P is not Pizza: Variables from grade 3 to 13

We expand the usual mathematical treatment of the syntax and semantics of variable to include consideration of the assignment of concrete referents in order to apply the mathematics to a real world situation. We then study the method of solution of several word problems from K-13 contexts to see how particular solution techniques affect the acquisition of the concept of variable.

P is not Pizza: Variables from grade 3 to 13

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Developing the concept of variable seems to be a major obstacle for students in learning algebra (Koedinger & Nathan, 2004; Moses & Cobb, 2001; Lochhead & Mestre, 1999; Usiskin, 1988). Earlier mathematical experience may help or hinder this development. In this article we consider how the study of several problems can influence student understanding of the notion. In order to situate our discussion we outline, in the context of high school algebra, the explication of 'variable' developed by 20th century logicians. Using this framework we then analyze student struggles with variables in a number of contexts. These vignettes, with one exception, do not involve careful collection of data. Thus, we propose a series of more specific further research questions to address the issues raised here.

These problems are studied in specific contexts. The 'pizza problem' (Problem A) was adapted by Baldwin from an example in (Lochhead & Mestre, 1999) and was used in a secondary math methods course. The 'chicken problem' (Problem B) is presented as a cartoon; a group of students were asked to find the weight of three chickens knowing the weight of each pair. We consider both the role of language and age in interpreting the problem and compare possible solution methods. The 'fence problem' (Problem C) is an old standard. We contrast two correct solutions with one abortive attempt by a student in a first university course for future elementary teachers. Finally, the sink problem (Problem D) is a standard sort of mixture problem. We discuss the approach taken to it in a first year university 'intermediate algebra' course. In each of these examples we will stress the importance of a precise specification of what the variable represents. We complete the introduction and clarify further the goals of comparing these problems after explaining the connections between algebraic expressions and numbers in the next section. We are grateful for the helpful comments of many colleagues at the conference, at Simon Fraser University, and at UIC, particularly Mara Martinez.

What is a variable?

In this first section we first describe in the context of high school algebra one of the insights of modern logic: the notion of variable can cogently be explained by describing both a context of formal expressions involving variables and the interpretation of these expressions in number systems. Connecting this mathematical notion with concrete problems requires the connecting of the expressions and the numbers with specific entities. We explore this connection in the second section.

The term 'variable' is used in many ways. For example, the words independent and dependent variable are introduced to describe the argument and range of a function. This notion of variable developed since the 18th century in an attempt to explicate calculus. Alternatively a variable can be viewed as symbol that can be replaced in the formal statement by a name for a number. This last usage, which we refer to as the *substitutional approach* appears much earlier, stemming from at least the 14th century. The next paragraphs explain the sense in which it encompasses the first and is actually necessary to describe the computation of an arithmetical function.

The crux of our analysis in that 'variable' cannot be understood in isolation. We describe the use of variable in terms of one syntactial and two semantical contexts. Syntax refers to a formal language of mathematics involving symbols (inscriptions). Mathematical semantics attaches mathematical objects to these symbols. Some of these symbols represent numbers; different sorts of formal expressions involving the symbols describe numbers, functions, and truth values. 'Real world' semantics attaches these symbols and numbers to objects in the world (e.g. the number or cost of pizza). The first section of this paper expounds syntax and mathematical semantics; the interface with real world semantics occupies the rest of the paper.

We describe below, specifically for the algebra of the real numbers, how to interpret various uses of variable in terms of the substitutional approach. In this note, variable refers (as in most mathematics) to a symbol such as $x, y, \ldots$. There are three components to the use of such a symbol: an expression or equation containing the symbol, a range of numbers for the variable to represent, and the assignment of a concrete quantity that the numbers measure (Mary's age in years). The last component is intrinsically tied to the choice of units.

The ensuing description is one instance of a general procedure found in any undergraduate text in symbolic logic. And rather than restricting to number we can consider other structures such as geometries, graphs etc. This analysis is the result of investigations by such philosophers and mathematicians as Peirce, Frege, Hilbert, Löwenheim, Skolem, Gödel, and Tarski. There is no thought that a fully formal explanation of the meaning of variables as begun in this section is part of the K-12 curriculum; rather it is a way to describe one aspect of that curriculum. Note however that the description we give below of the interpretation of expressions and equations is implicit in many high school algebra books, e.g. chapter 2 of (EDC, 2009).

