Constructing ω-stable Structures: Rank k-fields

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Answering an old question of Berline and Lascar, we showed in [1] and [2] that there is an expansion of an algebraically closed field by a unary predicate which has Morley rank 2. Recently, Boris Zilber asked whether this result extended from 2 to arbitrary $k < \omega$. It does. While no essentially new methods are needed in the proof there were some significant new cases to consider. We have taken the opportunity to lay out the proof for all k uniformly.

In presenting the argument there is a tension between an exposition following the method of discovery, making the motivation clear, and one emphasizing uniformities in the proofs of various aspects of the complicated arguments, showing why it is true. We have chosen to err in the latter direction, hoping with comments to keep the goal in sight.

We begin by defining a predimension $\delta(\overline{a}) = k \cdot t.d(\overline{a}) - |\overline{a} \cap P|$ on expansions of algebraically closed fields by a unary predicate P (bicolored fields). We fix a finite-to-one function μ limiting the number of realizations of 'primitive' extensions. In the first section we show that $(\overline{\boldsymbol{K}}_0^{\mu}, \leq)$ the class of such expansions with hereditarily nonnegative dimension has strong separation of quantifiers and amalgamation. We denote by $T^{k,\mu}$ the first order theory of the resulting generic model, which (by Section 1 of [1]) is ω-saturated; it follows that $T^{k,\mu}$ is ω-stable. In the second section we show the universe of a model of $T^{k,\mu}$ has Morley rank k.

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If $k > 2$ one new type of argument is introduced in the proof that the theory has strong separation of quantifiers. We can categorize the minimal strong extensions $\overline{b} \leq \overline{b}\overline{a} \leq \langle \overline{b}\overline{a} \rangle \in \boldsymbol{K}_0$ according to the possibilities for $\delta(\overline{a}/\overline{b})$.

1. $\delta(\overline{a}/\overline{b}) = k$; $\overline{a}$ is a white singleton independent from $\overline{b}$.

2. $\delta(\overline{a}/\overline{b}) = k - 1$; $\overline{a}$ is a black singleton independent from $\overline{b}$.

3. $\delta(\overline{a}/\overline{b}) = m$ with $1 \leq m < k - 1$; $\overline{a}$ is a black tuple with transcendence degree of $\overline{a}/\overline{b}$ equal to $\frac{\lg(\overline{a})}{k} + m$.

4. $\delta(\overline{a}/\overline{b}) = 0$; $\overline{a}$ is a black primitive.

Cases 1), 2), and 4) can be handled exactly as those of the same description are handled in [2]; this corresponds to $\delta(\overline{a}/\overline{b})$ is 2,1, 0 respectively in the earlier argument. However, case 3) is new. Like Cases 1 and 2, in Case 3, the structure $C[\overline{g}]$ is always in $\overline{\boldsymbol{K}}_0^{\mu}$. Unlike cases 1 and 2 and like case 4, the hypothesis that the function μ is finite-to-one must be applied. That is, a formula $\beta(\overline{y})$, chosen in the construction, depends not only on the formula τ but on the primitive codes $\{\boldsymbol{d}_i : i < \ell\}$ such that $\mu(\boldsymbol{d}_i) < 3\lg(\overline{a})$. These observations led to a reorganization of the argument in which this case structure is no longer explicit.

Our notation follows that in [1] and [2] (with the latter dominating as some terms have been updated over time). A careful reading of this one will require having those papers in hand. However, we have attempted to describe the argument so that a person familiar with the earlier papers can grasp the new points. In particular, we rely on the notion of strong submodel, the notation $I(\overline{y})$ for a type guaranteeing its realizations are strong and the notions of primitive codes and δ-formulas. The underlying field language is denoted L_f and L is the expansion of L_f by one unary predicate P. In some cases we must make a distinction between the field part of formula and the L-formula. Recall that in [1] $E_{\boldsymbol{d}}(\overline{x})$ indicated the presence of $\lg(\overline{x})$ 'sufficiently independent' realizations of the code $\boldsymbol{d}$. Here, we write $E^f_{\boldsymbol{d}}$ for the field part of $E_{\boldsymbol{d}}$ (omitting the assertion that certain points are black).

1 Strong Separation of quantifiers

In [1], we defined the notion of a complete-δ formula for a pair $\overline{a}/\overline{b}$. Among the important properties of such a formula $\phi(\overline{x};\overline{y})$ is that if $\phi(\overline{a}';\overline{b}')$ holds, $\delta(\overline{a}'/\overline{b}') \leq \delta(\overline{a}/\overline{b})$. This implies that the assertion that $\overline{b} \leq M$ (read $\overline{b}$ is strong in M and meaning for every $\overline{c} \in M$, $\delta(\overline{c}/\overline{b}) \geq 0$) is type definable. To avoid constant repetition of 'finite conjunction of', we slightly modify the definition from [1]. We require now that the type $I(\overline{y})$ denote the collection of *finite conjunctions of formulas* $(\forall \overline{x})\neg\phi(\overline{x};\overline{y})$ where $\phi(\overline{x};\overline{y})$ is a complete δ-formula for a minimal intrinsic extension of $\overline{b}$. Similarly $I^*(\overline{b})$ denotes the collection of *finite conjunctions of formulas* from $I(\overline{y})$ and the quantifier-free L-diagram of $\overline{b}$.

We showed in [1] that if the class $(\boldsymbol{K}_0, \delta)$ *admits strong separation of quantifiers* in (a minor variant of) the following sense then the generic model is ω-saturated.

