Three Red Herrings around Vaught's Conjecture

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September 17, 2014

Abstract

We give a model theoretic proof that if there is a counterexample to Vaught's conjecture there is a counterexample such that every model of cardinality $\aleph_1$ is maximal (strengthening a result of Hjorth's). In the process we analyze three examples of sentence characterizing $\aleph_1$. We also give a new proof of Harrington's theorem that any counterexample to Vaught's conjecture has models in $\aleph_1$ of arbitrarily high Scott rank below $\aleph_2$.

The three red herrings[1] are false leads towards solving Vaught's conjecture. Here is one strategy for establishing Vaught's conjecture that there is no sentence of $L_{\omega_1,\omega}$ that has exactly $\aleph_1$ countable models. Hjorth [10, 9], using descriptive set theoretic results of Mackey and others [4], has established that if there is a counterexample then there is one that has no model in $\aleph_2$. On the other hand, unpublished results of Harrington[2] show that every counterexample has models with arbitrarily large Scott ranks below $\aleph_2$. This supports the notion that one might construct a model of an arbitrary counterexample that has cardinality $\aleph_2$. The resulting contradiction would yield the conjecture. In this paper we show that this proof strategy and the published arguments for Hjorth's theorem give rise to some misleading thoughts about which are the crucial issues for investigating the conjecture: the three red herrings. After a brief review of Vaught's conjecture that fixes some terminology we state our results in more detail and make the red herrings more specific.

Recall that Vaught's conjecture [22] concerns the number of countable models of a countable first-order theory, or more generally, of a sentence in the infinitary logic $L_{\omega_1\omega}$, where countable conjunctions and disjunctions but only finite strings of quantifiers are allowed. The conjecture states:

Vaught Conjecture. If φ is a sentence of $L_{\omega_1\omega}$ then φ either has countably many or continuum-many countable models up to isomorphism.

*Research partially supported by Simons travel grant G5402 and Austrian Science Fund (FWF).

†Research supported by FWF (Austrian Science Fund) Grant P24654-N25.

‡Research supported by the Austrian Science Fund (FWF) Lise Meitner Grant M1410-N25

§Partially supported by NSF grant DMS-1308546

[1] See `http://en.wikipedia.org/wiki/Red_herring` for a fairly comprehensive account of the meaning of the red herring idiom.

[2] Notes of Harnik dating from the 70's are the most comprehensive source known to us on Harrington's proof. Marker [21] has on-line lecture notes. Knight, Montalban, and Schweber have another argument in [15].

The more "absolute" version replaces "continuum-many" by "a perfect set of " in the conclusion, where a *perfect set of countable models* is a perfect set of reals, each of which codes a countable model, such that distinct reals in the perfect set code non-isomorphic models. We say a sentence φ of $L_{\omega_1\omega}$ is *scattered* if it does *not* have a perfect set of countable models. Morley [23] defined scattered as: for every countable fragment F of $L_{\omega_1\omega}$ only countably many F-types are realized in a model of φ. He proved that such a theory has at most $\aleph_1$-countable models and so is scattered as defined here. We note the converse in Lemma 3.4. Even more, he showed that any sentence with fewer than $2^{\aleph_0}$ countable models is scattered. Thus the absolute version of Vaught's conjecture states:

Absolute Vaught Conjecture. If φ is scattered then φ has only countably many countable models.

In this terminology a counterexample to Vaught's conjecture is a sentence that is scattered and has models of arbitrarily high countable Scott rank.

A τ-sentence ϕ of $L_{\omega_1,\omega}$ is *complete* if for every τ-sentence ψ of $L_{\omega_1,\omega}$, $\phi \vdash \psi$ or $\phi \vdash \neg\psi$. A variant on Scott's theorem (characterizing countable models in $L_{\omega_1,\omega}$) shows every complete sentence is $\aleph_0$-categorical. Clearly counterexamples to Vaught's conjecture are not complete.

We explore here in more detail complete sentences θ of $L_{\omega_1,\omega}$ that characterize $\aleph_1$ (have models in $\aleph_1$ but no larger). We discuss three such examples due to Julia Knight [13], Laskowski-Shelah [19] and Hjorth [10]. For the last two examples we show by variants on the Fraïssé construction that there is a definable set X of 'absolute indiscernibles': every permutation of X extends to an automorphism of the countable model of θ. Such a set of 'absolute indiscernibles' imply θ can be 'merged' with any sentence ψ of $L_{\omega_1,\omega}$ to create a sentence which has no model in $\aleph_2$ but whose countable models are essentially the same as those of ψ. Hjorth showed that if M is the countable model of his example, S_∞ divides $\mathrm{aut(M)}$; he then applied a result of descriptive set theory to obtain the absolute indiscernibles and thus that if there is a counterexample to Vaught's conjecture there is one with no model in $\aleph_2$. Our model theoretic framework[3] for the construction of absolute indiscernibles shows this detour through descriptive set theory is not needed.

It is well-known (see e.g. [2]) that the study of complete sentences ϕ in $L_{\omega_1,\omega}$ can be reduced to the study of atomic models of a first order theory T_ϕ with elementary embedding as the natural notion of embedding. We will rely on this reduction and use whichever representation is more convenient. While a related reduction holds for incomplete sentences it will not play a role here.

For any class of models of a sentence of $L_{\omega_1,\omega}$ or more generally in any abstract elementary class there is a fundamental relation between the extendibility of models in a cardinal κ and the existence of a model in cardinality κ^+. We fix some notation using the observation of the last paragraph to work for convenience with atomic models of a first order theory.

Definition 0.1. *M is an extendible atomic model in κ of T_ϕ if $|M| = \kappa$ and there is a proper elementary extension of M which is an atomic model T_ϕ. Equivalently, we say M is not maximal.*

M is extendible with no cardinal parameter means extendible in $|M|$.

Note if ϕ is κ-categorical (i.e. complete if $\kappa = \aleph_0$) then if there is an extendible model in κ there is a model in κ^+. We will give examples of sentences with no models in κ^+ but both maximal and extendible models in κ.

Recall ([3]) that any counterexample to Vaught's conjecture has a model in $\aleph_1$; indeed, as we discuss below, conjecturally all have $2^{\aleph_1}$ models in $\aleph_1$. We note that each of the central examples considered here has $2^{\aleph_1}$ models in $\aleph_1$ and each model in $\aleph_1$ is maximal. Thus to establish Vaught's conjecture it suffices

[3] Ackerman [1] independently made the observation for the specific example.

to establish that any putative counterexample has a pair of atomic models in $\aleph_1$ one of which is a proper elementary extension[4] of the other.

We provide another proof of the existence of models of arbitrary Scott rank below $\aleph_2$ for any counterexample to Vaught's conjecture. Like Harrington's argument, this proof yields information about the complexity of the models but nothing about the embeddability relation. Thus we have identified three red herrings, the first is methodological, the second two are false leads towards the proof.

1. Hjorth gives a descriptive set theoretic proof that if there is a counterexample to Vaught's conjecture there is one with a model in $\aleph_2$; we give a model theoretic proof.
2. Hjorth's proof is about the existence of a model in $\aleph_2$; we show it is about the embeddability relation on models in $\aleph_1$
3. Harrington's theorem that there are models in $\aleph_1$ with Scott rank unbounded in $\aleph_2$ is a step towards finding a model in $\aleph_2$; we show all of them are maximal, so Harrington's theorem does not address the crucial embeddability issue.

The first and fourth authors would like to thank the European Science Foundation (ESF) for its support through Infty short visit grants that allowed them to visit the other authors at the Kurt Gödel Research Center (KGRC) in Vienna. During that visit, much of the presented work has been accomplished. In addition, all authors acknowledge extremely helpful conversations with Tapani Hytinnen at the inception of the project and with Dave Marker as he wrote up and presented the material for his Fall 2013 logic course [20].

1 Red Herring I: Model Theory or Descriptive Set Theory?

In the first subsection we avoid the use of descriptive set theory in [10] and give a model theoretic treatment for finding models with absolute indiscernibles. In the second subsection we provide a procedure for combining certain complete sentences of $L_{\omega_1,\omega}$ with other (possibly incomplete) sentences. We then generalize Hjorth's argument to show that if there is a counterexample to Vaught's conjecture, there is one with only maximal models in $\aleph_1$.

1.1 Some variants on the Fraïssé construction

This section is a meditation on [10]. Hjorth used, in the context of a particular example, two important extensions of the method of Fraïssé constructions. We will focus on the role of one of these two techniques: exploiting the role of the disjoint amalgamation property in finding atomic models. The second, considering Fraïssé constructions over a given model, is expounded and extended in [28]. We set up a general framework which gives a common description of salient features of [19, 9].

Notation 1.1. *We will deal with a possibly infinite relational vocabulary τ.*

We have formulated the material below with a submodel relation $\prec_{\boldsymbol{K}}$ that might be other than substructure to emphasize that the arguments in this section apply in more generality. But in this paper it suffices that $\prec_{\boldsymbol{K}}$ *is always interpreted as substructure*.

Definition 1.2. *Let $\boldsymbol{K}$ be a countable collection $\boldsymbol{K}$ of finite structures that is closed under isomorphism. For $A, B \in \boldsymbol{K}$ we define $A \prec_{\boldsymbol{K}} B$ if $A \subset B$.*

[4] As we show in Section 2, 'elementary' embedding can be greatly reduced.