In the simplest sense a function is a rule that assigns to each member of its domain a unique value. Thus the domain might be the words (strings of letters) in English and the function f could assign to each word the number of distinct letters occurring in it. Frequently, we write $f(x)$ rather than f although the x adds no information. Karl Menger (Menger, 1953) argued powerfully but futilely against writing the x more than 50 years ago [1]. The development of intuition for the notion of function is an important subject for study but not one we address here. Rather we are more concerned with the transition to writing expressions for functions.

'Algebra' generally refers to contexts where a set of numbers (e.g. the reals), is equipped with operations (e.g. $+, \cdot$), mapping it to itself. One can describe these functions without variables. We might write A^3 for the 'add three' function. This kind of idea has been explored extensively for developing function intuition in children (e.g. David Page (Page & Chval, 1995) and Robert Moses

[1] Menger distinguishes 'scientific' and 'pure' concepts of variable; our discussion of verbal description and units corresponds to his 'scientific'

(Moses & Cobb, 2001)). But when the function is defined by a more complicated combination of the operations on the domain, it is useful to introduce a symbol such as x to represent the argument of the function. We illustrate the versatility of this notion in the following examples.

In arithmetic, expressions are formal strings of symbols that are either names for numbers, or names for the fundamental arithmetical operations such as addition or multiplication. We explain below how to assign meaning to such expressions. In arithmetic we write expressions such as $1+1$ and equations[2] such as

$$3 + 4 = 5 + 2. \tag{1}$$

We have a set of numbers, say the real numbers, in mind and the symbols $1, 2, 3, \ldots, 1/2, 1/3, \pi \ldots$ naturally denote particular real numbers. And an equation is either true (Equation 1) or false:

$$3 + 4 = 5 + 3.$$

In studying algebra, we introduce a new group of symbols, called variables; they usually are letters such as $x, y, z, \ldots$.

This allows us to write new expressions [3] such as $x + 3$ or $3x^2 + 5x + 2$ and new equations such as

1. $y = x + 3$
2. $3x^2 + 5x + 2 = 0$
3. $x^2 + y^2 = 1$
4. $b = \frac{3}{4}d$

These equations appear similar but are used in different ways. We will discuss the four *kinds*[4] of use in turn. In each of these equations, the variable are *free* (not quantified). Equations with free variables determine relations on the real numbers (solution sets).

1. **Function arguments** Life is now more complicated than when we considered arithmetic. The expression $x + 3$ does not denote a number; for each particular value that is substituted for x, we get another number (the first plus 3). An expression like $x + 3$ determines (or represents) a function. In fact, we take advantage of this and write the equation $y = x + 3$. This equation is neither true nor false. Rather, it defines a subset of $\Re \times \Re$: the collection of pairs $\langle a, b\rangle$ such that $b = a + 3$. And so we compute the 'add 3' function by substituting a value for x and evaluating the expression.

2. **Unknowns** The *solution set* of an equation in one variable is a set of real numbers. That is, $3x^2 + 5x + 2 = 0$ defines the subset of those numbers a such that $3a^2 + 5a + 2 = 0$[5]. Now since the real numbers satisfy the distributive law: $3a^2 + 5a + 2 = (3a + 2)(a + 1)$. And since the real numbers satisfy the zero product property[6] $(3a + 2)(a + 1) = 0$ implies that $3a + 2 = 0$

[2]Technically, the 3 in Equation 1 is a numeral, a name for a number. Trying to make this distinction in the lower grades was one of the notorious follies of the 'new math'. But it is essential in algebra to distinguish between expressions or equations which are formal statements that either represent numbers, functions, or have a truth value.

[3]In fact, if we introduce x^n as an abbreviation for the product of n x's, we have defined the class of polynomials as done in high school algebra.

[4]Our classification is close but not identical to that in (Usiskin, 1988).

[5]One may write $\{a : 3a^2 + 5a + 2 = 0\}$ to denote this set.

[6]Mathematicians would say 'have no non-trivial zero divisors'

or $a + 1 = 0$. So the only two numbers that satisfy the given equation are $-2/3$ and -1. So $3x^2 + 5x + 2 = 0$ is a fancy way to describe the set $\{-2/3, -1\}$.
In this context, the word *unknown* is often used instead of variable. We are trying to find what values can be substituted for x to make the equation true.

3. **Curves** The equation $x^2 + y^2 = 1$, is a less trivial example. It defines the unit circle; all pairs of numbers (a, b) such that $a^2 + b^2 = 1$. In example (1) we have defined the graph of a function. Here we define the graph of a relation that is not a function.