Definition 1 *We say* $(\boldsymbol{K}_0, \delta)$ admits strong separation of quantifiers *if for any* $\overline{b} \leq \overline{a}\overline{b} \leq \langle \overline{a}\overline{b} \rangle \in \boldsymbol{K}_0$ *with* $\overline{a}$ *minimal strong over* $\langle \overline{b} \rangle$*, the following holds: For any formula* $\tau(\overline{x}; \overline{y})$ *in* $I(\overline{x}, \overline{y}) \cup Diag(\overline{a}, \overline{b})$ *there is a* $\beta(\overline{y}) \in I^*(\overline{b})$*, such that whenever* $\overline{b'} \subseteq C \in \boldsymbol{K}_0$ *and* $C \models \beta(\overline{b'})$*, there is* $D \in \boldsymbol{K}_0$ *with* $C \leq D$ *and* $\overline{a'} \in D$ *such that* $D \models \tau(\overline{a'}; \overline{b'})$*.*

We work here with $\overline{\boldsymbol{K}}_0^{\mu}$, the finitely generated models with hereditarily non-negative dimension that satisfy the μ-constraint, playing the role of $\boldsymbol{K}_0$ (Cf. [2]).

Notation 2 *1. Fix* $\overline{b} \leq \overline{a}\overline{b} \leq \langle \overline{a}\overline{b} \rangle \in \overline{\boldsymbol{K}}_0^{\mu}$ *with* $\overline{a}$ *minimal strong over* $\langle \overline{b} \rangle$ *and* $\tau(\overline{x}; \overline{y}) \in I^*(\overline{a}\overline{b})$*. Let* $\rho(\overline{x}, \overline{y})$ *be a complete* δ*-formula for* $\overline{a}$ *over* $\overline{b}$ *and let* $\rho^f(\overline{x}, \overline{y})$ *be the field part of* ρ*.*

2. *If* $\overline{a}$ *is primitive over* $\overline{b}$*, find a primitive code* $\boldsymbol{c}$ *and* $\overline{c} \subseteq \mathrm{acl}_f(\overline{b})$ *such that* $\overline{a}$ *is a generic solution of* $\phi^f_{\boldsymbol{c}}(\overline{x}; \overline{c})$*. Let* $\gamma(\overline{u}; \overline{y})$ *isolate the* L_f*-type of* $\overline{c}$ *over* $\overline{b}$*.*

We must show that there is $\beta(\overline{y})$ in $I^*(\overline{b})$ such that whenever $\overline{b}' \subseteq C \in \overline{\boldsymbol{K}}_0^{\mu}$, and $C \models \beta(\overline{b}')$, there is $D \in \overline{\boldsymbol{K}}_0^{\mu}$ with $C \leq D$ and $D \models (\exists \overline{x})\tau(\overline{x}; \overline{b}')$. Let $C[\overline{g}]$ be the white algebraic closure of $C\overline{g}$ where $\overline{g}$ is a realization of $\rho(\overline{x}, \overline{b}')$ which is independent from C over $\overline{b}'$. We will show that one of $C[\overline{g}]$ or C can be taken as D. We will choose a sequence of formulas β_i whose conjunction is the required β. $\beta_0(\overline{b}')$ asserts that the rank of $\rho^f(\overline{x}, \overline{b}')$ is the same as the rank of $\rho^f(\overline{x}, \overline{b})$. We require a minor variant of Lemma 2.6 of [2]:

Lemma 3 *Suppose* $\boldsymbol{d}$ *is a primitive code, and that* $\overline{e}_1, \ldots, \overline{e}_H \subseteq C[\overline{g}]$ *with* $E_{\boldsymbol{d}}(\overline{e}_1, \ldots, \overline{e}_H)$*, and* $H > \mu(\boldsymbol{d}) \geq q(\boldsymbol{d}) + m(\boldsymbol{d})$*. Then, either* $\overline{e}_i = \overline{g}$ *for some* i*, up to a possible reordering of variables, or* $H \leq 3l(\overline{g})$*.*

This lemma was proved in [2] only when $\overline{g}/C$ is primitive; it holds for $\overline{g}/C$ minimal strong by the same argument.

Notation 4 *1. Let* D *be the finite (since* μ *is finite to one) collection of primitive codes* $\boldsymbol{d}$ *with* $\mu(\boldsymbol{d}) < 3\lg(\overline{a})$*.*

2. *Fix a primitive code* $\boldsymbol{d} \in D$*. For each* σ*, a sequence of* $m(\boldsymbol{d})$ *distinct* $n(\boldsymbol{d})$*-tuples from* $\{1, \ldots, \lg(\overline{a}))\}$*, and sequence of constants* $\overline{a}$ *(variables* $\overline{x}$*)with length* $\lg(\overline{a})$*,* $\overline{a}^{\sigma}$ *or* $\overline{x}^{\sigma}$ *denotes the sequence from* $\mathrm{rg}(\overline{a})$ *or* $\mathrm{rg}(\overline{x})$ *indexed by* σ*.*

3. *Let* Z *be the finite set of pairs* $(\boldsymbol{d}, \sigma)$ *with* $\boldsymbol{d} \in D$ *and* σ *such a sequence of tuples.*

We have the following weakening of a key idea in proving the primitive case in [2]:

Lemma 5 *Suppose $\overline{a}$ is black.*

1. $T_0 \cup I^*(\overline{b}) \cup \{\rho(\overline{x};\overline{y})\} \vdash \mathrm{Diag}(\overline{a},\overline{b})$.