In the constructions at hand, we want to construct models with functions but demand that the model is locally finite. Following [9] we do this by formulating n-ary functions via $n+1$-ary relation symbols restricting our class $\boldsymbol{K}$ to those finite structures (neat for Hjorth) where these relations symbols define functions. Thus, we generalize Fraïssé *by not requiring that the class* $\boldsymbol{K}$ *be closed under substructure.*

Definition 1.3. $\boldsymbol{K}$ *satisfies*

1. Joint embedding *if for any* $B_1, B_2 \in \boldsymbol{K}$ *there is a* $C \in \boldsymbol{K}$ *with* $B_1 \prec_{\boldsymbol{K}} C$ *and* $B_2 \prec_{\boldsymbol{K}} C$.
2. Amalgamation: *if for any* $A \in \boldsymbol{K}$ *and for* $B_1, B_2 \in \boldsymbol{K}$ *with* $A \prec_{\boldsymbol{K}} B_1$ *and* $A \prec_{\boldsymbol{K}} B_2$ *there is a* $C \in \boldsymbol{K}$ *with* $B_1 \prec_{\boldsymbol{K}} C$ *and* $B_2 \prec_{\boldsymbol{K}} C$.
3. Disjoint amalgamation: *if for* $A \prec_{\boldsymbol{K}} B_1, B_2 \in \boldsymbol{K}$ *with* $B_1 \cap B_2 = A$, *there are embeddings of* B_1 *and* B_2 *into a* $B_3 \in \boldsymbol{K}$ *with a common restriction to* A *and the images of the maps intersect on the image of* A.

The intuition that 'everything that can happen does' cannot be expressed in the usual first order $\forall_2$-form here. This is because each structure in $\boldsymbol{K}$ fixes the algebraic closure of its elements. See Remark 2.16.

We write $B \cong_A B'$ if there is an isomorphism between B and B' that is the identity on A.

Definition 1.4. 1. *The model* M *is finitely* $\boldsymbol{K}$-homogeneous *or* rich *if for all* $A, B \in \boldsymbol{K}$, $A \prec_{\boldsymbol{K}} M, A \prec_{\boldsymbol{K}} B \in \boldsymbol{K}$ *implies there exists* $B' \prec_{\boldsymbol{K}} M$ *such that* $B \cong_A B'$.

2. *The model* M *is* generic *if* M *is rich and* M *is an increasing union of finite closed substructures.*

Hjorth calls generic structures 'full'. Of course, the following slight variant of the Fraïssé theorem is well-known.

Theorem 1.5. *Any* $\boldsymbol{K}$ *as in Definition 1.2 that has* $\aleph_0$ *members and that satisfies amalgamation and joint embedding generates a unique countably infinite generic* τ*-structure. Thus, a countable generic structure* M *is homogeneous in the sense that isomorphisms between finite substructures that are in* $\boldsymbol{K}$ *extend to automorphisms of* M.

We denote the Scott sentence of the generic by $\phi_{\boldsymbol{K}}$.

Definition 1.6. *An infinite set* I *is a set of* absolute indiscernibles *in* M *if every permutation of* I *extends to an automorphism of* M.

Now we seek more control over τ-structures to find absolute indiscernibles, by doing a further Fraïssé-style construction in an expanded language. While in general we follow the modern convention of using the same symbol for a model with all of its relations and the domain of that model, in cases where confusion may ensue, we will write $\mathbb{M}$ for a structure with its relations and $|\mathbb{M}|$ for the domain; in context $|\mathbb{M}|$ may mean the cardinality of the domain. If N is a unary predicate, $N(\mathbb{M})$ denotes the substructure with domain the interpretation of N in $\mathbb{M}$.

Notation 1.7. *For any vocabulary* τ, $\hat{\tau}$ *is obtained by adding a new unary predicate* Q *to* τ.

Lemma 1.8. *Let* $\boldsymbol{K}$ *be a* τ*-class that satisfies the hypotheses of Theorem 1.5 but with* disjoint *amalgamation. Then the generic model is extendible.*

Proof. Add a new unary predicate Q to τ to get $\hat{\tau}$. Set $\hat{\boldsymbol{K}}$ as the set of finite $\hat{\tau}$-structures A where $A\restriction\tau \in \boldsymbol{K}$ and Q is an arbitrary subset of A. Note that $\hat{\boldsymbol{K}}$ has disjoint amalgamation since $\boldsymbol{K}$ does. (The disjoint hypothesis is crucial here to obtain any sort of amalgamation in the expanded language and in fact yields disjoint amalgamation in the expanded language.) The definition of $\hat{\boldsymbol{K}}$ implies in particular that an $A \in \boldsymbol{K}$ can be expanded to a model in $\hat{\boldsymbol{K}}$ by putting all elements in Q. Similarly any extension of a member of $\boldsymbol{K}$ can be expanded to $\hat{\boldsymbol{K}}$ by putting every new element in Q. Thus if $\mathbb{M}$ is a generic $\hat{\tau}$-model there is a generic τ-model $\mathcal{N}$ contained in $Q(\mathbb{M})$ and the two are isomorphic. $\square_{1.8}$

The result of Theorem 1.8 can be achieved by a standard model theoretic method. First write[5] $\mathbb{M}$ as a union of an increasing chain $\langle A_i : i < \omega\rangle$ where the A_i are finite members of $\boldsymbol{K}$. Then choose another extension B_1 of A_0 and inductively construct B_{i+1} as a disjoint amalgamation of A_i and B_i over A_{i-1}. (See the diagram on page 135 of [2].)

The argument for Lemma 1.8 implies only that Q contains a model, not that it picks one out. As in Lemma 1.8 we use *disjoint* amalgamation in 1) of the next proof to ensure the amalgamation of two diagrams which have points in the 'ears' that are τ-isomorphic over the base but are in different fibers.

Notation 1.9. *Fix a vocabulary τ. τ_1 is obtained by adding new unary predicates U, V and a binary relation symbol P. The sentence θ_0 says U and V partition the universe and P is a projection of V onto U. If $\mathbb{M}$ is a τ_1-structure satisfying θ_0, we say it is a (κ, λ)-model if $|V(\mathbb{M})| = \kappa$ and $|U(\mathbb{M})| = \lambda$.*

Theorem 1.10. *Let $\boldsymbol{K}$ be a τ-class that satisfies the hypotheses of Theorem 1.5 but with* disjoint *amalgamation.*

1. *There is a countable generic τ_1-structure $\mathbb{M} \models \theta_0$ such that P defines a projection function p from $V(\mathbb{M})$ onto $U(\mathbb{M})$, $U(\mathbb{M})$ is a set of absolute indiscernibles in $\mathbb{M}$ and $V(\mathbb{M})\restriction\tau$ is isomorphic to the generic structure for $\boldsymbol{K}$.*

2. *There is a proper elementary extension $\mathbb{M}_1$ of $\mathbb{M}$ with $U(\mathbb{M}) = U(\mathbb{M}_1)$.*

3. *There is a proper elementary extension $\hat{\mathbb{M}}_1$ of $\mathbb{M}$ with $U(\mathbb{M}) \subsetneq U(\hat{\mathbb{M}}_1)$.*

Proof.

1. We require that the predicates U and V partition the universe and restrict the relations of τ to hold only within the predicate V. We set $\boldsymbol{K}_1$ as the set of finite τ_1-structures (V_0, U_0, P_0) where $V_0\restriction\tau \in \boldsymbol{K}$ and P_0 is the graph of a partial function from V_0 into U_0.

   To amalgamate, use disjoint amalgamation in the V-sort; extend the projection by the union of the projections. If the disjoint amalgamation contains new points, project them arbitrarily to U. Let $\mathbb{M}$ be the generic model for $\boldsymbol{K}_1$.

   To see that $U(\mathbb{M})$ is a set of absolute indiscernibles, consider a permutation σ of $U(\mathbb{M})$. Let $\mathbb{F}$ be the set of finite partial isomorphisms f between substructures (A, A') of $\mathbb{M}$ that are also in $\boldsymbol{K}_1$ and such that $f\restriction U(A) = \sigma\restriction U(A)$. We now show $\mathbb{F}$ is a back and forth system. Given an $f \in \mathbb{F}$ with domain and range a pair (A, A'), let $A \prec_{\boldsymbol{K}_1} B \prec_{\boldsymbol{K}_1} \mathbb{M}$ for some finite $B \in \boldsymbol{K}_1$. Let $B_0 = U(B) - U(A)$ and define B_0' as $\sigma(B_0)$. Observe $AB_0 \in \boldsymbol{K}_1$.

   Now $AB_0 \prec_{\boldsymbol{K}_1} B$ and $AB_0 \cong A'B_0'$ by some $g \in \mathbb{F}$ so by genericity there is a $B^* \in \boldsymbol{K}_1$ with $A'B_0' \prec_{\boldsymbol{K}_1} B^* \prec_{\boldsymbol{K}_1} \mathbb{M}$ with $B^* \cong B$ by a map g_1 extending g. This completes the forth argument; the back is similar. The union of this back and forth system is an automorphism of $\mathbb{M}$ extending σ.

[5] Model theorist often refer to a filtration or taking a resolution.

2. Apply a slight variant on Lemma 1.8, considering the class $\hat{\boldsymbol{K}}_1$ obtained by expanding τ_1 to $\hat{\tau}_1$ by adding Q. Require that $U(A) \subset Q(A)$ for each $\hat{\tau}_1$ structure $A \in \hat{\boldsymbol{K}}_1$.

3. Apply Lemma 1.8, considering the class $\hat{\boldsymbol{K}}_1$ obtained by expanding τ_1 to $\hat{\tau}_1$ by adding Q.