4. **Function families** How does the word 'vary' enter the picture? In the first context we *vary* the argument by choosing which number to substitute for x and then we compute the value of 'add 3' at that argument. Consider the bouncing ball experiment (Goldberg, 2000). A ball is dropped from a various heights and each time we measure the height to which it bounces back. We collect data and to analyze it we fix the following vocabulary. The 'manipulated variable', d, is drop height - the distance above the ground from which we drop the ball. The 'responding variable', b, is bounce height - the distance above the ground the ball rises to. Suppose that the data shows the bounce height is 3/4 of the drop height. How do we represent that information as an equation? We write

$$b = \frac{3}{4}d$$

and interpret this equation exactly as in kind 1). But this example illustrates the flexibility of our notation. $b = \frac{3}{4}d$, is the result of substituting $3/4$ for the variable k in the equation in three variables $b = kd$. For any particular ball, we find that the 'bounciness' (more formally 'coefficient of resiliency') k is constant. Thus we have a family of equations with the *parameter* k; we say the bounce height is *proportional* to the drop height.
So our analysis of the bouncing ball represents a more general phenomena. We have an equation in several variables (for simplicity: k, d, b); thus it defines a subset of $\Re^3$. For any particular choice (substitution) of a value for k, we get an equation with a 'manipulated' (or independent) variable d and a 'responding' (or dependent) variable b. To describe the graphs of these equations we consider substitutions of real numbers into the equation $b = kd$; these give us a subset of $\Re^2$.

Laws of Algebra The equation $x(y+z) = xy+xz$ is a problematic notation. If we interpret it in the same way as the examples above we see that it defines $\Re \times \Re \times \Re$. So we really meant to write:

$$(\forall x)(\forall y)(\forall z)x(y + z) = xy + xz.$$

This sentence is true because it is true no matter what triple of real numbers is substituted for the variables. We say that the universal quantifiers $\forall$ have *bound* the variables. Recall that the solution of a finite set (system) of equations is those tuples of numbers that satisfy each of the equations. In high school algebra only universal quantifications of systems of equations or systems of equations with no quantifiers appear[7]. And often the universal quantifiers are omitted for convenience despite the ambiguity.

[7]The discussion here is the first step in the definition of truth in first order logic. The further development does not appear in 'algebra' and so is omitted here.

Problems A-C (pizza, chicken, fence) fall into Kind 2); Problem D (Sink) unites Kind 2) and Kind 4).The last three problems can be represented as systems of linear equations. Indeed, the chicken and fence problems are represented in matrix form [8] by

$$AX = C$$

where the 3×3 matrix A is the same in both problems, although the constants C are different.

In all but Problem A, we examine arithmetic solution methods which do not require the use of a symbolic variable. We want to contrast the mental models (which are precursors of variables) involved in these concrete solution methods with the ones involved in algebraic solutions. Since the problems themselves are of little inherent use, we consider that the purpose for studying them in school is to develop intuitions that support solving a large class of problems. So our question is, 'How do the ways of thinking involved in the supposedly intuitive solutions help or impede the development of an understanding of variable that transfers to other situations?' This question underlies our study of each of the examples.

Having described mathematical syntax and semantics, we now consider some of the mathematical and psychological difficulties involved in properly attaching a concrete referent to variables. We also study how students' work with specific problems influences their development of these notions.

Variables in different problems

We now discuss the four problems and consider how the notion of variable is developed in each case.

Pizza problem

This problem adapted to a local situation an example in (Lochhead & Mestre, 1999) and has been given both on exams and for class discussion to several cohorts of students in Methods of Teaching Secondary Mathematics. These were either graduate students or advanced undergraduates. These future teachers are asked to analyze a student's reasoning; the exact problem the student was asked to solve is deliberately left vague.

Problem 0.1 I went to the Pompeii restaurant and bought the same number of salads and small pizzas. Salads cost two dollars each and pizzas cost six dollars each. I spent $40 all together. Assume that the equation $2S + 6P = 40$ is correct. Then,

$$2S + 6P = 40.$$

Since $S = P$, I can write

$$2P + 6P = 40.$$

So

$$8P = 40.$$

[8] Any system of linear equations can be described as a matrix equation.

The last equation says 8 pizzas is equal to \$40 so each pizza costs \$5.

What is wrong with the above reasoning? Be as detailed as possible. How would you try to help a student who made this mistake?