2. *For any σ and $\boldsymbol{d}$, if $\overline{e}$ is a realization of $\phi_{\boldsymbol{d}}(\overline{z}; F_d(\overline{a})^\sigma)$ that is independent from $\mathrm{acl}(\overline{a}\overline{b})$ over $F_d(\overline{a}^\sigma)$:*

$$T_0 \cup I^*(\overline{b}) \cup \{\rho(\overline{x},\overline{y}), \overline{z} \cap \overline{xy} = \emptyset, \phi^f_{\boldsymbol{d}}(\overline{z}, F_{\boldsymbol{d}}(\overline{x}^\sigma))\} \vdash \mathrm{Diag}^f(\overline{a},\overline{b},\overline{e}).$$

Proof. For the first part, it suffices to show $T_0 \cup I^*(\overline{b}) \cup \{\rho(\overline{x};\overline{y})\} \vdash \mathrm{Diag}(\overline{a},\overline{b})$ since ρ specifies that $\overline{a}$ is black. Suppose $\overline{a}'\overline{b}'$ satisfies $I^*(\overline{b}) \cup \rho(\overline{x};\overline{y})$. Then $\langle \overline{a}\overline{b}\rangle$ and $\langle \overline{a}'\overline{b}'\rangle$ are isomorphic as fields, since as a realization of $I^*(\overline{b})$, $\langle \overline{b}\rangle \approx \langle \overline{b}'\rangle$ and $\mathrm{td}(\overline{a}'/\overline{b}')$ must equal $\mathrm{td}(\overline{a}'/\overline{b}')$. Otherwise, noting that $\delta(\overline{a}/\overline{b}) \leq k-1$ (as $\overline{a}$ is black), we would have

$$\delta(\overline{a}'/\overline{b}') \leq k\cdot(\mathrm{td}(\overline{a}/\overline{b})-1) - \lg(\overline{a}') = \delta(\overline{a}/\overline{b}) - k < 0$$

contradicting $\overline{b}' \leq \overline{a}'\overline{b}'$.

To see the second, let $\overline{a}', \overline{b}'$ satisfy $T_0 \cup I^*(\overline{b}) \cup \{\rho(\overline{x},\overline{y})\}$. By the first part, $\langle \overline{a}\overline{b}\rangle$ and $\langle \overline{a}'\overline{b}'\rangle$ are isomorphic as fields. Consider an arbitrary black $\overline{e}'$ such that $\phi^f_{\boldsymbol{d}}(\overline{e}', F_{\boldsymbol{d}}(\overline{a}')^\sigma)$ and $\overline{e}' \cap \overline{a}'\overline{b}' = \emptyset$. If $\mathrm{td}(\overline{e}'/\overline{a}'\overline{b}') < R_M(\phi^f_{\boldsymbol{d}}(\overline{z}, F_{\boldsymbol{d}}(\overline{a}^\sigma))) = s$, then (since $ks = \lg(\overline{z})$)

$$\delta(\overline{e}'/\overline{a}'\overline{b}') = k\cdot \mathrm{td}(\overline{e}'/\overline{a}'\overline{b}') - \lg(\overline{z}) \leq k(s-1) - \lg(\overline{z}) = -k.$$

Since $\overline{a}$ is black, $\delta(\overline{a}'/\overline{b}') \leq \delta(\overline{a}/\overline{b}) \leq k-1$, so

$$\delta(\overline{e}'\overline{a}'/\overline{b}') = \delta(\overline{e}'/\overline{a}'\overline{b}') + \delta(\overline{a}'/\overline{b}') \leq -k + (k-1) < 0.$$

But this contradicts the hypothesis that $\overline{b}'$ satisfies $I^*(\overline{b})$. So every black realization of $\phi^f_{\boldsymbol{d}}(\overline{z}, F_{\boldsymbol{d}}(\overline{a}')^\sigma)$ is generic for $\phi^f_{\boldsymbol{d}}(\overline{z}, F_{\boldsymbol{d}}(\overline{a}')^\sigma)$. This implies $\langle \overline{e}\overline{a}\overline{b}\rangle$ and $\langle \overline{e}'\overline{a}'\overline{b}'\rangle$ are isomorphic as fields as required. $\square_5$

It is fairly straightforward to find β_i guaranteeing the following. Details are on page 10 of [2].

Lemma 6 *For $i = 1,2,3$, there are β_i, each a conjunction of members of $I^*(\overline{b})$, such that for every $(\boldsymbol{d},\sigma) \in Z$:*

1. *If $C \models \beta_1(\overline{b}')$ then for any r and sequence of r distinct $n(\boldsymbol{d})$-tuples $\overline{z}_1, \ldots \overline{z}_r$ from $\overline{xy}$, $E^f_{\boldsymbol{d}}(\overline{z}_1, \ldots \overline{z}_r)$ is realized in $\overline{g}\overline{b}'$ if and only $E^f_{\boldsymbol{d}}(\overline{y}_1, \ldots \overline{y}_r)$ is realized in $\overline{a}\overline{b}$. In particular, this holds when $\overline{z}_1, \ldots \overline{z}_r$ is $\overline{x}^\sigma$.*

2. *If $C \models \beta_2(\overline{b}')$ and if $F_{\boldsymbol{d}}(\overline{a}^\sigma) \subseteq \mathrm{acl}(\overline{b})$, then $F_{\boldsymbol{d}}(\overline{g}^\sigma) \subseteq \mathrm{acl}(\overline{b}')$.*

3. In the case that $\overline{a}$ is primitive over $\overline{b}$, we have, if $C \models \beta_3(\overline{b}')$ then

$$C \models \forall \overline{uz}\ [\gamma(\overline{u}; \overline{b}') \wedge \phi^f_{\boldsymbol{d}_0}(\overline{z}; \overline{u}) \rightarrow \tau(\overline{z}; \overline{b}')].$$

We also need the following preparatory lemma. Its proof is the bottom half of page 9 of [2].