$\square_{1.10}$

As we have done the construction for 1), the reduct of $V(\mathbb{M})$ to τ is a generic model for τ; each fiber will contain such a generic model but unless $\boldsymbol{K}$ is closed under substructures, some fibers will not be models of the generic[6]. Moreover, if every atomic model of the theory of the $\boldsymbol{K}$-generic model in $\aleph_1$ is maximal, as in the Examples 2.5 and 2.7), each of the elementary submodels of the τ-reduct of the $(\aleph_1, \aleph_0)$ model is countable. Here is a further variant. Add the requirement that each finite subset of each fiber is contained in a member of $\boldsymbol{K}$ contained in that fiber and there are no relations across the fibers; each fiber will be a generic model but $V(\mathbb{M})$ will not be. In the cases considered in Section 2 the maximality of the models in $\aleph_1$ (the fact that every formula is equivalent to an existential formula), make it impossible to get both $V(\mathbb{M})$ and the fibers to be models of the Scott sentence. See page 12 of [9].

1.2 Applications to Vaught's conjecture

In this section we use the methods developed in Section 1.1 along with the existence of receptive sentences that characterize $\aleph_1$ to show if there are any counterexamples to Vaught's conjecture there are counterexamples that characterize $\aleph_1$. In Section 2, we explore the existence of such sentences. We employ the vocabulary τ_1 with predicates, U, V, P as in Notation 1.9 and look at models of θ_0. Further we will consider a sentence ψ in a vocabulary τ'; τ_2 denotes $\tau_1 \cup \tau'$.

Definition 1.11. *Let θ be a complete τ_1-sentence of $L_{\omega_1,\omega}$, $U(x)$ a predicate, such that θ implies θ_0 (from Notation 1.9) and let ψ an arbitrary (possibly incomplete) τ'-sentence of $L_{\omega_1,\omega}$.*

- *The* merger *$\chi_{\theta,U,\psi}$ of the pair (θ, U) is the conjunction of θ and ψ^U (where the latter is the relativization of ψ to the set defined by U). Thus $\chi_{\theta,U,\psi}$ is a τ_2-sentence.*

- *If U defines an infinite absolutely indiscernible set in the countable model of θ, we call the pair (θ, U)* receptive*. We call θ receptive if there is an U such that (θ, U) is receptive and in that case we also call the countable model of θ a receptive model.*

Below, we write $\boldsymbol{I}(\chi, \lambda)$ to denote the number of models of an $L_{\omega_1,\omega}$-sentence χ in the cardinality λ.

Theorem 1.12. *Let (θ, U) be receptive and ψ a sentence of $L_{\omega_1,\omega}$.*

1. *The merger $\chi_{\theta,U,\psi}$ is a complete sentence if and only if ψ is complete.*

2. *There is a* 1-1 *isomorphism preserving function between isomorphism types of the countable models of ψ and the isomorphism types of countable models of the merger $\chi_{\theta,U,\psi}$.*

3. *For every cardinal λ, $\boldsymbol{I}(\chi_{\theta,U,\psi}, \lambda) = \max(\boldsymbol{I}(\theta, \lambda), \boldsymbol{I}(\psi, \lambda))$.*

[6]The fibers are not the union of members of $\boldsymbol{K}$ since some members of $\boldsymbol{K}$ overlap several fibers.

Proof. The first statement is a direct consequence of the assumption of receptiveness. To see 2., take any countable $\mathbb{M} \models \psi$, let $\mathbb{M}' \models \chi_{\theta,U,\psi}$ be countable such that the set defined by U contains a copy of $\mathbb{M}$. By absolute indiscernability of the set defined by U and the completeness of θ, the assignment $\mathbb{M} \mapsto \mathbb{M}'$ is well-defined and 1-1 on the isomorphism types. $\square_{1.12}$

It is well-known that any counterexample to Vaught's conjecture must have an uncountable model [7]. In Section 2, we will (with the help of Theorem 1.5) find receptive pairs with a complete sentence that characterizes $\aleph_1$ and moreover (Examples 2.5 and 2.7) has only maximal models in $\aleph_1$.

Corollary 1.13. *Let θ be a complete sentence such that every model of cardinality $\aleph_1$ is maximal and let (θ, U) be receptive. If ψ is a counterexample to Vaught's conjecture then the merger $\chi_{\theta,U,\psi}$ is one as well, which moreover has only maximal models in $\aleph_1$ and so characterizes $\aleph_1$.*

To see that $\chi_{\theta,U,\psi}$ has only maximal uncountable models, note that any extension of the receptive piece must, because of the projection, also extend the model of θ but θ has only maximal models in $\aleph_1$. We can also get examples of sentences with no models in $\aleph_2$ which have long strictly increasing sequences of models in $\aleph_1$; see Corollary 2.10

We discuss now a notion which plays a central role in [10] but has been replaced by model theoretic arguments in our account.

Definition 1.14. S_∞ divides *the topological group H if there is a continuous homomorphism from a closed subgroup of H onto S_∞.*

Recall from Theorem 1.8 that $\mathbb{M}$ denotes the generic structure involving (V, U) with the projection function P from V onto U.

Corollary 1.15.

1. *For any structure $\mathcal{N}$, if X is a set of absolute indiscernibles in $\mathcal{N}$, then S_∞ divides* $\mathrm{aut}(\mathcal{N})$.
2. *Let $\mathbb{M}$ be the structure built as in Theorem 1.5 and where $\hat{\mathbb{M}}$ is the relativized reduct of $\mathbb{M}$ to $M(\mathbb{M})$ (so a τ-structure). Then,* $\mathrm{aut}(\mathbb{M})$ *projects onto S_∞ and also S_∞ divides* $\mathrm{aut}(\hat{\mathbb{M}})$.

Proof. 1. Each permutation of X (thus S_∞) extends to a member of $\mathrm{aut}(\mathcal{N})$ by the definition of absolute indiscernibility and restriction maps $\mathrm{aut}(\mathcal{N})$ onto S_∞.

2. Now, $A_1 = \mathrm{aut}(\mathbb{M} \restriction \mathrm{V}(\mathbb{M}))$ is a closed subgroup of $\mathrm{aut}(\hat{\mathbb{M}})$ and A_1 projects onto S_∞ by mapping $\hat{\alpha} \in \hat{\mathbb{M}}$ to $\alpha \restriction U(\mathbb{M})$ for any $\alpha \in \mathrm{aut}(\mathbb{M})$ with $\alpha \restriction U(\mathbb{M}) = \hat{\alpha}$. (The choice of α does not matter as $\hat{\alpha}$ respects the equivalence relation induced by the projection p.) $\square_{1.15}$

Remark 1.16. Clearly, Knight's example 2.4 does not have an infinite set of absolute indiscernibles since the example is linearly ordered and so the automorphism group of any infinite subset is a proper subset of S_∞. This does not tell us that S_∞ does not divide the automorphism group of Knight's example. Hjorth [8] shows the latter result by considering the topological Vaught conjecture.

The material Hjorth quotes from Becker-Kechris [4] to justify the existence of absolute indiscernibles appears to imply: If S_∞ divides $\mathrm{aut}(\mathrm{N})$ for some countable τ-structure N then it is possible to expand N to a receptive τ_2 structure. Is there a model theoretic proof of this proposition?

2 Red Herring II: $\aleph_1$ or $\aleph_2$?

In Subsection 1.1 and 1.2, we presented an abstract method to transfer from a counterexample to Vaught's conjecture to one with no model in $\aleph_2$. In this section, we show the model theoretic methods of these sections also allow the construction of receptive sentences characterizing $\aleph_1$. Indeed all models in $\aleph_1$ of these sentences are maximal.

In fact, all known complete sentences of $L_{\omega_1,\omega}$ that characterize $\aleph_1$ are composed by trivial means from three prototypic examples (Knight, Laskowski-Shelah, Hjorth) which have no extendible model in $\aleph_1$. In the examples non-extendibility will be much stronger. There will be no proper atomic $\exists_1$-extension of M which satisfies T_ϕ. We next establish the combinatorics behind this phenomena. This section has minimal connection with Vaught's conjecture; rather, we give a fine analysis of how a complete sentence can characterize $\aleph_1$ and analyze the connections among the three examples.

As a side-note, we get:

Remark 2.1. A trivial trick shows: If there is a counterexample to Vaught's conjecture σ and ψ_κ is a complete sentence which characterizes κ, there is a counterexample to Vaught's conjecture which characterizes κ. Just take a disjoint union of a model of ϕ_κ and a model of the sentence $\chi_{\theta,\phi,\sigma}$ where (θ, ϕ) is receptive and θ characterizes $\aleph_1$.

We first identify a combinatorial principle that accounts for the maximality of the models in $\aleph_1$ of the Knight and Laskowski-Shelah examples. We write $P_\kappa(X)$ for the collection of all subsets of X which have cardinality $< \kappa$.

