$ so d_1d_2H, $d_1a_2d_2$ and thus $d_1d_2a_2H$ are collinear in π. But then since a_1a_2H is a line, a_1 is also on this line. Similarly $d_1d_3a_3a_1V$ is also a line. But they meet in both a_1 and d_1. So $a_1 = d_1$ contrary to a_1 not on ℓ_α.

For the second claim, continue with the case $k = 1$. Suppose the line $s = \langle s_1, s_2, s_3\rangle \in \alpha$ is through b, c, but s_1 is not defined. Then $b_1 = c_1$. But $O_{a,b}$ lies on a_1b_1 and $O_{a,c}$ lies on a_1c_1, so $b_1 = c_1$ is on ℓ_α contradicting part 1.

By Lemma 4.3.14, we can choose any d on s and assume α is generated by d and bc, yielding iii) from ii). $\square_{4.3.15}$

We need a bit more information about the composition of lines.

Lemma 4.3.16 *Let $r = \langle r_1, r_2, r_3 \rangle$ be a line with $r_k \wedge \omega$ defined, say $k = 1$. For any point b_1 on π, there is a unique triple $\langle b_1, b_2, b_3 \rangle$ on r.*

Proof. By the inventory of lines, Remark 4.3.2, we can choose a k, (say 2 for concreteness) so that $r_2 \neq r_1$. Consider first the case that r is of type 1 or type 2. Then, again by Remark 4.3.2, H is not on r_1 nor r_2. Let $b_2 = b_1 H \wedge r_2$. If b_1 is not on ω then neither is b_2 and $b_3 = r_3 \wedge b_2 D$. Then H is not on either r_1 or r_2. If b_1 is on ω then so is b_2. But $r_2 \wedge H b_1 = b_2$ so neither b_1 nor b_2 is H. By Lemma 4.3.3, we can choose b_3 as required. If r is of type 3, we are in case Remark 4.3.2.3c and $\langle b_1, b_2, b_3 \rangle = \langle V, D, * \rangle$. $\square_{4.3.16}$

Lemma 4.3.17 *If a, b, c are regular, any two regular lines in the plane α generated by abc in sense of Definition 4.3.6 intersect.*

Proof. Let r and $s = \langle s_1, s_2, s_3 \rangle$ be two such lines. Lemma 4.3.14 allows us to assume a is on r and so r has the form $y'a$ for some y' on ℓ_α. We need to show that $y'a \wedge s$ is non-empty. By Lemma 4.3.15.2, there is a k such that s_k is defined. For simplicity, assume $k = 1$. Let $d_1 = y'a_1 \wedge s_1$. By Lemma 4.3.16 there is a unique triple $d = \langle d_1, d_2, d_3 \rangle$ on s. We claim d is also on $y'a$. By the definition of α, O_{da} lies on ℓ_α. But d_1 is on $a_1 y'$ so $O_{da} = y'$. Thus $d = y'a \wedge bc$ as required. $\square_{4.3.17}$

Lemma 4.3.18 *If a, b, c are regular, the plane α generated by abc in sense of Definition 4.3.6 is the same as that given by Definition 4.3.13:*

$$\alpha = \bigcup_{x \in bc} \ell_{xa}$$

where $\ell_{xa} \in \mathcal{L}$ is through the regular triangles x and a.

Proof. Let d be a regular triangle on bc. We must show $O_{a,d}$ lies on $\ell_\alpha = O_{a,c} O_{a,b} \in \pi$. By Lemma 4.3.11, $O_{b,c}$ is on ℓ_α. Since d is on bc, $O_{b,d} = O_{b,c} \in \ell_\alpha$. By Lemma 4.3.11 again, $O_{a,d}$ is on ℓ_α as required.

For the converse, let $y \in \alpha$. Then y is on a line $y'ay$ with y' on ℓ_α and we need to show that $y'a \wedge bc$ is non-empty. Apply Lemma 4.3.17 with $r = y'a$ and $s = bc$. $\square_{4.3.18}$

4.4 The plane at infinity

The points of Ω are a very special plane in $\mathcal{S}$. In this subsection, we study the lines on this plane and conclude with Lemma 4.4.7 verifying that Ω is in fact a plane in the sense of Definition 4.3.13.

Remark 4.4.1 Note that if b, c are regular triangles, the point $O_{b,c} \in \pi$ is on every plane containing b, c. Moreover if b, c are in each of two planes α and β the line through b, c is the intersection of α and β. As, if there were some regular d in the intersection but not on that line, Lemma 4.3.14 and the argument in the first paragraph of Lemma 4.4.2 shows $\alpha = \beta$.