In each case only about one-half of the students identified the source of the difficulty: P is the *number of pizzas* I bought; not the cost of the pizza, and not 'pizzas'. This problem proved an effective way of drawing the students attention to the need to identify the variable verbally (e.g. number of salads) and determine what set of numbers it ranges over.

Note that after applying the distributive law P is number of 'meals' bought. One advantage of algebra is that we do *not* have an assigned meaning for each variable that remains the same throughout the computation. And indeed, the example shows that this is often impossible, even in very simple situations.

To elaborate a bit, in the given equation $2S + 6P = 40$, the coefficients 2 and 6 indicate unit prices of salads and pizza. 2 and 6 are not 2 dollars and 6 dollars; they represent 2 dollars/salad and 6 dollars/pizza. So, the addition of them doesn't result in 8 dollars; it is 8 dollars/(salad and pizza), or meals as we wrote above. For arithmetical reasoning, it is important to know that the sum of the unit prices of different units requires a new unit for the sum. If we continue this fine analysis, in the equations $2P + 6P = 40$ and $8P = 40$, each P refers to a different unit: the number of salads, the number of pizzas, and the number of (salads and pizza) in order. Again, one power of algebra is that we do not have to worry about such matters. If our formal algebraic reasoning is correct and we have assigned the proper meaning to P at the beginning of the calculation, the value of P that we find has that meaning, the number of pizzas. Of course, this power is not needed for all problems. Koedinger and Nathan (Koedinger & Nathan, 2004) argue that for certain problems intuitive argument is more effective for algebra learners than symbolic manipulation.

The pizza problem involves two equations (one somewhat hidden); the others concern systems of three equations. But we began with it to demonstrate the difficulty relative experts have with interpreting word problems and properly describing the variables involved. *We explore whether this difficulty may be exacerbated by the use of problems as those below with 'intuitive solutions' that rely on inappropriate concretization and do not support a proper understanding of variable.*

Further Research Questions 1 *Study in more detail the reactions of students to this problem. Does failure to state the problem posed to the high school student introduce a distraction from the real question? Are practicing algebra teachers more able to deal with the issues raised here? How does using first letters for variables interfere with/afford students abilities to assign correct meanings to variables?*

Chicken problem

CEMELA (Center for the Mathematics Education of Latinos) is a consortium of four universities. The group in Chicago runs after-school program at a grade school with primarily bilingual students in Chicago. The students meet twice a week after school. There is no attempt at a unified curriculum; rather the students are allowed to choose from a group of fifteen or twenty mathematically rich activities. Their work is facilitated by an undergraduate. The discussion below concerns roughly three minutes of conversation among four 3rd graders in the CEMELA after-school program who are trying to solve this problem. The problem (in the appendix) was presented in both English and Spanish. It provides three cartoons showing the weight of each pair from three chickens and asks for the weight of the three chickens and the individual students. This group of students did not solve the problem; we discuss several obstacles they encountered. We intend in a later paper to analyze this discourse in much more detail and to discuss the work of some other groups who were somewhat more successful with the problem.

The facilitator tries to lead the group towards the strategy: add the weights in the three pictures to get 24; divide by 2 to get 12. He repeatedly asks, 'How many chickens?' (He wants the answer of six; the number of images in the first three pictures.)

Objects versus images: These very young students have trouble with the distinction between 'chickens' and 'images of chickens'. At various times, they answer the question by 3, 6 and 9. The student who answers 9 clearly points to all 9 images on the page. This same confusion arises around the number of big chickens. It appears that these third graders don't understand the conventions for reading cartoons. They conflate the 'number of chickens' with the 'number of images of chickens'. The teacher might address that problem by having the students interpret the individual cartoons before trying to solve the problem. Or one might feel that the problem is simply 'too hard' for students of this age.

Weight: At various times different students report the sum of the weights as 24, 34, and 44. The 34 apparently arose by adding in the 10 twice. Careful analysis of the video shows the student who answers 44 had misread the¡6 and ¡8 as 16 and 18. The 34 apparently arose by adding in the 10 twice. These were all mistakes of understanding, not computational errors.

Weight vrs object: The more important issue in our context concerns the question "How many chickens are there?". This question may be a reasonable strategy to elicit a solution to the problem. Since six images of chickens appear in the cartoons whose weights total 24 kg, we have to divide 24 by 2. But it produces a confusing doubling of each chicken. There is no evidence the students distinguished between 'two chickens' and 'twice the weight of the chicken'. In this case this question seemed to exacerbate the image/object confusion. We explore this issue further below as we examine a different situation leading to a similar mathematical problem.