Definition 2.2. *Let $f : \mathcal{P}_\omega(X) \mapsto \mathcal{P}(X)$. We say $A \in \mathcal{P}_\omega(X)$ is f-independent if for every $A' \subseteq A$ and $a \in A'$, $a \notin f(A' - \{a\})$.*

Lemma 2.3. *For every $k \in \omega$ and for every ordinal α, if $|X| = \aleph_{\alpha+k}$ and $f : \mathcal{P}_\omega(X) \to \mathcal{P}_{\aleph_\alpha}(X)$ then X contains an f-independent set of size $k + 1$.*

Proof. We prove this by induction on k. For $k = 0$ and $|X| = \aleph_\alpha$, any element of $X \setminus f(\emptyset)$ suffices. Suppose the result holds for k and consider a set X_1 with $|X_1| = \aleph_{\alpha+k+1}$ and $f : \mathcal{P}_\omega(X_1) \to \mathcal{P}_{\aleph_\alpha}(X)$. Choose any subset Y_0 of X_1 with cardinality $\aleph_{\alpha+k}$ and close it under f (via ω-iterations) getting a set Y with $|Y| = \aleph_{\alpha+k}$. Fix any element $a \in X_1 \setminus Y$. Define g on $\mathcal{P}_\omega(Y)$: for $A \in \mathcal{P}_\omega(Y)$, $g(A) = f(\{a\} \cup A) \cap Y$. By induction, there is a g-independent set $B \subset Y$ of size $k + 1$ and thus $B \cup \{a\}$ is an f-independent set of size $k + 2$. (Note $f(a) = g(\emptyset)$ which contains no element of the g-independent set B). $\Box_{2.3}$

Our Lemma 2.3 just abstracts from the proof in [19] by weakening the requirement that f be a closure operator. The proof of the lemma actually shows that if we know that for some X with $|X| = \aleph_{\alpha+k}$ and $f : \mathcal{P}_\omega(X) \to \mathcal{P}_{\aleph_\alpha}(X)$, X does not contain an f-independent set of size $k + 2$, then no $Y \subsetneq X$ with $|Y| = |X|$ can be closed under f. In particular under these assumptions, if X is a model of an $L_{\omega_1,\omega}$-sentence and f has the property that any submodel of X is closed under f, X can have no proper submodel with the same cardinality.

In the two following examples, f will be closure under certain functions in the vocabulary of the sentences, and the described combinatorics will imply that every proper submodel of a model in $\aleph_1$ has to be countable, or equivalently that no uncountable model will be extendible. This implies that, provably in ZFC, the sentences characterize $\aleph_1$.

Example 2.4 (Knight)**.** In [13] Julia Knight constructed by an inspired ad hoc procedure a complete sentence ϕ_K in $L_{\omega_1,\omega}$ in the vocabulary containing $<$ and unary functions g_n ($n < \omega$) such that if $M \models \phi_K$, M is linearly ordered by $<$ and all predecessors of any $a \in M$ are definable from a by some g_n. So the order is $\aleph_1$-like. While, of course, it is evident that ϕ_K has no model in $\aleph_2$, note that the result follows from Lemma 2.3: f assigns to a finite set its closure under the g_n (which is the smallest initial segment containing it). The assigned sets are countable and there are no independent sets of size 2.

Example 2.5 (Laskowski-Shelah)**.** In [19] Laskowski-Shelah constructed by a generalized Fraïssé construction, that is easily seen to satisfy disjoint amalgamation, a complete sentence ϕ_{LS} in $L_{\omega_1,\omega}$ whose countable model is receptive. In this case, the function f for Lemma 2.3 is closure under certain functions which is locally finite on models of ϕ_{LS} and the sentence implies that there is no f-independent set of cardinality 3. Thus also this example has no proper pair of models in $\aleph_1$.

Now we examine Hjorth's example, which uses a different combinatorial principle. The following definition is a special case of a notion introduced by Souldatos [27] in a detailed study of characterization of cardinals.

Definition 2.6. *A complete $L_{\omega_1,\omega}$ sentence ϕ* homogeneously characterizes $\aleph_1$ *if ϕ characterizes $\aleph_1$ and the countable model of ϕ contains an infinite set of absolute indiscernibles.*

Example 2.7 (Hjorth)**.** In [9] Hjorth constructed by a Fraïssé construction *two* complete sentences, $\phi_{H'}, \phi_H$ that each characterize $\aleph_1$, but only the second provides a homogeneous characterization. Unlike Examples 2.4 and 2.5, one cannot explain the maximality by Lemma 2.3. There is no clearly identifiable closure relation which has finite dimension.

We sketch this construction in the framework of Section 1.1. The vocabulary τ for H contains binary relations S_n ($n < \omega$) and $k+2$-ary relations $T_k(x_0, x_1, y_0, \ldots y_{k-1})$ ($k < \omega$). The S_n are thought of as colored edges (so disjoint).

We require that for any model M there is a function $f : M^2 \mapsto \omega$ (which is not in the vocabulary) such that

1. for every pair a, b $M \models S_{f(a,b)}(a, b)$ and
2. $M \models T_k(a, b, c_0, \ldots, c_{k-1})$, exactly if $\{c_0, \ldots c_{k-1}\}$ is the set of points on which $f(a, *) = f(b, *)$.

The class $\boldsymbol{K}$ is the class of finite structures F satisfying the conditions described and for any distinct $a, b \in F$, the c_i from condition 2. also belong to F. It is easy to check that this $\boldsymbol{K}$ has disjoint amalgamation and joint embedding and so a generic model by Theorem 1.5. (In fact, one can disjointly amalgamate on the union of the models in the amalgamation diagram.) There are only countably many elements in $\boldsymbol{K}$ because the quantifier-free type of any element of $\boldsymbol{K}$ is determined by a finite subvocabulary.

Let H' be the generic model with Scott sentence $\phi_{H'}$. Clearly H' does not admit a set of indiscernibles. (Any such set would have to be a complete graph for one S_n. But then, the local finiteness imposed by T_k would be contradicted.)

To remedy this, we construct ϕ_H. Apply the construction of Theorem 1.10, adding the projection P and the predicates M and N. Since $\boldsymbol{K}$ for H' has disjoint amalgamation, Theorem 1.10 yields a generic receptive model H with Scott-sentence ϕ_H. ϕ_H homogeneously characterizes $\aleph_1$ and the proof of Theorem 1.15 shows that S_∞ divides $Aut(H')$ as well as $Aut(H)$.

It is easy to see that the class of models of ϕ_H does not satisfy amalgamation (for $\exists_1$-extensions) in $\aleph_0$. Fix a_i ($i < \omega$) in a model M and note that there are consistent types over M of elements c, c' such that for infinitely many distinct n_i we have $S_{n_i}(a_i, c), S_{n_i}(a_i, c')$ but for some $d \in M$ the types and new

$m \neq m'$ the types require $S_m(d,c), S_{m'}(d,c')$. This implies that the algebraic closure of c, c' is infinite if they are distinct and lie in a common model; but the closure of any pair of distinct elements is finite. So any amalgamation must identify c and c'; the second condition forbids this identification.

Theorem 2.8. *No uncountable model of either Hjorth example is extendible.*

Suppose for contradiction that there were a pair of models of ϕ_H in $\aleph_1$ with M_1 a strict submodel of M_2. Then fixing any c in $M_2 - M_1$, note that for each $n \in \omega$, there is at most one $a \in M_1$ such that $f(a,c) = n$. But then M_1 must be countable. $\square_{2.8}$

Recall Definition 1.9 of the two cardinal models in this context.

Lemma 2.9. *Hjorth's example ϕ_H has both $(\aleph_1, \aleph_0)$ models and $(\aleph_1, \aleph_1)$ models. No model of the first type can be embedded in a model of the second type. Every extension of a countable model must extend each fiber.*

Proof. Let M_0 be the countable model of ϕ_H. We obtain an $(\aleph_1, \aleph_0)$ model by iterating the construction in Theorem 1.10.2. For an $(\aleph_1, \aleph_1)$ model, iterate Theorem 1.10.3 $\aleph_1$ times, noting that the generic extension of M_0 both extends each fiber and adds fibers. The second assertion follows from our main result that no model in $\aleph_1$ can be a $\exists_1$-extension of another. For the third, note that ϕ_H implies that for any three fibers and any n there exist elements a, b, c in distinct fibers such that $S_n(a,c)$ and $S_n(b,c)$. This is impossible if a, b are in a proper elementary submodel of a structure with c in a new fiber. Thus the extension we obtained first in this proof must extend each fiber. $\square_{2.9}$

We can now provide the examples of sentences with no model in $\aleph_2$ but for any $\beta < \aleph_2$, chains of length β of models in $\aleph_1$. The construction in the example does not involve the Fraissé ideas except as input for the model N. Thus the same argument shows that if there for an arbitrary sentence ϕ such that every model of ϕ with cardinality κ is maximal then there is a sentence θ_ϕ such that for every ordinal $\beta < \kappa^+$ there is an increasing chain of models of θ_ϕ of length β whose last element is maximal.

Example 2.10. Consider the vocabulary τ of any of the examples ϕ of $L_{\omega_1,\omega}$-sentences such that all uncountable models are maximal, and form τ_1 as in Notation 1.9. Let θ_1 be the complete sentence that says

- θ_0 (as defined in Notation 1.9), which expresses that P is a projection of V onto U
- U is a model of ϕ
- every P-fiber in V is a model of ϕ, and there are no τ-relations between fibers or between V and U

θ_1 is a complete sentence that obviously characterizes $\aleph_1$. For every $\beta < \omega_2$, there is an increasing, continuous chain of models of θ_1 of length β: start with a model where U has size $\aleph_1$ (which means U is non-extendible), and where every P-fiber is countable. Now enumerate the fibers in order-type β and define a chain $(M_\alpha | \alpha < \beta)$ as follows. In limits take unions. Given some M_α, define $M_{\alpha+1}$ from it such that all fibers are unchanged, except for the α-th fiber which is properly extended (it does not even matter if it stays countable or is made uncountable). Note that if we always extend to an uncountable model the final structure is maximal.

Now we get a stronger conclusion about models in $\aleph_1$ that holds for all three examples. Namely, there are $2^{\aleph_1}$ models in $\aleph_1$. We show this follows from the fact that all models are maximal. We need a definition and theorem. The results are originally due to Shelah [24] but this proof is from [26] and our direct references are to [2] where the results are formulated for abstract elementary classes. For simplicity we work with atomic models of a first order theory and $\prec$ means elementary submodel. We want to identify two different kinds of proper pairs of countable models.