Lemma 7 *Let $\epsilon(\overline{z}, \overline{x}, \overline{y})$ be any field formula such that for any $\overline{a}', \overline{b}'$, $MR(\epsilon(\overline{z}, \overline{a}', \overline{b}')) < \frac{\lg(\overline{z})}{2}$. There is a formula β_ϵ in $I^*(\overline{b})$ such that if $C \models \beta_\epsilon(\overline{b}')$ then $C[\overline{g}] \models (\forall \overline{z})[\overline{z} \cap \overline{g}\overline{b}' = \emptyset \wedge \overline{z} \subset P \rightarrow \neg\epsilon(\overline{z}, \overline{g}, \overline{b}')]$.*

With the aid of this lemma we choose two more of the β_i. The next Lemma is immediate since $\tau(\overline{x}, \overline{y})$ has the form $(\forall \overline{z})[\neg\tau''(\overline{z}, \overline{x}, \overline{y})]$ where $\tau''(\overline{z}, \overline{x}, \overline{y})$ satisfies the conditions on ϵ in Lemma 7. We apply the result first to show $C[\overline{g}] \models \tau(\overline{b}')$ and again to choose β_5.

Lemma 8 *Let $\tau''(\overline{z}; \overline{xy})$ be a complete δ-formula for a minimal intrinsic extension of $\overline{xy}$. There is a formula $\beta^\tau(\overline{y}) \in I^*(\overline{b})$ such that if $C \models \beta^\tau(\overline{b}')$ then $C[\overline{g}] \models (\forall \overline{z})\neg\tau''(\overline{z}; \overline{xy})$. I.e. $C[\overline{g}] \models \tau(\overline{g}\overline{b}')$.*

Proof. Either $\tau(\overline{x}, \overline{y})$ has the form $(\forall \overline{z})[\neg\tau''(\overline{z}, \overline{x}, \overline{y})]$ where $\tau''(\overline{z}, \overline{x}, \overline{y})$ is $\overline{z} \subset P \wedge \epsilon(\overline{z}, \overline{x}, \overline{y})$ and $\epsilon(\overline{z}, \overline{x}, \overline{y})$ satisfies the conditions in Lemma 7 (and we finish) or is in $\mathrm{Diag}(\overline{a}\overline{b})$. In the second case the result is trivial if $\overline{a}$ is a white singleton (our original case 1) and the result is immediate by Lemma 5 1) otherwise. $\square_8$

For any $\boldsymbol{d} \in D$ we want to control the realizations $\boldsymbol{d}$ in $C[\overline{g}]$. In the next two lemmas we forbid first with those that don't intersect $\overline{g}\overline{b}'$ and then with those that split over $\overline{g}\overline{b}'$.

Lemma 9 *There is a $\beta_4 \in I^*(\overline{b})$, such that if $C \models \beta_4(\overline{b}')$ and $F_{\boldsymbol{d}}(\overline{a}^\sigma) \not\subseteq \mathrm{acl}_f(\overline{b})$ then for every $(\boldsymbol{d}, \sigma) \in Z$, every solution in $C[\overline{g}]$ of $\phi_{\boldsymbol{d}}(\overline{z}; F_{\boldsymbol{d}}(\overline{g}^\sigma))$ intersects $\overline{g}\overline{b}'$ nontrivially.*

Proof. As usual we work for each $(\boldsymbol{d}, \sigma)$ separately and take the conjunction of the results as β_4. Let $\overline{e}$ satisfy the unique L-type $q \in S(\mathrm{acl}_f(\overline{a}\overline{b}))$ of a realization of $\phi^f_{\boldsymbol{d}}(\overline{z}, F_{\boldsymbol{d}}(\overline{a}^\sigma))$ with $\mathrm{td}(\overline{e}/\overline{a}\overline{b}) = R_M(\phi^f_{\boldsymbol{d}}(\overline{z}, F_{\boldsymbol{d}}(\overline{a}^\sigma)))$. That is, $F_{\boldsymbol{d}}(\overline{a}^\sigma)$ is the canonical base of $\mathrm{tp}(\overline{e}/\overline{a}\overline{b})$. By standard properties of canonical bases (2.26 2 of [4]) or by the special argument on page 9 of [2], $\overline{e}$ depends on $\overline{a}$ over $\overline{b}$. Thus, there are l, strictly less than the rank of $\rho(\overline{x}; \overline{b})$, and an L_f-formula $\psi(\overline{x}; \overline{yz})$ with $\psi(\overline{a}; \overline{b}\overline{e})$ (i.e. $\psi(\overline{a}, \overline{b}, \overline{z}) \in q$), such that for any $\overline{rs}$ the rank of $\psi(\overline{x}; \overline{rs})$ is l. By part 2) of Lemma 5, there is a $\beta(\overline{y})$, which is a conjunction of formulas from $I^*(\overline{b})$ so that

$$T_0 \cup \{\beta(\overline{y})\} \cup \{\rho(\overline{x}, \overline{y}), \overline{z} \cap \overline{xy} = \emptyset, \phi^f_{\boldsymbol{d}}(\overline{z}, F_{\boldsymbol{d}}(\overline{x}^\sigma))\} \vdash \psi(\overline{x}, \overline{y}, \overline{z}).$$