This vignette indicates that when such problems are used in elementary school to develop students understanding of algebra, the teacher must have a very sensitive understanding of exactly what is happening in a solution. It seems very important that these are *third* graders; the materials were taken from a 6th grade curriculum but have been used with all ages.

In particular, Gail Burrill (Burrill, 1998, 200x) has reported several diverse reactions to the chicken problem. Sixth grade students approach the problem intuitively and work out the answer without setting up equations; so do research mathematicians. In a secondary math methods class, the students set up a system of linear equations as below. School administrators were apprehensive about the need to set up equations and had trouble approaching the problem. Here are the appropriate equations:

$$A + B = 6 \tag{2}$$

$$B + C = 10 \tag{3}$$

$$A + C = 8 \tag{4}$$

where A, B, C are the weights of the small, medium, and large chicken, respectively. Even at this point, where we have reduced to equations, there are two alternatives. Students tend to solve these equations by blindly applying the method of Gaussian elimination, eliminating in pairs. But, recognizing the symmetry of the situation (each variable occurs twice), one might first see that the twice the sum of the three variable is 24 so the sum of the variables is 12 and then, e.g. since $A + B = 6$, C must be 6. This would represent insightful manipulation of the 'naked math' representation. It is crucial for this method that $2A$ is twice the weight of a small chicken, not two small chickens.

Further Research Questions 2 *Study the work of older children on the chicken problem. Does the issue of counting the chickens arise? Does this lead to confusion between 'chicken' and 'weight' as the variable? More generally, does the role of different images as a precursor of variable (more concretely showing that something is being doubled) have the unintended consequence of reinforcing the idea that the variable is 'chicken' rather than 'weight of chicken'? What would happen if the children used a scale and properly weighted model chickens? Does the focus on paper and pencil misrepresent the understanding of these young children?*

Work Problems

Zalman Usiskin (Usiskin, 1980, 2007) inveighs at length against traditional word problems in his articles on 'What should not be in the algebra curriculum ...!'

He writes, 'The traditional word problems (coin, age, mixture, distance-rate-time, and digit) are in the curriculum because of a very valuable goal, the goal of translating from the real world into mathematics. But except for mixture problems, they do not help achieve that goal. In fact, they convince students that there are no real applications of algebra because they are so ridiculous.'

We largely agree with this critique. But, while 'translating from the real world into mathematics' is one purpose of these problems, there is a second: 'giving easily accessible examples of the use of variables'. That is, rather than demonstrations of the power of algebra, these problems can be seen as ways of making algebraic representations accessible. Of course, this goal is also defeated by unreasonable examples. Our last pair of examples are variants of a traditional work problem.

The Fence problem[9].

This problem appeared (in more generic form) in a course for future elementary school teachers from the text (Beckman, 2008). We call the intended solution method, which is extensively used in the Singapore Curriculum, (Education, 2001) the 'strip method'. In each of these courses there is extended development of the strip method for problems of various sorts of which such work problems are among the most complicated. Abramovich and Nabors (Abramovich & Nabors, 1997; ?, ?) elaborate the use of similar methods, which they dub enactive, using spreadsheet software. They also suggest how interesting questions about the divisibility properties of natural numbers can be developed in this context: by asking when the solution is an integral number of hours.

[9]This example was developed in the spring of 2008 when John McCain was clearly the Republican candidate for president but Hillary Clinton and Barack Obama were still contending for the Democratic nomination.

Problem 0.2 Hillary and Barack can paint a fence in one hour.

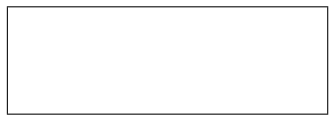
So can Barack and John.

But Hillary and John take two hours.

How long does it take Hillary, Barack and John to paint the fence if they work together?

Solution 0.3 (Strip method) The diagrams in Figure 1 solve the problem using the strip method.

Hillary and Barack paint one fence in one hour.



John and Barack paint one fence in one hour.

Hillary and John paint one fence in two hours.
So, Hillary and John paint 1/2 fence in one hour.

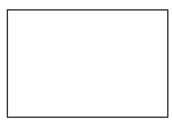


Figure 1. Strip Method

So if we had two each of Hillary, Barack and John they would paint 2 1/2 fences in one hour.