We need to describe the line through two points on Ω: some effort is needed to define these lines. The key is to establish Remark 4.4.1 for special triangles.

Lemma 4.4.2 *Suppose $b, c \in \Omega$. There is a point t on ω such that t is in every plane α that contains a regular point of $\mathcal{S}$ and b, c.*

If α is not contained in Ω, $\{t\} = \alpha \wedge \omega$.

Proof. Consider two distinct planes α and β which both contain b, c. Say $\alpha = \langle a_1, b, c\rangle$ and $\beta = \langle a_2, b, c\rangle$. Then $\ell_\alpha \wedge \ell_\beta = t$ is a point (degenerate triangle) on π. If t is not on ω, choose a regular triangle t' on tb in $\mathcal{S}$. By two applications of Lemma 4.3.14 (replacing either a_1 or a_2 by t'), $\alpha = \langle a_1, b, c\rangle = \langle t', b, c\rangle = \langle a_2, b, c\rangle = \beta$. So any plane γ containing b, c meets α in $\ell_\alpha \wedge \ell_\gamma = \{t\}$.

If α is not contained in Ω, we can generate α as $\langle a, b, c\rangle$ for any regular $a \in \alpha$. But then neither O_{ab} nor O_{ac} is on ω so $\omega \cap \alpha$ must be singleton (otherwise α would contain all of π). $\square_{4.4.2}$

Since $t = O_{at} \in \ell_\alpha = O_{ab}O_{ac}$, we have:

Corollary 4.4.3 *Suppose $b, c \in \Omega$ and α is a plane containing b, c. Then for any regular point a on α, $\alpha = \langle a, b, c\rangle = \langle a, c, t\rangle = \langle a, b, t\rangle$ where $t = \ell_\alpha \wedge \omega$.*

Now we show that the intersection of a plane with Ω is determined by any pair of points in that intersection.

Lemma 4.4.4 *Suppose α, β are planes that are not contained in Ω and $b, c \in \Omega$, $b, c \in \alpha \cap \beta$. If $d \in \Omega \cap \alpha$, then $d \in \Omega \cap \beta$.*

Proof. By Lemma 4.4.2, α and β intersect in a point $t \in \omega$. Choose any regular point $a \in \beta$, so $\beta = \langle a, b, c\rangle$. By Lemma 4.4.2, $t \in \beta$ since $b, c \in \alpha \wedge \beta$. Applying Corollary 4.4.3, $\beta = \langle a, b, c\rangle = \langle a, c, t\rangle$.

Let γ be the plane generated by a, d, c. Then $t \in \gamma$ since α and γ are planes containing a and d. Again by Corollary 4.4.3, $\gamma = \langle a, d, c\rangle = \langle a, t, c\rangle = \beta$. So $\gamma = \beta$ and $d \in \beta$ as required. $\square_{4.4.4}$

Lemma 4.4.4 justifies the following Definition 4.4.5 of a line through two points in Ω.

Definition 4.4.5 *For $a, b \in \Omega$ the line through a, b is the intersection of some (any) regular plane that contains a and b with Ω. Note this line includes the point t from Lemma 4.4.2.*

Lemma 4.4.6 *Any two regular planes in $\mathcal{S}$ intersect in a line.*

Proof. Let α and β be two regular planes. If there are two points b, c in their intersection the intersection is the line through b, c by Remark 4.4.1. In particular we are finished if $\ell_\alpha = \ell_\beta$.

So we must show two regular planes α, β intersect in at least two points. One point is $e = \ell_\alpha \wedge \ell_\beta$. To find a second, let r be a regular line in β. We will prove that r intersects α is a point different from e. For this, choose $a \in \alpha$ but not on ℓ_α. Let γ

be the plane generated by r and a. Then γ contains both r and the line $s = ae$. By Lemma 4.3.17, r and s meet in a point, which since s lies on α, must be in α. Thus, $r \wedge s$ is our second point.

$\square_{4.4.6}$

Lemma 4.4.7 *Ω is a plane.*

Proof. Let a, b, c lie on Ω. Let α, β, γ three planes, each containing a regular point and containing ab, bc, ac respectively. For any $d \in \Omega$, Definition 4.3.13 shows it suffices to that $ad \wedge \boldsymbol{c}\neg\emptyset$. Let δ be a regular plane which intersects Ω in the line ad. By Lemma 4.4.6 or Definition 4.4.5 δ and β intersect in a line m. Then m and bc are both in β. So by Lemma 4.3.18 and Lemma 4.3.17, $m \wedge bc$ is the required point of intersection. $\square_{4.4.7}$

4.5 Verifying the axioms

For concreteness, we specify the axioms as formulated in [39]. The term plane is not primitive.

Definition 4.5.1 (Basic axioms) **Axiom 1** *There are at least two distinct points.*

Axiom 2 *Two distinct points A and B determine one and only one line on both A and B.*

Axiom 3 *If A and B are distinct points, there is at least one point distinct from A and B on the line M_1M_2.*

Axiom 4 *If A and B are distinct points, there is at least one point not on the line AB.*

First we verify the basic axioms.

Lemma 4.5.2 *$(\mathcal{S}, \mathcal{L}, \epsilon)$ satisfies the basic axioms.*

Proof.

Axiom 1 Any two distinct points a_1, a_2 in $\pi - \omega$ determine a unique regular triangle $a_1a_2a_3 \in \mathcal{S}$ by drawing the required lines through V and D.

Axiom 2 Clearly the converse of Desargues implies any two triangles in $\mathcal{S}$ with at most one on Ω are centrally perspective by some point. The lines of perspectivity have a unique intersection so there is a single point of perspectivity which indexes the line. The axiom holds for pairs of points in Ω using Definition 4.4.5 and Lemma 4.4.4.

Axiom 3 Given two regular triangles a, b which are in perspective from a point O and so are on a line in $\mathcal{S}$, choose any c_1 on $\ell_1 = 0a_1b_1$ and let c_2 be the intersection of c_1K and $\ell_2 = 0a_2b_2$; now the converse of Desargues theorem shows, since the triangle $c_1c_2c_3$ generated from c_1c_2 by the procedure in the proof of Axiom 2 is axially perspective with $a_1a_2a_3$, they are centrally perspective. So O is on c_1a_1.

Axiom 4 Modify the proof for Axiom 3. Given two regular triangles a, b which are in perspective from a point O, choose $P \neq O$ and draw ℓ_1, ℓ_2, ℓ_3 through O and each of the a_i. Now repeat the rest of the argument for Axiom 3, using the new choices for ℓ_1, ℓ_2 to find c not on the line $ab \in \mathcal{S}$.

$\square_{4.5.2}$

Two more axioms in [39] relate to the 3 dimensionality. For convenience in the proofs, we label the points in Pasch's axiom with lower case letters.

Definition 4.5.3 *(Pasch's axiom:)*

Axiom 5 *Suppose A, B, C are three non-collinear points, D is a distinct point on BC and E is a distinct point on CA. Then there is a point F on the intersection of AB and DE.*

Lemma 4.5.4 $(\mathcal{S}, \mathcal{L}, \epsilon)$ *satisfies Pasch's axiom.*

Proof. Fix a, b, c as in Pasch's axiom and suppose they generate the plane α. If these three points are in Ω, then $\alpha = \Omega$ and we must show two lines r, s in Ω intersect. For some regular planes α and β, $\alpha \wedge \Omega = r$ and $\beta \wedge \Omega = s$. By Lemma 4.5.7 $\alpha \wedge \beta$ intersect in a line q and $q \wedge \Omega$ is the required intersection of r and s.

So we may assume a is regular. And in fact, if b or c is in Ω by replacing b by b' a regular point on ab (or c' a regular point on ac), we generate the same plane since $\ell_\alpha = O_{a,b}O_{a,c} = O_{a,b'}O_{a,c'}$. But then by Lemma 4.3.18, the plane α is also the plane generated by a, b, c in sense of Robinson. By Lemma 4.3.17, we have Pasch's axiom except if d and e are in Ω. But consider the regular line ab. Note by Fact 4.3.5 that every regular line intersects Ω and we finish by Definition 4.4.5. $\square_{4.5.4}$

To specify the geometry is three dimensional, De Robinson adds:

Definition 4.5.5 *(3-space axioms)*

Axiom 6 *If A, B, C are three non-collinear points, there is at least one point D not on the plane ABC.*

Axiom 7 *Any two distinct planes have a line in common.*

Remark 4.5.6 *Axiom 6, the assertion that there is a point off each plane, is easily checked by considering the cases by which a plane is generated.*

Lemma 4.5.7 *Axiom 7: Any two planes in $\mathcal{S}$ intersect in a line.*

Proof. Let α and β be arbitrary planes. For regular planes this is Lemma 4.4.6. If there are two points b, c in their intersection the intersection is the line through b, c by Remark 4.4.1 and Lemma 4.4.4 (in the case one of the planes is Ω). For the $\beta = \Omega$ case, choose any a on α and two distinct lines through a. By Fact 4.3.5 they intersect in (necessarily distinct) points.

$\square_{4.5.7}$

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