Now suppose $C \models \beta(\overline{b}')$ and $\overline{e}$ is solution in $C[\overline{g}]$ of $\phi^f_{\boldsymbol{d}}(\overline{z}, F_{\boldsymbol{d}}(\overline{g}^\sigma))$ that is disjoint from $\overline{g}\overline{b}'$. Then $\overline{e} \in C - \overline{b}'$ so $\overline{e}$ is independent from $\overline{g}$ over $\overline{b}'$. On the other hand $C[\overline{g}] \models \psi(\overline{g}, \overline{b}', \overline{e})$, so $MR(\overline{g}/\overline{b}'\overline{e}) < MR(\overline{g}/\overline{b}')$. This contradiction yields the result. $\square_9$

Lemma 10 *There is a $\beta_5 \in I^*(\overline{b})$, such that for every $(\boldsymbol{d}, \sigma) \in Z$, if $C \models \beta_5(\overline{b}')$ no solution in $C[\overline{g}]$ of $\phi_{\boldsymbol{d}}(\overline{z}; F_{\boldsymbol{d}}(\overline{g}^\sigma))$ splits over $\overline{g}\overline{b}'$.*

Proof. A solution of $\phi_{\boldsymbol{d}}(\overline{z}; F_{\boldsymbol{d}}(\overline{g}^\sigma))$ that splits over $\overline{g}\overline{b}'$ determines a nontrivial partition of $\overline{z}$ into $\overline{z}_1\overline{z}_2$ and a subsequence, which we denote $(\overline{xy})^\tau$ of $\overline{xy}$ with the same length as $\overline{z}_2$ such that

$$C[\overline{g}] \models (\exists \overline{z})[\overline{z} \in P \wedge \overline{z}_1 \cap \overline{g}\overline{b}' = \emptyset \wedge \phi_{\boldsymbol{d}}(\overline{z}_1(\overline{g}\overline{b}')^\tau; F_{\boldsymbol{d}}(\overline{g}^\sigma))].$$

Now for each choice of $\boldsymbol{d}, \sigma, \tau$, taking $\phi^f_{\boldsymbol{d}}(\overline{z}_1(\overline{xy})^\tau; F_{\boldsymbol{d}}(\overline{x}^\sigma))$ as $\epsilon(\overline{z}_1; \overline{x}, \overline{y})$ satisfies the hypothesis of Lemma 7 so there is a $\beta_{\boldsymbol{d},\sigma,\tau}(\overline{y})$ such that if $C \models \beta_{\boldsymbol{d},\sigma,\tau}(\overline{b}')$ then

$$C[\overline{g}] \models (\forall \overline{z})[(\overline{z} \in P \wedge \overline{z}_1 \cap \overline{g}\overline{b}' = \emptyset) \rightarrow \neg\phi_{\boldsymbol{d}}(\overline{z}_1\overline{g}\overline{b}')^\tau; F_{\boldsymbol{d}}(\overline{g}^\sigma)].$$

Now let β_5 be the conjunction of the $\beta_{\boldsymbol{d},\sigma,\tau}$. $\square_{10}$

Theorem 11 *Fix $\overline{b} \leq \overline{a}\overline{b} \leq \langle \overline{a}\overline{b}\rangle \in \overline{\boldsymbol{K}}^\mu_0$ with $\overline{a}$ minimal strong over $\langle \overline{b}\rangle$ and $\tau(\overline{x}; \overline{y}) \in I^*(\overline{a}\overline{b})$. There is a formula $\beta(\overline{y}) \in I^*(\overline{b})$ such that whenever $\overline{b}' \subseteq C \in \overline{\boldsymbol{K}}^\mu_0$, and $C \models \beta(\overline{b}')$,*

1. *$C[\overline{g}] \models \tau(\overline{g}, \overline{b}')$ and*
2. *either $C[\overline{g}] \in \overline{\boldsymbol{K}}^\mu_0$ or $C \models (\exists \overline{v})\tau(\overline{v}, \overline{b}')$.*

Thus, $\overline{\boldsymbol{K}}^\mu_0$ has strong separation of variables.

Proof. Part 1 is immediate by Lemma 8. By Lemma 3, D is the finite collection of codes which can prevent $C[\overline{g}] \in \overline{\boldsymbol{K}}^\mu_0$. Now let $\beta(\overline{y})$ be the conjunction of β_τ and the $\beta_i(\overline{y})$, depending on $\overline{a}/\overline{b}$, for $i < 6$ defined above. Now suppose $\overline{e}_0, \ldots, \overline{e}_r$ is a sequence of maximal length contained in $C[\overline{g}]$ such that $C[\overline{g}] \models E_{\boldsymbol{d}}(\overline{e}_0, \ldots, \overline{e}_r)$ with $\boldsymbol{d} \in D$. We aim to show $r < \mu(\boldsymbol{d})$ or $C \models \tau(\overline{b}')$.

Case 1. $C[\overline{g}] \models E_{\boldsymbol{d}}(\overline{g}^\sigma)$ for some σ. By the choice of β_1, $\langle \overline{a}\overline{b}\rangle \models E^f_{\boldsymbol{d}_i}(\overline{a}^\sigma)$.

Case 1a). $F_{\boldsymbol{d}}(\overline{g}^\sigma) \subseteq \mathrm{acl}_f(\overline{b}') \subseteq C$. Since, $C \leq C[\overline{g}]$ and $\phi_{\boldsymbol{d}}(\overline{e}_i, F_{\boldsymbol{d}}(\overline{g}^\sigma))$, the $m(\boldsymbol{d})$ of the $\overline{e}_i$ which are proper subsets of $\overline{g}$ contradict that $\overline{g}/C$ is a minimal strong extension.