Thus, the actual three can paint 5/4 of a fence in an hour.

And so it takes them 4/5 of an hour to paint the fence.

This is a concrete method of solution. Notice again that, like the chicken problem, it relies on a doubling of the characters. Even after the use of the fraction strip method, there are two difficult steps in this solution. The first is the assertion that if the 'doubled' actors can do the task in 2 1/2 hours it takes three people 5/4 of an hour. The second is the decision to invert the fraction in the last step. One can (Singapore does) teach arithmetic so this step is automatic. But it is a separate and difficult intuition.

More important from our standpoint, there aren't really two Hillarys. Just as in the chicken problem, the mental doubling is problematic. There are two difficulties with such a doubling. One is that we are asking a student to develop an intuition by imagining something that can't happen; there are not two chickens or two Hillarys. But more important, it repeats the student's mistake in the pizza problem. If one thinks of two copies of the object, one is not thinking of that which is actually doubled. In the first case the weight of each chicken is doubled. In the fence case, as we will see in analyzing Solution 4.4, under the strip solution the amount of fence done in an hour is doubled. The authors of this paper have different reactions to this situation; one finds it is easy for her to think of doubling the amount (by mentally doubling Hillary). Another finds this notion profoundly confusing. Later, we will formulate this disagreement as a research question.

We will discuss below the special conditions on problems that make such a concrete solution possible. But first let's see what happens from a too-quick jump to forming equations.

Solution 0.4 (time equations) A pre-service elementary teacher was given the problem in 'a job of work form'. She attempted the problem by writing the following system of equations.

$$A + B = 1 \tag{5}$$
$$B + C = 1 \tag{6}$$
$$A + C = 2 \tag{7}$$

What is the difficulty? Based on the right-hand side representing the number of hours, the student has apparently taken the variables to represent the amount of time taken by each person. But such a deduction might assume a consistency in units the student does not have. Perhaps the student has more of the mindset of those doing the pizza problem incorrectly: 'Hillary plus Barack equals 1'. This suggests a further specific investigation.

Further Research Questions 3 *Interview students at this level who make this sort of mistake to determine their reasoning.*

Solution 0.5 (From strips to equations) The strips approach leads naturally to the equations:

$$A + B = 1 \tag{8}$$
$$B + C = 1 \tag{9}$$
$$A + C = 1/2 \tag{10}$$

What do the variables represent in this solution? The *amount of fence* each person does *in one hour*. The unit 'fences' is determined by inspecting the right hand side of the equation. We will consider these equations again after our next set of examples.

Solution 0.6 (Rational function method) In many algebra books problems of this type are posed as an excuse to study rational expressions. Following the scheme of solution suggested for a somewhat easier problem in (Sobel & Lerner, 1987), we would define the variable x to be the time it requires the three cooperating politicians to paint the fence. Then $\frac{1}{x}$ would be the amount of fence the three of them paint in one hour. Then, as we worked out above, $\frac{2}{x}$ is the amount each politician and his double can paint. So we have

$$1 + 1 + \frac{1}{2} = \frac{2}{x} \tag{11}$$
$$\frac{5}{2} = \frac{2}{x} \tag{12}$$
$$x = \frac{5}{4} \tag{13}$$

At first glance, the choice of time as a variable, thereby introducing a variable in the denominator, seems perverse. There is the gain that there is only one variable and that 'time' is a simpler unit than rate.

Sink Problems.

The next series of problems are from a paired course in basic chemistry and intermediate algebra. The course was taught by a mathematician and a chemist. They attempted to put mixture problems into a coherent framework through the use of functions; we will see below that work problems fit into the same framework. Note that Intermediate Algebra has essentially the content of Algebra II in the American high school. Nevertheless, it is the largest single course at many universities (even though frequently offered for no credit). The functions approach described below is not in general use in Intermediate Algebra, but was central in this course.

The problems and solutions described followed a lot of work on linear functions and then on linear equations. The function notation had been used in the class for some time before this problem was presented. When this problem was presented to a conference containing unilingual Spanish elementary teachers, several of them had trouble realizing that the proffered solution to the next problem was actually intended as a solution. This may have been just a language difficulty. But as they were talking with a translator, I think it more likely they foundered on the notation $H(t)$.

Problem 0.7 (A rate problem: Filling Sinks I) The hot water tap delivers 3 quarts per minute; the cold water tap delivers 4 quarts per minute. If both taps are turned on how long does it take to fill a sink that holds 12 quarts?