Definition 2.11. *Fix a first order theory T. (M, a, N) is a* maximal triple *if $M \prec N$ are atomic models of T, $a \in N \setminus M$ and if for every pair of atomic models $M' \prec N'$ with $M \prec M'$, $N \prec N'$, if $M \neq M'$ then $a \in M'$.*

Definition 2.12. *$M \prec N$ is a* cut-pair *in λ if $|M| = |N| = \lambda$ are atomic models and there exist atomic models N_i for $i < \omega$ such that $M \prec_{\boldsymbol{K}} N_{i+1} \prec N_i \prec N$ with N_{i+1} a proper elementary submodel of N_i and $\bigcap_{i<\omega} N_i = M$.*

The following is proved as Lemma 7.8 of [2].

Lemma 2.13. *Suppose the class of atomic models of T is λ-categorical. If T has a cut-pair in cardinality λ and it has a maximal triple in λ, then $I(\lambda^+, \boldsymbol{K}) = 2^{\lambda^+}$.*

Theorem 7.4 of [2] implies that if a theory has no maximal triples in $\aleph_0$, it has an extendible model in $\aleph_1$. This means that if every model in $\aleph_1$ is maximal (as is the case in the three examples give above), a maximal triple exists and since generally every complete sentence in $L_{\omega_1,\omega}$ has a countable cut-pair, Lemma 2.13 implies:

Theorem 2.14. *If a complete sentence ϕ of $L_{\omega_1,\omega}$ has uncountable models but no uncountable model of ϕ is extendible then ϕ has $2^{\aleph_1}$ models in $\aleph_1$.*

Question 2.15. *Is there a model $\mathbb{M}'$ in $\aleph_1$ such that $N(\mathbb{M}')$ is absolutely indiscernible in $\mathbb{M}'$?*

Note that if $\mathbb{N}(\mathbb{M}')$ is absolutely indiscernible then the fibers must be isomorphic.

Remark 2.16. Looking at these examples from a first order perspective leads to some misleading ideas of how to distinguish them. The first order theory of Knight's example has the strict order property and each of the Hjorth and Laskowski-Shelah examples has the independence property. But this is misleading when restricting to atomic models of the theories. In the latter cases, it is clear that the formula $S_n(x, y)$ can arbitrarily partition an arbitrarily large *finite* set of indiscernibles. But suppose there were an infinite set of points I such that for each $X \subset I$, there is an a_x such that for $c \in I$, $S_n(a_x, c)$ iff $c \in X$. Then each a_x is in the (traditional) algebraic closure of I in the model of the first order theory. But only countably many of the a_x can appear in an atomic model.

3 Red Herring III: Scott Rank Unbounded in ω_2 or Embeddability?

In this section, we provide an account of Harrington's result that a counterexample to Vaught's conjecture has models of size $\aleph_1$ with Scott rank unbounded in $\aleph_2$. Other accounts of the result are in [21], [20] and [18], and another (different) proof in [15]. Let us start with a reminder about some classical notions and facts. For background see [11, 6, 5].

3.1 Scott rank and Morley analysis

We need to consider the notion of Scott rank in $L_{\infty,\omega}$.

Definition 3.1. *Let M be an L-structure and let $\overline{a}, \overline{b}$ be n-tuples in M. By induction over the ordinal α, we define the notion of α-equivalence of $\overline{a}$ and $\overline{b}$, denoted by $\overline{a} \equiv_\alpha \overline{b}$:*

- *$\overline{a} \equiv_0 \overline{b}$ if $\overline{a}$ and $\overline{b}$ satisfy the same quantifier-free L-formulas.*
- *For limit α, $\overline{a} \equiv_\alpha \overline{b}$ if $\overline{a} \equiv_\beta \overline{b}$ for all $\beta < \alpha$*
- *$\overline{a} \equiv_{\alpha+1} \overline{b}$ if*
  - *for all $c \in M$ there exists some $d \in M$ such that $\overline{a}c \equiv_\alpha \overline{b}d$ and*
  - *for all $d \in M$ there exists some $c \in M$ such that $\overline{a}c \equiv_\alpha \overline{b}d$*

The Scott rank *of M is the minimal α such that α-equivalence implies $(\alpha+1)$-equivalence for all tuples in M.*

Note that the relations $\equiv_\alpha$ are a refining sequence of equivalence relations and the Scott rank of a structure M is an ordinal of cardinality at most the cardinality of M.

Using the standard proof of Scott's isomorphism theorem (e.g. page 1 [12]) one defines $\Theta^{(M,\overline{b},\alpha)}(\overline{x})$ (in $L_{|\alpha|^+,\omega}$) for any tuple $\overline{b} \in M$ that are true of some $\overline{a} \in M$ if and only if $\overline{a} \equiv_\alpha \overline{b}$.

Consider the special case where $\overline{b}$ is the empty tuple and thus $\Theta^{(M,\emptyset,\alpha)}$ is a sentence. This sentence is unique up to the ordering of the conjunctions. In particular, it has a well-defined quantifier rank. An L-structure N satisfies it if and only if we can realize back-and-forths of length α between M and N. A simple induction shows that this is equivalent to M and N satisfying the same $L_{\infty,\omega}$-sentences of quantifier rank at most α. In this case, we say that M and N are α-equivalent and write $M \equiv_\alpha N$.

Definition 3.2. *Let M be a countable L-structure of Scott rank α. The* canonical Scott sentence *σ_M of M is the $L_{\omega_1,\omega}$-sentence:*

$$\Theta^{(M,\emptyset,\alpha)} \wedge \bigwedge_{\overline{a} \in M^{<\omega}} \forall \overline{x}(\phi^{M,\overline{a},\alpha}(\overline{x}) \to \phi^{M,\overline{a},\alpha+1}(\overline{x}))$$

The sentence σ_M is true of exactly those structures that are back-and-forth equivalent to M. Since countable structures are back-and-forth equivalent if and only if they are isomorphic, σ_M describes completely the isomorphism type of M and so axiomatizes a complete $\aleph_0$-categorical $L_{\infty,\omega}$ theory.

Recall the definition of a *scattered* sentence given in the introduction. There is a more concrete and useful way of defining that notion in terms of α-equivalence and Scott ranks.

Theorem 3.3. *The following are equivalent:*
(a) φ is scattered.
(b) For each countable α, there are only countably many α-equivalence classes of models of φ.
(c) For any countable α, there are only countably many models of φ of Scott rank less than α.

Proof sketch. By Silver's theorem concerning Borel (even coanalytic) equivalence relations, for each α the equivalence relation of α-equivalence (which is Borel for countable α) has either countably many or a perfect set of equivalence classes. In the latter case we get a perfect set of non-isomorphic countable models of T. So (a) implies (b). And (b) implies (c), as models of Scott rank less than α are isomorphic iff they are α-equivalent. Now assume (c) and suppose that there were a perfect set of countable models of φ, given by a perfectly splitting tree $\mathcal{T}$. Let A be a countable transitive model of ZFC$^-$ (ZFC without the power set axiom) containing codes for φ and $\mathcal{T}$. Then we can form a perfectly splitting subtree $\mathcal{T}^*$ of $\mathcal{T}$ such that every branch through $\mathcal{T}^*$ is Cohen-generic over A. But each branch x through $\mathcal{T}^*$ codes a model of φ whose Scott sentence belongs to $M[x]$ and therefore has Scott rank less than Ord$(M[x])$ = Ord(M), contradicting (c). $\square$

As there are only ω_1 possible Scott ranks, it follows from (c) of the previous theorem that a scattered theory has at most ω_1 many countable models.

A *countable fragment* F of $L_{\omega_1\omega}$ is a countable set of formulas in $L_{\omega_1\omega}$ containing all first-order formulas and closed under subformulas, finite Boolean combinations, quantification and change of free variables. Of course any countable set of formulas in $L_{\omega_1\omega}$ is contained in a least countable fragment. An *F-type* is a set of the form $p(x) = \{\psi(x) \mid \psi(x) \in F$ and $M \vDash \psi(m)\}$, for some model M and finite tuple m from M. We say that $p(x)$ is *realized in* M.

Lemma 3.4. *Suppose that φ is scattered. Then for every countable fragment F containing φ, there are only countably many F-types realized in models of φ.*

Proof. Note that if $p(x)$ is an F-type then the sentence that says that $p(x)$ is realized has quantifier rank at most the sup of the ranks of the formulas in F plus one. Let α bound the ranks of these sentences. Then α-equivalent models realize the same F-types and therefore by 3.3(b), as φ is scattered, there are only countably many F-types realized in models of φ. $\square$

We now review the standard Morley analysis and explicitly constructing the tree of all theories appearing in that analysis. Suppose that φ is scattered and choose a countable fragment F_0 containing φ. *Level* 0 of the Morley tree, denoted by $\mathcal{T}_0$, consists of all complete F_0-theories containing φ, i.e., all sets of the form $\{\psi \mid \psi$ is a sentence of F_0 and $M \vDash \psi\}$ for some model M of φ. There are only countably many such theories as there are only countably many F_0-types realized in models of φ.

Now for each F_0-type $p(x)$ realized in a model of φ consider the formula $\wedge_{\psi(x)\in p(x)}\psi(x)$ and let F_1 be the least fragment containing F_0 as well as all of these formulas. As there are only countably many F_0-types realized in models of φ, F_1 is a countable fragment. Now for each theory T in $\mathcal{T}_0$ we define the extensions of T in $\mathcal{T}_1$, *level* 1 *of* $\mathcal{T}$: If T is $\aleph_0$-categorical, i.e., all of its countable models are isomorphic, then T is a dead node and has no extensions in $\mathcal{T}_1$. Otherwise the extensions of T in $\mathcal{T}_1$ are the complete F_1-theories containing T. Again by scatteredness, there are only countably many such F_1-theories.

Now suppose for some $\alpha < \omega_1$, F_α and $\mathcal{T}_\alpha$ has been constructed. Define level $\alpha+1$ of $\mathcal{T}$ by enlarging the fragment F_α to the least fragment $F_{\alpha+1}$ containing F_α and the conjunctions of the F_α-types realized in models of φ and extend each theory T in $\mathcal{T}_\alpha$ which is not $\aleph_0$-categorical to the complete $F_{\alpha+1}$ theories containing T. For limits δ we let F_δ be the union of the fragments F_α, $\alpha < \delta$ and let $\mathcal{T}_\delta$, the δ-th level of $\mathcal{T}$ be the unions along paths cofinal through $\mathcal{T}_{<\delta}$.

Now we connect the rank of the canonical Scott sentence of a model with height assigned by Morley's analysis.

Lemma 3.5. *Let $M \models \varphi$ be countable of Scott rank β. Then there is a sentence in the fragment $F_{\beta+3}$ which is equivalent to the canonical Scott sentence of M.*

Proof. Fix some $\overline{a} \in M$ and for any α, let $\Psi^{M,\overline{a},\alpha}(\overline{x})$ the conjunction of all F_α-formulas true of $\overline{a}$ in M. By definition, $\Psi^{M,\overline{a},\alpha}(\overline{x})$ belongs to $F_{\alpha+1}$. We show that $\Psi^{M,\overline{a},\alpha}(\overline{x}) \models \Theta^{M,\overline{a},\alpha}(\overline{x})$ by induction over α (for all possible countable models M simultaneously). For $\alpha = 0$ and α a limit ordinal, this follows immediately from the definitions. Now suppose we know that $\Psi^{M,\overline{a},\alpha}(\overline{x}) \models \Theta^{M,\overline{a},\alpha}(\overline{x})$. Let N be any countable model and let $\overline{b} \in N$ satisfy $\Psi^{M,\overline{a},\alpha+1}(\overline{x})$. We want to verify $N \models \Theta^{M,\overline{a},\alpha+1}(\overline{b})$. By definition,

$$\Theta^{M,\overline{a},\alpha+1}(\overline{x}) \equiv \forall y \bigvee_{c\in M} \Theta^{M,\overline{a}c,\alpha}(\overline{x}, y) \wedge \bigwedge_{c\in M} \exists y \Theta^{M,\overline{a}c,\alpha}(\overline{x}, y)$$

and by induction, it will be enough to show that $\overline{b}$ satisfies both conjuncts with the occurrences of Θ replaced with Ψ.

To see that $N \models \forall y \bigvee_{c\in M} \Psi^{M,\overline{a}c,\alpha}(\overline{b}, y)$, take any $d \in N$ (for y) and find a corresponding $c \in M$ which makes this statement true. If $p_\alpha(\overline{x}, y)$ is the F_α-type of $(\overline{b}, d)$ in N, the formula $\psi(\overline{x}) \equiv \exists y p_\alpha(\overline{x}, y)$ belongs to the $F_{\alpha+1}$-type of $\overline{b}$ in N and thus also to the $F_{\alpha+1}$-type of $\overline{a}$ in M (since we assume $N \models \Psi^{M,\overline{a},\alpha+1}(\overline{b})$). Any witness $c \in M$ for y in of $\psi(\overline{a})$ will be such that $N \models \Psi^{M,\overline{a}c,\alpha}(\overline{b}, d)$. The argument for the second conjunct is similar. This finishes the induction.

Recall that if α is at least the Scott rank of M, the formula $\Theta^{M,\overline{a},\alpha}(\overline{x})$ expresses back-and-forth equivalence with $(M, \overline{a})$ and thus isolates the complete $L_{\infty,\omega}$-type of $\overline{a}$ in M. Thus for α at least that large, the formulas $\Theta^{M,\overline{a},\alpha}(\overline{x})$ and $\Psi^{M,\overline{a},\alpha}(\overline{x})$ are in fact equivalent. Therefore, using Ψ in place of Θ gives a sentence equivalent to the canonical Scott sentence of M. By carefully examining the definition of the canonical Scott sentence, we will find an equivalent of it in $F_{\beta+3}$, where β is the Scott rank of M. $\Box_{3.5}$

Note that the bound $\beta + 3$, where β is the Scott rank of M is not optimal. For example, any countable model of the first order theory of a successor function (using a single binary relation) has Scott rank ω but is already $\aleph_0$-categorical in its F_1-theory.

Proposition 3.6. *(a) For each limit $\delta < \omega_1$, each node T of $\mathcal{T}_\delta$ is a satisfiable (F_δ-complete) theory.*

(b) Each theory appearing in the Morley tree is an atomic theory*, i.e. if T lies in the fragment F then each F-formula which is T-consistent is implied by a formula which is T-complete. Equivalently, T has a model which realizes only principal types of the theory T.*

(c) Suppose that T lies on level α of the Morley tree and α is a limit ordinal. Then any model of T has Scott rank at least α.

(d) Every countable model M of φ is the unique model of some theory on a terminal node of the Morley tree of φ.

(e) φ is a counterexample to the (absolute) Vaught conjecture iff $\mathcal{T}$ has uncountable height.

Proof. (a) A model of it can be constructed as the union of a chain $(M_{\alpha_i}|i < \omega)$, where $(\alpha_i|i < \omega)$ is cofinal in δ, M_{α_i} is the prime model of $T \restriction F_{\alpha_i}$, and M_{α_i} is F_{α_i}-elementary embedded into $M_{\alpha_{i+1}}$ for all $i < \omega$.
(b) This is simply because T has only countably many types. If some T-consistent formula were not implied by any T-complete formula then we could build a perfect tree of distinct types for T.
(c) Let M be a countable model of T and suppose it has Scott rank $\beta < \alpha$. By Lemma 3.5, the theory of M in the fragment $F_{\beta+3}$ is $\aleph_0$-categorical, so there can be no extension of M on level $\beta + 4$, and so certainly none on level α, contradicting our assumption that T is on level α
(d) Given a countable model M and $\alpha < \omega_1$, let $\mathrm{Th}_\alpha(M)$ be the complete F_α-theory of M. With increasing

α, the $\mathrm{Th}_\alpha(M)$ form a path through the Morley tree which terminates at a countable level by (c), ending with a node at some level α that makes $\mathrm{Th}_\alpha(M)$ $\aleph_0$-categorical.
(e) Since all levels of the Morley-tree are countable, this follows immediately from (d). $\square_{3.6}$

Remark 3.7. *We make essential use of the countability of δ in proving part (a). If we take the union of theories along an uncountable path, we cannot guarantee satifiability by the above argument because we would have to pass countable limit stages δ where we cannot be sure that the union would be the prime-model at level δ.*

3.2 The generic and extended Morley trees

Is it possible to extend the construction of the Morley tree beyond ω_1? We can form the union T_{ω_1} of an ω_1-branch $(T_i | i < \omega_1)$ through the Morley tree, but it is no longer clear that this theory has a model. But let's use a bit of set theory.

Definition 3.8 (The Generic Morley tree)**.** *Enlarge the universe V by making the ω_1 of V countable, with a standard Lévy collapse to a forcing extension $V^* = V[G]$. Now as the scatteredness of φ is absolute (it is Π^1_2) we can build $\mathcal{T}^*$ for φ in V^*.*

This tree will have height $\omega_1^{V^*}$, the ω_2 of V, again by absoluteness (the statement that $\mathcal{T}$ has uncountable height is again Π^1_2). We will call this tree the generic Morley tree.

Theorem 3.10 implies that the generic Morley tree is independent of the choice of the generic G used to define V^*.

One crucial point is that $\mathcal{T}^*$ does in fact belong to V. We will now construct in V a sequence of $L_{\omega_2,\omega}$-fragments $\widetilde{F}_\alpha$ of size at most $\aleph_1$ and a tree $\widetilde{\mathcal{T}}$ of height $\aleph_2$ of theories in these fragments and later show that this tree coincides with $\mathcal{T}^*$.

Definition 3.9 (Extended Morley Tree)**.** *Let $\mathbb{P}$ be the set of all finite partial functions from ω to ω_1, ordered by reverse inclusion. We define simultaneously fragments $\widetilde{F}_\alpha \subset L_{\omega_2,\omega}$ and collections $\widetilde{\mathcal{T}}_\alpha$ of $\widetilde{F}_\alpha$-theories by induction over $\alpha < \omega_2$:*

- *Let $\widetilde{F}_0 = F_0$, the same countable fragment containing ϕ that we used for the standard Morley tree at level zero.*
- *Given $\widetilde{F}_\alpha$, let $\widetilde{\mathcal{T}}_\alpha$ be the collection of all sets $A \subset \widetilde{F}_\alpha$ such that*
  - *$\phi \in A$*
  - *there is some $p \in \mathbb{P}$ with $p \Vdash$ "A is a satisfiable, $\widetilde{F}_\alpha$-complete theory and no $A \restriction \widetilde{F}_\beta$ is $\aleph_0$-categorical for $\beta < \alpha$"*
- *Given $\widetilde{\mathcal{T}}_\alpha$, define $\widetilde{F}_{\alpha+1}$ as the smallest $L_{\omega_1,\omega}$-fragment containing $\widetilde{F}_\alpha$ and all formulas of the form $\bigwedge t$ where for some $p \in \mathbb{P}$, $p \Vdash$ "t is a complete $\widetilde{F}_\alpha$-type (over the empty set) realized in a model of ϕ".*
- *if α is a limit ordinal, let $\widetilde{F}_\alpha$ be the union of all $\widetilde{F}_\beta$ for $\beta < \alpha$.*

Finally we set $\widetilde{\mathcal{T}} = \bigcup_{\alpha<\omega_2} \widetilde{\mathcal{T}}_\alpha$ and call it the extended Morley tree.

Recall that the generic Morley tree $\mathcal{T}^*$ is defined as the (standard) Morley tree in a generic extension V^* of the universe V obtained by forcing with $\mathbb{P}$. We will write F^*_α for the α-th fragment of the standard Morley tree from the point of view of V^*.

Theorem 3.10. *$\widetilde{\mathcal{T}}$ equals the generic Morley tree $\mathcal{T}^*$. In particular, $\mathcal{T}^*$ is an element of V. Moreover, $\widetilde{\mathcal{T}}$ contains $\mathcal{T}$ (the standard Morley tree in V) as an initial segment.*

Proof. First we show that if F^*_α belongs to V then any $T \in \mathcal{T}^*$ on level α does too. Suppose not and let $T \in \mathcal{T}^*$ be a counterexample and $\dot{T}$ be a name for it in V. In particular, no element of $\mathbb{P}$ decides exactly what formulas belong to $\dot{T}$ and which do not, which will allow us to build a perfect tree of forcing-conditions, whose paths each force a different interpretation of $\dot{T}$. For that, let B be a countable elementary submodel of some transitive $A \models \mathrm{ZFC}^-$ such that B contains $\mathbb{P}$, ϕ, $\dot{T}$ and F^*_α as elements. We construct the tree inside the (Mostowski-) collapse $\overline{B}$ of B in such a way that those perfectly many paths f are each contained in a filter $G_f \in V$ which is $\overline{\mathbb{P}}$-generic over $\overline{B}$, where $\overline{\mathbb{P}}$ is the image of $\mathbb{P}$ under collapse (i.e. we make sure each path hits the countably many $\overline{\mathbb{P}}$-dense sets of $\overline{B}$). Since B knows that $\dot{T}$ is forced to be satisfiable (due to its belonging to $\mathcal{T}^*$), it follows that $\overline{\dot{T}}$ (the image of $\dot{T}$ under collapse) is forced to be satisfiable and we find a model M_f of T_f, the interpretation of $\overline{\dot{T}}$ given by G_f, in $\overline{B}[G_f] \subset V$. Also, the sentence ϕ belongs to T_f as it is not moved under collapse (due to its countability). But by the absoluteness of the satisfaction-relation, all those models (from $\overline{B}[G_f]$'s point of view) are also models from V's point of view. So we have found a perfect set of models of ϕ in V, contradicting scatteredness.

Now we show by induction over α that $\widetilde{\mathcal{T}}_\alpha$ and $\mathcal{T}^*_\alpha$, as well as the corresponding fragments, and $\widetilde{F_\alpha}$ and F^*_α, coincide. The limit stages are immediate by taking unions.

We begin by showing that if the fragments $\widetilde{F}_\alpha$ and F^*_α at some level $\alpha = \beta+1 < \omega_2$ coincide, then $\widetilde{\mathcal{T}}_\alpha = \mathcal{T}^*_\alpha$. Let $T \in \widetilde{\mathcal{T}}_\alpha$. By definition, there is some $p \in \mathbb{P}$ that forces that T is a complete (for $\widetilde{F}_\alpha$), satisfiable theory, not categorical in any preceding fragment. Now using homogeneity of $\mathbb{P}$ (see e.g. [17], exercise (E1), pp. 244, 245), there is also some q in the generic filter used to define V^* that forces these properties, so they are true in V^*, which means that T satisfies in V^* the properties required to belong to $\mathcal{T}^*_\alpha$. Conversely, if $T \in \mathcal{T}^*_\alpha$, this means that it is complete (for F^*_α) and satisfiable and not categorical in any earlier fragment. Thus there must be some forcing-condition $p \in \mathbb{P}$ forcing these properties. By the first argument in this proof, T is known to be an object in V, and p witnesses that it belongs to $\widetilde{\mathcal{T}}_\alpha$.

To complete the induction step, we have to show that the fragments at level $\alpha + 1$ coincide, now knowing that $\widetilde{\mathcal{T}}_\alpha = \mathcal{T}^*_\alpha$. This follows from the fact that all (fragment-) types realized in V^* in models of theories in $\mathcal{T}^*$ belong already to V, which is true by the same argument as in the beginning of this proof, applied to names of types $\dot{p}$ rather than names of theories $\dot{T}$.

For the *moreover*-part, we simply observe that the fragments and theories in question are already countable in V, and thus we have absoluteness of satisfiability of the theories, as well as prime-models in V, which gives us, for $\alpha < \omega_1^V$, precisely the same theories we have on the standard Morley-tree in V. $\Box_{3.10}$

From the construction of the generic Morley tree, we use the following property of any T in the generic Morley tree.

Definition 3.11. *Let F be an $L_{\omega_2,\omega}$-fragment of size at most $\aleph_1$ and T a collection of F-sentences. T is* generically F-atomic *if in V^*, T is a satisfiable F-atomic $L_{\omega_1,\omega}$-theory*

Immediately from Theorem 3.10 we have:

Lemma 3.12. *In V, for any $\alpha < \omega_2$, any theory $T \in \widetilde{T}_\alpha$ is generically F_α-atomic.*

As in the proof of Theorem 3.10, this means there is some $p \in \mathbb{P}$ that forces "$\dot{T}$ is a complete (for $\dot{F}_\alpha$), satisfiable, theory that is $\dot{F}_\alpha$-atomic". This fact is key in the proof of Lemma 3.23.

3.3 Direct limits of fragments, theories and models

Our goal in this section is to show:

Theorem 3.13 (Model Existence theorem)**.** *If T is a theory on $\widetilde{\mathcal{T}} = \mathcal{T}^*$ then T has a model.*

The proof of this theorem will be immediate from Lemmas 3.21 and 3.23.

To prove these lemmas we need some further machinery. We begin with some standard notions.

We consider here directed systems indexed by ordinals. Recall that a *directed system of sets, indexed by an ordinal* α consists of (X_i, f_{ij}) where for each $i < j < k < \alpha$, the X_i's are sets, $f_{ij} : X_i \to X_j$, and satisfy $f_{ii} = id$ and $f_{ik} = f_{jk} \circ f_{ij}$.

Given any directed system (X_i, f_{ij}), we denote the direct limit by X^*. Additionally, for each $i < \alpha$, we let $f_i : X_i \to X^*$ denote the canonical map.

Definition 3.14. *We say that a directed system (X_i, f_{ij}) indexed by α is* continuous *if, for all non-zero, limit ordinals $\beta < \alpha$, we have X_β equal to the direct limit of $(X_i, f_{ij})_{i<\beta}$ and, for each $i < \beta$, the canonical map f_i is equal to $f_{i\beta}$.*

Consider the theory T^*_α in the fragment F^*_α. This proof is uniform in α so we write T for T^*_α and F for F^*_α. We will construct the model of T using the following directed system.

Definition 3.15. *Let A be a transitive model of ZFC$^-$ of size ω_1 which contains T, F, each τ-symbol, and ϕ as elements. Let $(A_i \mid i < \omega_1)$ be a continuous increasing chain of countable elementary submodels of A such that T, F, each τ-symbol and ϕ are elements of A_0. For each $i < \omega_1$ let $p_i : A_i \to \overline{A}_i$ be the Mostowski collapse of A_i. If $i < j$, we have an elementary embedding $\pi_{ij} : \overline{A}_i \to \overline{A}_j$ given by $\pi_{ij} = p_j \circ p_i^{-1}$.*

To motivate the next set of definitions and arguments let us examine what happens to an F-formula $\bigwedge_{x \in X} \chi_x$ where each $\chi_x \in F$ and $|X| = \aleph_1$. First note that each χ_x is in some A_i. But some χ_x may themselves be uncountable conjunctions and then some of the conjuncts will be missing from A_i (and so from $\overline{A}_i$). So while each π_{ij} is the identity on $L_{\omega,\omega}(\tau)$ an infinite conjunction (disjunction) will gain elements as we pass from $\overline{A}_i$ to $\overline{A}_j$. This is the case of clause 3 in Definition 3.16.

In the following we consider fragments F_α in vocabularies τ_α. In the first order application of the construction, the F_α will always be $L_{\omega,\omega}$ and the vocabularies will vary. In the application to Harrington's theorem, the vocabulary is fixed but the fragments grow.

Definition 3.16. *A* directed system of fragments *of length $\beta \leq \omega_1$ is a continuous directed system (F_i, π_{ij}) where for $i < \beta$ each F_i is a countable fragment of $L_{\infty,\omega}(\tau_i)$ and the maps π_{ij} satisfy the following for each $i < j < \beta$:*

- *π_{ij} is the identity on atomic formulas;*
- *π_{ij} commutes with each of $\neg, \wedge, \vee, \exists$; and*
- *for each $\theta(\mathbf{x}) \in F_i$,*

– *θ and $\pi_{ij}(\theta)$ have the same free variables;*
– *θ is a disjunction (conjunction) if and only if $\pi_{ij}(\theta)$ is a disjunction (conjunction); and*
– *ϕ is a disjunct (conjunct) of θ if and only if $\pi_{ij}(\phi)$ is a disjunct (conjunct) of $\pi_{ij}(\theta)$.*

Fact 3.17. *Any continuous directed system of fragments (F_i, π_{ij}) of length β has a limit which is a fragment F^* of $L_{\infty,\omega}(\tau^*)$ (where τ^* is the union of the τ_α).*
That is, for each $i < \beta$ there is an $\pi_i : F_i \to F^*$ such that for any $i < j$ and $\phi \in F_i$, $\pi_i(\psi) = \pi_j(\pi_{ij}(\phi))$.

Definition 3.18. *Suppose that (F_i, π_{ij}) is a continuous directed system of countable fragments of length $\beta \leq \omega_1$ and that for each i, M_i is an τ_i-structure.*

1. *A mapping $\sigma_{ij} : M_i \to M_j$ is π_{ij}-*elementary *if, for all $\theta(\mathbf{x}) \in F_i$ and all $\boldsymbol{a} \in M_i^{lg(\mathbf{x})}$,*

$$M_i \models \theta(\boldsymbol{a}) \qquad \textit{if and only if} \qquad M_j \models \pi_{ij}(\theta)(\sigma_{ij}(\boldsymbol{a})).$$

2. *A directed system $(F_i, \pi_{ij}, M_i, \sigma_{ij})$ of fragments and models is a pair consisting of a directed system of fragments (F_i, π_{ij}) and a directed system of τ_i-structures (σ_{ij}, M_i) such that for each $i < j < \beta$, σ_{ij} is π_{ij}-elementary.*

The following is evident from the definition of direct limit.

Lemma 3.19. *Suppose $(F_i, \pi_{ij}, M_i, \sigma_{ij})$ is directed system of fragments and models. There is a direct limit $(F^*, \pi_i, M^*, \sigma_i)$, where σ_i is a τ_i embedding such that:*

1. *$\sigma_i = \sigma_j\sigma_{ij}$ for $i < j < \beta$.*
2. *Every element of M^* is in the image of σ_i for all sufficiently large $i < \beta$.*
3. *For $\psi \in F_i$ and $\boldsymbol{a} \in M_i$,*

$$M_i \models \psi(\boldsymbol{a}) \Leftrightarrow M^* \models \pi_i(\psi)(\sigma_i(\boldsymbol{a}))$$

Definition 3.20. *The directed system $(F_i, \pi_{ij}, M_i, \sigma_{ij})$ is* atomic *if each M_i is an F_i atomic model* and *a formula $\theta(\mathbf{v}) \in F_i$ is F_i-complete if and only if $\pi_{ij}(\theta(\mathbf{v})) \in F_j$ is F_j-complete.*

We check a crucial point.

Lemma 3.21. *Suppose $(F_i, \pi_{ij}, M_i, \sigma_{ij})$ is an* atomic *directed system with direct limit $(F^*, \pi_i, M^*, \sigma_i)$. Then M^* is atomic and $\theta(\mathbf{v}) \in F_i$ is F_i-complete if and only if $\pi_i(\theta(\mathbf{v})) \in F^*$ is F^*-complete.*

Proof. If $\theta(\mathbf{v}) \in F_i$ is not F_i-complete then since π_i preserves finite boolean operations $\pi_i(\theta(\mathbf{v})) \in F^*$ is not F^*-complete by taking the image of the witness to incompleteness. Conversely, suppose $\theta(\mathbf{v}) \in F_i$ is F_i-complete, and $\chi \in F^*$. For some $j > i$ and some $\psi \in F_j$, $\pi_j(\psi) = \chi$. Since $\pi_{ij}(\theta)$ is complete,

$$M_j \models (\forall \mathbf{v})\pi_{ij}(\theta)(\mathbf{v}) \to \psi(\mathbf{v}) \text{ or } M_j \models (\forall \mathbf{v})\pi_{ij}(\theta)(\mathbf{v}) \to \neg\psi(\mathbf{v}).$$

Without loss of generality assume the first holds. Then

$$M^* \models (\forall \mathbf{v})\pi_i(\theta)(\mathbf{v}) \to \chi(\mathbf{v})$$

as required. $\square_{3.21}$

This method gives a new proof of a result obtained independently by Knight [14], Kueker [16] and Shelah [25] (in "Various results", chapter IV).

Corollary 3.22. *Suppose T is a complete first order theory in a vocabulary τ of cardinality $\aleph_1$. Then T has an atomic model in $\aleph_1$.*

First, well order the symbols of τ as a sequence with order type ω_1. For each $i < \omega_1$, let τ_i contain those symbols that appear within the first i on the list. And let F_i be $L_{\omega,\omega}(\tau_i)$. Since the isolated types are dense, for every τ-formula ϕ that is consistent with T, there is a complete formula ψ such that $T + \psi \vdash \phi$. It is easily seen that the set

$$C = \{i < \omega_1 : \text{for every consistent } F_i\text{-formula, there is a complete } \psi \in F_i\}$$

is club in ω_1. Thus, by reindexing, we may assume that our original listing has this feature. Now take $\pi_{i,j} = id$ for all $i < j < \omega_1$. Put $T_i := T \cap F_i$. Because of our reindexing, each T_i is a countable theory for which the isolated types are dense, so we can choose a countable, atomic $M_i \models T_i$. The existence of an atomic model M^* of T with cardinality $\aleph_1$ follows immediately from Theorem 3.21. $\square_{3.22}$

And now we show this machinery can be applied to theories on the extended Morley tree.

Lemma 3.23. *Let F be a fragment of $L_{\omega_2,\omega}$ with cardinality $\aleph_1$ and suppose the F-complete theory T is generically atomic. Then there is a directed system $((F_i, T_i, \pi_{ij}) : i < \omega_1)$ where T_i is a theory in the fragment F_i such that the direct limit of $((F_i, T_i, \pi_{ij}) : i < \omega_1)$ is (F, T).*

Further, for each i, T_i is an atomic theory so has an atomic model M_i and an embedding σ_{ij} into M_j so $(F_i, \pi_{ij}, M_i, \sigma_{ij})$ is an atomic directed system and the limit of $(M_i, \sigma_{ij} : i < \omega_1)$ is a model of T of cardinality $\aleph_1$.

Proof. Recall from Definition 3.15 that we have constructed a continuous chain $\langle A_i | i < \omega_1\rangle$ of elementary submodels of a transitive model of ZFC$^-$ of size ω_1. Each of T, F and each τ-symbol, and ϕ are elements of the initial model A_0. For each $i < \omega_1$, let $p_i : A_i \to \overline{A}_i$ be the Mostowski collapse of A_i. If $i < j$, we have an elementary embedding $\pi_{ij} : \overline{A}_i \to \overline{A}_j$ given by $\pi_{ij} = p_j \circ p_i^{-1}$.

The verification that $(F_i, \pi_{ij} : i, j < \omega_1)$ is a direct system of fragments is routine. Suppose for example that $\theta = \bigwedge_{x\in X} \chi_x \in F_i$. The assertion that θ is a conjunction is clearly preserved by elementary embedding. Now $\overline{A}_i \models \chi_x \in X$ for each $x \in p_i(X \cap A_i)$ so since $\pi_{i,j}$ is an elementary embedding $\overline{A}_j \models \pi_{i,j}(\chi_x) \in \pi_{i,j}(X)$ (i.e. $\pi_{i,j}(\chi_x)$ is a conjunct of $\pi_{i,j}(\theta)$ for each $x \in p_i(X \cap A_i)$).

Let $T_i = p_i(T) \in \overline{A}_i$. We have assumed that T is a generically atomic F-theory; by the definability of forcing this property is preserved by elementary equivalence (in set theory) so for each i, T_i is generically atomic in $\overline{A}_i$. Since $\overline{A}_i$ is countable we can build (in V) an $\overline{A}_i$-generic G for $\mathbb{P}^{\overline{A}_i}$. In $\overline{A}_i[G]$, T_i is an atomic theory with an atomic model M_i. But M_i was built in V. Let $\sigma_{ij} : M_i \to M_j$. ($\sigma_{ij}$ exists as T_j extends the complete atomic F_i-theory T_i.) Since π_{ij} is elementary

$$M_i \models \theta(\boldsymbol{a}) \qquad \text{if and only if} \qquad M_j \models \pi_{ij}(\theta)(\sigma_{ij}(\boldsymbol{a})).$$

Similarly, for any $\psi \in F_i$, $\psi \in T_i$ if and only $\pi_{ij}(\psi) \in T_j$. Crucially, since being an atom is elementary, if $\theta \in F_i$ is an F_i-atom in T_i, then $\pi_{ij}(\theta)$ is not only an F_i atom but an F_j-atom in T_j. (This is because $F \in \overline{A}_0$ is coextensive with F_i in $\overline{A}_i$ and F_j in $\overline{A}_j$.) Thus $(F_i, \pi_{ij}, M_i, \sigma_{ij})$ is an atomic directed system. By Lemma 3.21, there is a direct limit M^* which is an atomic model of $T^* = T$. $\square_{3.23}$

3.4 Conclusion: Harrington's theorem

Theorem 3.24 (Harrington)**.** *If ϕ is counterexample to Vaught's conjecture then ϕ has models of Scott rank α for arbitrarily large $\alpha < \omega_2$.*

Proof. For any $\alpha < \omega_2$, choose a theory T_α of height α on the generic Morley tree. By Lemma 3.12, T_α is generically atomic. By Lemma 3.23, T has a model of cardinality $\aleph_1$. And Lemma 3.5 shows that every model of T_α has Scott rank at least α. $\square_{3.24}$

We conclude with two questions.

The first is highly unlikely. Can the proof of Theorem 3.24 be modified to construct two models in $\aleph_1$, one properly contained in the other? We say unlikely because by the the results of the first two sections this would imply Vaught's conjecture.

The second is more plausible. Baldwin [3] observed that deep results of Shelah yield that any first order counterexample to Vaught's conjecture has $2^{\aleph_1}$ models in $\aleph_1$. We have just shown any $L_{\omega_1,\omega}$ counterexample to Vaught's conjecture has $\aleph_2$ models in $\aleph_1$. Can this be extended to $2^{\aleph_1}$?

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