Case 1b) $F_{\boldsymbol{d}}(\overline{g}^\sigma) \not\subseteq \mathrm{acl}_f(\overline{b}')$. Then, by the choice of β_2, $F_{\boldsymbol{d}}(\overline{a}^\sigma) \not\subseteq \mathrm{acl}_f(\overline{b})$. By our choice of $\beta_4(\overline{y})$, every solution of $E_{\boldsymbol{d}}(\overline{z}, \overline{g}^\sigma)$ in C intersects $\overline{b}'$. But by the choice β_5, none of them split over $\overline{g}'\overline{b}$. So all r of the $\overline{e}_i$ are in $\overline{b}'\overline{g}$. Applying β_1 again this would imply $\rangle \overline{a}\overline{b}\rangle \not\in \overline{\boldsymbol{K}}^\mu_0$. From this contradiction, we deduce $C[\overline{g}] \in \overline{\boldsymbol{K}}^\mu_0$.

Case 2. $C[\overline{g}] \models \neg E_{\boldsymbol{d}}(\overline{g}^\sigma)$ for every σ. By Lemma 2.3 of [2], either at most $q(\boldsymbol{d})$ lie in or split over C and we finish since $q(\boldsymbol{d}) + m(\boldsymbol{d}) < \mu(\boldsymbol{d})$ or none split and $F_{\boldsymbol{d}}(\overline{e}_1, \ldots \overline{e}_m) \in C$. Then no $\overline{e}_i$ can be a proper subset of $\overline{g}$ so $C[\overline{g}] \in \overline{\boldsymbol{K}}^\mu_0$ or $\overline{g}$ must be one of the $\overline{e}_i$ and we have $C[\overline{g}] \models \phi_{\boldsymbol{d}}(\overline{g}, F_{\boldsymbol{d}}(\overline{e}_1, \ldots \overline{e}_m))$. Since $C \leq C[\overline{g}]$, $\overline{g}$ is a generic solution of $\phi_{\boldsymbol{d}}(\overline{z}, F_{\boldsymbol{d}}(\overline{e}_1, \ldots \overline{e}_m))$.

We also have $\overline{g}$ is a generic solution of $\phi_{\boldsymbol{c}}(\overline{x}, \overline{c})$ (see Notation 2). By uniqueness of codes and parameters $\boldsymbol{d}$ is $\boldsymbol{c}$ up to order of variables, so without loss of generality $\boldsymbol{d}$ is $\boldsymbol{c}$. By the choice of β_3 in Lemma 6, $C \models \tau(\overline{e}_1, \overline{b}')$ and we finish. $\square_{11}$

To summarize, we briefly return to the organization of the proof in the earlier papers around the (now) four types of minimal strong extension. In the first three (non-primitive) cases, we are in case 1 of the proof as organized here and $C[\overline{g}] \in \overline{\boldsymbol{K}}_0^{\mu}$. But when $\overline{a}/\overline{b}$ is primitive we may be in the situation that C attains the maximum number of realizations of the code for $\overline{a}/\overline{b}$. Then adding one more takes $C[\overline{g}]$ out of $\overline{\boldsymbol{K}}_0^{\mu}$ but C still satisfies τ and the proof succeeds.

To emphasize that strong separation of quantifiers was an additional condition beyond amalgamation (which might fail for some choices of μ which satisfied amalgamation), in [1], we proved amalgamation first. It is actually simpler to return to the scheme of [3] and derive amalgamation from strong separation of quantifiers as follows.

Corollary 12 *If $M \leq N_1, N_2$ and all are in $\overline{\boldsymbol{K}}_0$ then there is an $N \in \overline{\boldsymbol{K}}_0$ with $N_1, N_2 \leq N$ (by maps which agree on M).*

Proof. If for every $\overline{g} \in N_2 - M$, $\delta(\overline{g}/M) > 0$, we take N as the white algebraic closure of the free amalgamation of N_1 and N_2 over M. Moreover, by expanding M, we may assume there is no $\overline{g} \in N_2 - M$ which can be mapped by an M-embedding into $N_1 - M$. Now fix $\overline{g} \in N_2 - M$ which satisfies a primitive code $\boldsymbol{c}$ and $\phi_{\boldsymbol{c}}(\overline{g}; \overline{c})$ for some $\overline{c} \in M$. By Theorem 11, either $N_1[\overline{g}] \in \overline{\boldsymbol{K}}_0$ and we finish or there are $\mu(\boldsymbol{c}) + 1$ realizations of $\boldsymbol{c}$ in N_1. But there are at most $\mu(\boldsymbol{c}) \in M$ (since $N_2 \in \overline{\boldsymbol{K}}_0$) and none split over M since $M \leq N_1$ so there must be one in $N_1 - M$. But then we can embed $M[\overline{g}]$ into N_1 after all.

$\square_{12}$

We now know that the class of expansions of algebraically closed fields with strong submodel ($\leq$) induced from the given predimension, such that the number of realizations of any primitive is bounded by the function μ, has both the amalgamation property and strong separation of variables. It follows from [1] that there is a generic model, which is ω-saturated; we denote the theory of this model by $T^{k,\mu}$. In the next section we show it has Morley rank k.

2 Computing Rank

The proof that the dimension function $\delta(\overline{a}) = k] \cdot t.d(\overline{a}) - |\overline{a} \cap P|$ yields a theory with Morley rank k follows the argument in [2] with a couple of variants. The use of the μ-function guarantees that if $\delta(\overline{a}/A) = 0$, $\text{tp}(\overline{a}/A)$ is algebraic. We need the following general remark, Lemma 3.3 of [2]:

Fact 13 *Let $N \leq M \models T^{k,\mu}$. If $\overline{a}$ and $\overline{a}'$ are disjoint from N, $N\overline{a} \leq M$, $N\overline{a'} \leq M$, $\overline{a}'$ satisfies a complete δ-formula, $\phi(\overline{x}, \overline{n})$, for $\overline{a}$ over N with base $\overline{n}$ and $\delta(\overline{a}/N) = \delta(\overline{a}'/N)$ then $tp(\overline{a}/N) = tp(\overline{a}'/N)$.*

With this in hand we get the upper bound by a simpler version of the upper bound argument for infinite rank, Lemma 3.6 of [2].

Lemma 14 *Let $N \prec M \models T^{k,\mu}$, $\overline{a} \cap N = \emptyset$ and suppose that $\phi(\overline{x}; \overline{m})$ is a complete δ-formula for $\overline{a}$ over N based on $\overline{m}$, then*

$$R_M(\phi(\overline{x}; \overline{m})) \leq \delta(\overline{a}/\overline{m}).$$

Proof. We proceed by induction on $\delta(\overline{a}/\overline{m})$. Fix $\phi, \overline{m}, \overline{a}, k$ and assume that for all $\overline{m}' \in N$ and all $\phi', \overline{a}'$ for which $\delta(\overline{a}'/\overline{m}') < \delta(\overline{a}/\overline{m})$: $R_M(\phi'(\overline{x}'; \overline{m}')) \leq \delta(\overline{a}'/\overline{m}')$.

By Fact 13 there is at most one type $q(\overline{x}) \in S(N)$ such that $\phi(\overline{x}; \overline{m}) \in q$ and for some (hence any) $\overline{b}$ realizing q, $\delta(\overline{b}/N) = \delta(\overline{a}/\overline{m})$, $N\overline{b} \leq$ M, and $\overline{b} \cap N = \emptyset$. We will show all other complete types over N containing $\phi(\overline{x}; \overline{m})$ also contain a formula of rank strictly less than $\delta(\overline{a}/\overline{m})$. The lemma follows immediately.

Fix $\overline{a}'$ satisfying $\phi(\overline{x}; \overline{m})$ and suppose that the type of $\overline{a}'$ over N is not of the form q described above. Set $\overline{b}' = \overline{a}' - N$. Now $R_M(tp(\overline{a}'/N)) = R_M(tp(\overline{b}'/N))$, so it suffices to show that $tp(\overline{b}'/N)$ contains a formula of rank strictly less than $\delta(\overline{a}/\overline{m})$.

If $\delta(\overline{b}'/N) < \delta(\overline{a}/\overline{m})$, we are done by the inductive hypothesis. If $N\overline{b}' \not\leq$ M, let $\overline{d}$ denote $\mathrm{icl}(\overline{b}'/N)$, whence, since $\overline{d} \subseteq \mathrm{acl}(\overline{b}'N)$, $R_M(\mathrm{tp}(\overline{d}/N)) = R_M(\mathrm{tp}(\overline{b}'/N))$. But $\delta(\overline{d}/N) < \delta(\overline{b}'/N) \leq \delta(\overline{a}/\overline{m})$, so we may again appeal to the inductive hypothesis.

In the remaining case, $\delta(\overline{b}'/N) = \delta(\overline{a}/N)$ and $N\overline{b}' \leq$ M but $\overline{b}' \neq \overline{a}'$. Write $\overline{a}' = \overline{b}'\overline{c}'$, so that $\overline{c}' = \overline{a}' \cap N$, and let $\overline{a} = \overline{b}\overline{c}$ be the corresponding partition of $\overline{a}$.

Since $\overline{a}'$ satisfies the *complete* δ-formula $\phi(\overline{x}; \overline{m})$ for $\overline{a}$ over $\overline{m} \leq N$, we get the second inequality in the following expression.

$$\delta(\overline{a}/N) = \delta(\overline{b}'/N) \leq \delta(\overline{b}'/\overline{c}'\overline{m}) \leq \delta(\overline{b}/\overline{cm}) \leq \delta(\overline{b}/\overline{cm}) + \delta(\overline{c}/\overline{m}) = \delta(\overline{a}/N), \tag{1}$$

so $\delta(\overline{c}/\overline{m}) = 0$. (We knew $\delta(\overline{c}/\overline{m}) \geq 0$ since $N \leq N\overline{a}$.) But this implies (since $\overline{m} \leq \overline{mc}$) that $\mathrm{tp}(\overline{c}/\overline{m})$ is algebraic, whence $\overline{c} \in N$. So $\overline{c}$ is empty, $\overline{a} = \overline{b}$, and thus $\overline{a}' = \overline{b}'$. This contradiction concludes the proof. $\square_{14}$

For the proof of the lower bound argument we must again make slight variations on the proof of the lower bound for infinite rank in [2]. Informally, a primitive decomposition of a set writes it as an ascending union with each component primitive over its predecessor; the formal notion and the basic properties are in Section 3 of [2]. We modify the definitions of (j, m) types and 'ample' for the finite rank case.

Definition 15 *Fix $B \subseteq M$, $b \in M$ and let $\overline{a} = \mathrm{icl}(b/B)$. We say $\mathrm{tp}(b/B)$ is a $(j, m)^*$-type if $B \leq B\overline{a}$, and for $\overline{a} = \overline{b}\overline{c}$, where $\overline{c}$ is maximal in $\overline{a}$ with $\delta(\overline{c}/B) = 0$,*

1. *$\overline{c}$ has a primitive decomposition with m steps.*

2. *$\delta(\overline{a}/\overline{c}B) = j$.*

3. *$\overline{a}$ is minimal strong over $\overline{c}B$.*

Definition 16 *We say $\boldsymbol{K}_0$ (or the theory T of the generic of $\boldsymbol{K}_0$) is *-ample if for every $\mathrm{tp}(b/B)$, if $\overline{a} = \mathrm{icl}(b/B)$ is minimal strong with $\delta(\overline{a}/B) = j$, then for every $m < \omega$, there are $B_m \supset B$ and b_m such that b_m realizes $\mathrm{tp}(b/B)$ and $\mathrm{tp}(b_m/B_m)$ is a $(j-1, m)^*$-type.*

The key fact in the definition of *-ample is that if $\mathrm{tp}(b/B)$ is a $(j, m)^*$-type then for any $C \supseteq B$ such that b and C are independent over B, $\mathrm{tp}(b/C)$ is also a $(j, m)^*$-type. Thus, $\mathrm{tp}(b_m/B_m)$ must be a forking extension of $\mathrm{tp}(b/B)$ and without loss of generality, B_m is elementary in the universe.

Theorem 17 *Suppose $\boldsymbol{K}_0$ is *-ample, $n = max\{\delta(a) : \ a \in N \in \boldsymbol{K}_0\}$, and T is the theory of the generic model M.*

1. *Let $N \prec M$, if $\mathrm{tp}(e/N)$ is a $(j, m)^*$-type then $U(e/N) = j = R_M(e/N)$.*

2. *In T, $R_M(x = x) = k$ is the maximal U-rank of a one type.*

Proof. 1) We show by induction that if $\mathrm{tp}(b/N)$ is a $(j, m)^*$-type then $U(\mathrm{tp}(b/N)) \geq m$. For $j = 0$ and any m, the result follows directly by Lemma 3.9 of [2]. Now suppose we have the result for $j' < j$ and any m. Consider $\mathrm{tp}(b/N)$ where $\overline{a} = \mathrm{icl}(b/N)$, $\overline{a} = \overline{b}\overline{c}$, $\overline{c}$ is maximal in $\overline{a}$ with $\delta(\overline{c}/N) = 0$, $\overline{a}$ is minimal strong over $N\overline{c}$ and $\delta(\overline{a}/N\overline{c}) = j$. Apply the fact that $\boldsymbol{K}_0$ is *-ample to $p = \mathrm{tp}(b/N\overline{c})$: for every $m < \omega$, there exist $N_m \supset N\overline{c}$ and b_m such that b_m realizes $\mathrm{tp}(b/N\overline{c})$ and $p_m = \mathrm{tp}(b_m/N_m)$ is a $(j-1, m)^*$-type. Then p_m is a forking extension of p and by induction $U(p_m) = j - 1$. Thus, $U(p) \geq j$. Note $U(b/N\overline{c}) = U(\overline{a}/N\overline{c})$. By Lemma 3.10 of [2], $U(b/N)) = U(\overline{a}/N) = U(\overline{a}/N\overline{c}) + m \geq j + m$. But by Lemma 14 this is also an upper bound on the Morley rank and, in general, U rank is at most Morley rank so we finish.

2) Since an independent white point has type $(k, 0)$, it has U-rank k by part 1; this is the upper bound by Lemma 14. $\square_{17}$

Now we can conclude the analysis.

Theorem 18 *The theory $T^{k,\mu}$ is ω-stable with Morley rank k.*

Proof. By Lemma 11 and the comment above it, $T^{k,\mu}$ has strong separation of variables whence the generic model is saturated. It then follows immediately from Lemma 14 that $T^{k,\mu}$ is ω-stable and the exact computation of the rank follows from Lemma 17 providing we verify the theory is $*$-ample. This is verified in detail for $k = 3$ in the last example of [2] and the other cases are analogous. (Technically the example showed ample – not *-ample. But in fact we showed only the existence of primitive decompositions of length m for each m. For ample these must be nonalgebraic and were in [2] since without the μ-function no primitives are algebraic. Here we need each primitive is algebraic and that is ensured by the μ-function.) $\square_{18}$

This argument for the rank doesn't give the description of models of the theory as the algebraic closure of the black points that we provided in [1]. For this, a routine translation of the argument in Theorem 2.9 of [1] shows that if $\overline{a}$ is minimal strong over $\overline{b}$ with $\delta(\overline{a}/\overline{b}) = 1$ (determined by $\rho(\overline{a};\overline{b})$) then, writing $\overline{x}$ as $x\overline{x}'$, $(\exists\overline{x}')\rho(x,\overline{x}';\overline{b})$ is strongly minimal. Since all fields of finite Morley rank are almost strongly minimal (Proposition 2.12 [5]), the universe is in the algebraic closure of any strongly minimal set and a fixed set of constants. By the proof of this result every element must be the sum of ℓ elements from (a multiplicative translate of) the set. Rank computations show that for $T^{k,\mu}$, $\ell = k$.

References

[1] J.T. Baldwin and K. Holland. Constructing ω-stable structures: Rank 2 fields. *Journal of Symbolic Logic*, 65:371–391, 2000.

[2] J.T. Baldwin and K. Holland. Constructing ω-stable structures: Infinite rank. *Fundamenta Mathematica*, 170, 2001.

[3] Kitty Holland. Model completeness of the new strongly minimal sets. *J. Symbolic Logic*, 64:946–962, 1999.

[4] A. Pillay. *An introduction to stability theory.* Clarendon Press, Oxford, 1983.

[5] Bruno Poizat. *Groupes Stables.* Nur Al-mantiq Wal-ma'rifah, 82, Rue Racine 69100 Villeurbanne France, 1987.