Solution 0.8 (Functional solution) In the this case we are given the two rates:

cold water: 3 quarts per minute

hot water: 4 quarts per minute

So in any t minutes, the cold water delivers $3t$ quarts:

$$C(t) = 3t$$

and the hot water delivers $4t$ quarts:

$$H(t) = 4t.$$

We are asked how long it takes for them together to fill one sink which holds 12 quarts. Let $T(t)$ be the total amount of water delivered in t minutes. Then

$$C(t) + H(t) = T(t).$$

And we are asked for what t, is $T(t) = 12$. That is,

$$C(t) + H(t) = 12$$

So, we must solve:

$$3t + 4t = 12.$$

Easily, $t = 12/7$ minutes.

The solution here combines the uses of variables in classes 4) and 2) of Section . First we have identified functions H, C, T that denote the amount of water delivered by the various taps after a given amount of time. Then, we have introduced a variable and formula to represent these functions symbolically. Finally, with this formalism, we use variable as in class 2) to complete the solution of the problem.

Sink problem I, with given rates was used as transition between rate problems in one variable and a method of solution of general work problems of the fence problem type. Here is an example of the general situation.

Problem 0.9 (A work problem: Filling Sinks II) The hot water tap can fill the sink in 3 minutes; the cold water tap can fill it in 4 minutes. If both taps are turned on how long does it take to fill the sink.

Solution 0.10 (Work problem: rate solution) To solve this problem, we need to be creative about rates. Instead of using 'natural' rates like 'quarts per minute', we invent a unit of sinks per minute.

Then we have the rates:

cold water: 1/4 sink per minute

hot water: 1/3 sink per minute

So in any t minutes, the hot water fills $t/3$ sinks and the cold water fills $t/4$ sinks. We are asked how long it takes for the two taps together to fill one sink.

$$C(t) + H(t) = 1$$

$$t/4 + t/3 = 1$$

$$t(1/4 + 1/3) = 1$$

$$\frac{7}{12}t = 1$$

$t = 12/7$ minutes.

This problem required the insight (recall of similar situations) to realize that the trick is to create an artificial rate of sinks per minute. With this insight it reduces to the previous case.

We return to the fence problem using the focus on rates from the sink problem. But now we reverse the roles and take the rates as the variables.

Solution 0.11 (Fence problem: rate approach) Let A, B and C be the rates in fences per hour at which Hillary, Barack, and John respectively paints.

Now

$$A \cdot 1 + B \cdot 1 = 1 \quad (14)$$

$$B \cdot 1 + C \cdot 1 = 1 \quad (15)$$

$$A \cdot 2 + C \cdot 2 = 1 \quad (16)$$

This of course yields an equivalent system to the fraction strip approach; but the method is uniform. In contrast to equations (8)-(10), the coefficients determined by the times given for the painting are explicitly represented.

The approach in the Intermediate Algebra Course differs from the earlier problems and solutions in several important ways. The course was aimed at more advanced students and functional notation was stressed. One of Usiskin's complaints about word problems was addressed by using as actors the hot and cold water taps which can easily cooperate. We have presented an explicitly algebraic solution where the time variable is clearly identified.

The distinctive effect of the rate approach is seen clearly when it is applied to the fence problem. Instead of some 'copying of people', the amount of time that a person works is doubled. We have a very natural and concrete situation: we are describing what happens if the candidates just continue working for a longer time. Similarly, this approach to the chicken problem makes explicit that it is the weight of the chicken that is doubled, not the chicken.

When the fraction-strip approach is expressed with variables, and solved algebraically we have the following situation. Instead of some 'copying of people,' the left hand-side of the equation $2A + 2B + 2C = \frac{5}{2}$ describes the abstract notion of twice the amount of fence that each person paints in each hour.

In short,

- In the strip approach the variable was: the amount done *in* a unit time.
- In the functions approach the variable was: the amount done *per* unit time.

The difference between these two notions is often blurred in speech– even by mathematicians. But the distinction is essential. The units of the variables (fences; fences per minute) are different. By identifying the variable as a rate the fence problem is made part of a general pattern. The strip method also solves a family of problems; but a more restricted one for those problems in which one can make concrete sense of what is done in unit time.

Summary

We have discussed three problems: chicken, fence, and sink that can be solved as systems of linear equations. We argue that the problems we are considering are placed in the curriculum for a number of reasons: 1) to show that algebra connects to the real world and to prior arithmetic understandings and 2) to develop students understanding of variable and how to set up equations.

We have analyzed various solution strategies for these problems and made some mathematical observations. We have seen that via the use of rates the last two become instances of the same general problem. We have also presented other more arithmetical solution techniques. Some of these solutions are very particular to the problem in two senses. The context of the problem is very helpful in choosing a solution technique. The exact choice of numerical coefficients is essential to the solution method. For example, the fact that each variable occurs twice with coefficient 1 is crucial to the arithmetic solution technique for the chicken and fence problems. In solving the fence problem by the rate approach, Solution 0.11, this computational trick is not available unless equation 16 is divided by 2.

Much effort has been fruitfully spent in the last few decades in stressing that many problems have different techniques of solution. It is equally important to realize that not all solutions are equal. Some are more efficient, more insightful, more general, or more beautiful than others. And some may be more appropriate for students at different stages in their mathematical understandings.

Further Research Questions 4 *We suggest that further investigations of student learning should be directed to understanding how various solution strategies affect the ability of students to develop a strong notion of variable. What is the effect of the use of transitional devices such as the images of chickens or strips in developing the understanding that variables represent numbers? Are there certain techniques in presenting such problems that are especially effective? In particular, what is the effect of 'doubling' on students learning of the notion of variable? How is student learning affected by considering examples that do or do not have concrete models for the variables? Do students over-generalize from solution techniques that are dependent on the choice of coefficients? Is there a significant difference in student understanding of 'amount in unit time' versus 'rate' (see the last paragraphs before the summary)?*

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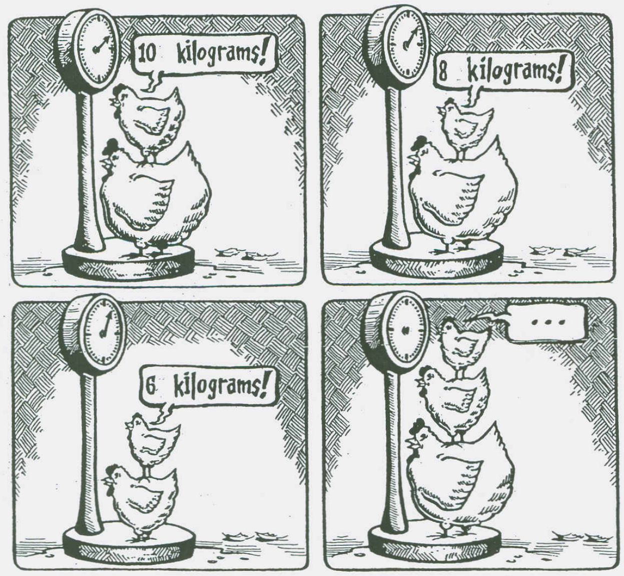
Chickens

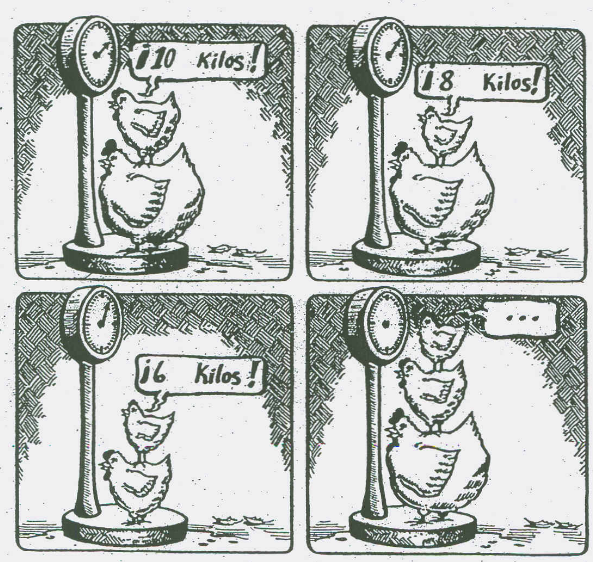
Three chickens weighed themselves in different groupings.



1. What should the scale read in the fourth picture?
2. Now you can find out how many kilograms each chicken weighs. Show how.

Pollos

Tres pollos se pesaron en tres diferentes grupos.



1. ¿Qué peso debaría marcar la báscula en la cuatra figura?
2. Ahora puedes averiguar cuántos kilos pesa cada pollo. Explica: