Gerald Sacks, Model Theory and Differentially closed fields

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annotated bibliography, What I learned about model theory and
geometry, number theory ...
great reliance on talks/slides/papers by Marker, Freitag, Nagloo

1 Axiomatic Analysis

2 Definable Analysis

3 Implicit Analysis

We don't discuss non-standard analysis as it deals directly with higher order objects.

Saturated Model Theory

Saturated Model Theory

GERALD E. SACKS
Massachusetts Institute of Technology

1972
W. A. Benjamin, Inc.
ADVANCED BOOK PROGRAM
Reading, Massachusetts

Saturated Model Theory

Sacks

The least misleading example of a totally transcendental theory is the theory of differentially closed fields of characteristic 0 (DCF_0).

[Sac72, p 5]

Axiomatic Analysis

Axiomatic Analysis

Axiomatic analysis

1. Study the behavior of fields of functions with operators
2. *without* explicit attention in the formalism to continuity but rather to the algebraic properties of the functions.
3. The function symbols of the vocabulary act on the functions being studied;
4. the functions are elements of the domain of the model.

The principal example here is differentially closed fields.

Another is the theory of Transseries [AvdDvdH17],

Asymptotic Differential Algebra and Model Theory of Transseries by Aschenbrenner, van den Dries, and van der Hoeven.

Differential Fields

A differential field is a field K of characteristic 0 with a derivation
$D : K \mapsto K$

$$D(a + b) = D(a) + D(b) \text{ and } D(ab) = aD(b) + bD(a).$$

A differential polynomial in variables $X_1, ..., X_n$ over K is an element of
the ring $K\{X_1, ..., X_n\}$ which is

$$K[X_1, ..., X_n, D(X_1), ..., D(X_n), ..., D^{(m)}(X_1), ..., D^{(m)}(X_n), ...].$$

The order of $f \in K\{X_1, ..., X_n\}$ is the largest m such that some $D^{(m)}$
occurs. The constant subfield of K is $C_K = \{x \in K : D(x) = 0\}$.

Differentially Closed Field

Robinson

Abraham Robinson introduced the notion of a differentially closed field, analogous to an algebraically closed field in 1959.

A differential field K is existentially closed if for any finite system Σ of polynomial differential equations having a solution in some $L \supseteq K$ already has a solution in K.

Robinson [Rob59]

1. DCF_0 is the theory of existentially closed differential fields.
2. complicated axiomatization
3. DCF_0 is complete and model complete (Each formula is equivalent to an existential formula.)

Conceptual Axiomatization

Blum

Axioms

$(K, D) \models \mathrm{DCF}_0$ if

1. K is an algebraically closed field of characteristic zero;

2. If $f(X), g(X) \in K\{X\}$ are nonzero and $\mathrm{order}(f) > \mathrm{order}(g)$, then there is an $x \in K$ such that $f(x) = 0 \wedge g(x) \neq 0$.

Theorem ([Blu68])

DCF_0 is a complete theory with quantifier elimination axiomatizing the theory of existentially closed differential fields.

Differential Closure

Theorem

1. *[Blu68]:* DCF_0 *is ω-stable and so has prime models over any subset A of any model.*
2. *By [She72], this model is unique over A.*
3. *But it is* not *minimal [Ros74, She73] and Kolchin.*

Vaught's Conjecture:

Theorem (Shelah 1981)

A complete ω-stable theory has either $\leq \aleph_0$ or $2^{\aleph_0}$ countable models.

Theorem (Hrushovski-Sokolović 1992)

There are $2^{\aleph_0}$ countable differentially closed fields of characteristic zero.
(Because it has eni-dop.)

[Har84, HS93, Mar, Pil06, PZ03]

STRONGLY MINIMAL

Lachlan **Marsh**

$a \in \mathrm{acl}(B)$ if $\phi(a, \mathbf{b})$ and $\phi(x, \mathbf{b})$ has only finitely many solutions.

Definition

A definable set D is strongly minimal if and only if it is infinite and every definable subset of D is finite or cofinite.

This implies

1. algebraic closure induces a pregeometry on D;
2. any bijection between acl-bases for $D(M)$ extends to an automorphism of $D(M)$

These two conditions assign a unique dimension on D
The complex field is strongly minimal.

Integrating in Finite Terms

Painlevé termed 'irreducible' a differential equation that could not be solved as a combination of 'known' functions. (See [Ros72] for an introductory account.)

Key Idea: (Nagloo-Pillay)

The solution set of an ODE is strongly minimal iff it is irreducible.

This is something of a 'Turing Thesis'. As the order of the equation goes up, what solutions of lower order equations should be considered classical?

See work of Unemura, [NP17].

Painlevé Property   Painlevé

Definition

A differential equation has the Painlevé property if the movable singularities are ordinary poles.

Painlevé, Gambier, and Fuchs proved that any order 2 ODE with the Painlevé property can be mapped by fractional linear transformation to one of 50 normal forms. 44 of these are solvable in finite terms (previously known special functions).

Here are the remaining 6.

Painlevé equations

General problem: classify the (differential) algebraic relations over $\mathbb{C}(t)$ between solutions of Painlevé equations.

$$P_I : \qquad \frac{d^2y}{dt^2} = 6y^2 + t$$

$$P_{II}(\alpha) : \qquad \frac{d^2y}{dt^2} = 2y^3 + ty + \alpha$$

$$P_{III}(\alpha, \beta, \gamma, \delta) : \qquad \frac{d^2y}{dt^2} = \frac{1}{y}\left(\frac{dy}{dt}\right)^2 - \frac{1}{t}\frac{dy}{dt} + \frac{1}{t}(\alpha y^2 + \beta) + \gamma y^3 + \frac{\delta}{y}$$

$$P_{IV}(\alpha, \beta) : \qquad \frac{d^2y}{dt^2} = \frac{1}{2y}\left(\frac{dy}{dt}\right)^2 + \frac{3}{2}y^3 + 4ty^2 + 2(t^2 - \alpha)y + \frac{\beta}{y}$$

$$P_V(\alpha, \beta, \gamma, \delta) : \qquad \frac{d^2y}{dt^2} = \left(\frac{1}{2y} + \frac{1}{y-1}\right)\left(\frac{dy}{dt}\right)^2 - \frac{1}{t}\frac{dy}{dt}$$
$$+ \frac{(y-1)^2}{t^2}\left(\alpha y + \frac{\beta}{y}\right) + \gamma\frac{y}{t} + \delta\frac{y(y+1)}{y-1}$$

$$P_{VI}(\alpha, \beta, \gamma, \delta) : \qquad \frac{d^2y}{dt^2} = \frac{1}{2}\left(\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-t}\right)\left(\frac{dy}{dt}\right)^2$$
$$- \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{y-t}\right)\frac{dy}{dt}$$
$$+ \frac{y(y-1)(y-t)}{t^2(t-1)^2}\left(\alpha + \beta\frac{t}{y^2} + \gamma\frac{t-1}{(y-1)^2} + \delta\frac{t(t-1)}{(y-t)^2}\right)$$

Irreducibility of Painlevè equations

Painlevè believed that, at least for general values of the parameters, the set defined by the equations would be 'irreducible'– not integrable in terms of 'classical' functions.

A series of papers by K. Okamoto, K. Nishioka, M. Noumi, H. Umemura, and H. Watanabe (survey in [Oka99]) resolved various parts of this intuition.

Identifying irreducibility with strong minimality powers further progress.

The trichotomy Theorem   Hrushovski

Zilber conjectured the following trichotomy for the combinatorial geometry of a strongly minimal set.

Zilber Conjecture/Hrushovski-Sokolovic̀ Theorem for DCF_0

Every strongly minimal geometry is one of

1. (geometrically) trivial: $\mathrm{cl}(X) = \bigcup_{x \in X} \mathrm{cl}(x)$.

2. locally modular: X is essentially a vector space. Possibly after adding some constant symbols, X is nonorthogonal to the Manin kernel $A^\sharp$ of some simple abelian variety A that does not descend to the field of constants.

3. field-like: non-orthogonal to the field of constants (admits a differential Galois theory in the sense of Kolchin).

S.M. sets rule

Itai

Definition

Two strongly minimal sets D_1 and D_2 are *non-orthogonal* ($D_1 \not\perp D_2$) if there is a definable *n-m*-ary binary relation definable on $D_1 \times D_2$.

Theorem (Hrushovski-Sokolovič)

If $X \subseteq K^n$ is strongly minimal and non-locally modular, then $X \not\perp C$, the field of constants.

Theorem [HI03]

An *algebraically* closed differential field K is *differentially* closed if every strongly minimal formula over K has a solution in K.

The Painlevè conjectures

Nagloo   Pillay

Theorem [Pillay-Nagloo]

For generic values of the parameters, each of the equations of the Painlevé families defines a strongly minimal, geometrically trivial, and ω-categorical set.

E.g. in P_{II}, if α is transcendental the solution set is strongly minimal and (geometrically) trivial.

K. Nishioka [Nis04] proved the strong minimality (irreducibility) for P_I using differential algebra.

Geometric triviality and even irreducibility when the coefficients are not generic has very different behavior depending on the constants. [FN22] for latest.

Geometric triviality and ω-categoricity

Theorem: [Hru95] If the solution set Y of an order 1 ODE is strongly minimal and geometrically trivial, then Y is ω-categorical.

Conjecture: If Y is the solution set of an ~~order 1~~ ODE that is strongly minimal and geometrically trivial, then Y is ω-categorical.

Theorem: [FS18] REFUTED

(Freitag-Scanlon) The solution set of the 3rd order differential equation satisfied by the j-function is strongly minimal and geometrically trivial, but not ω-categorical.

Group Actions

Fact

$\mathrm{PSL}_2(\Re)$ acts on $\mathbb{H} = \{z \in \mathbb{C} : \mathrm{im}(z) > 0\}$ by fractional linear transformations: for
$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ and $\tau \in \mathbb{H}$, $A(\tau) = \left(\frac{a\tau + b}{c\tau + d} \right)$.

This is precisely the group of orientation preserving isometries of $\mathbb{H}$.

Fact

For any discrete subgroup $\Gamma \subseteq \mathrm{PSL}_2(\Re)$, the quotient $\Gamma \setminus \mathbb{H}^*$ is a compact Hausdorff space that can be given the structure of a Riemann surface. Therefore if Γ' is of finite index in Γ, the quotient $\Gamma' \setminus \mathbb{H}^*$ is a compact Riemann surface, and is therefore algebraic by the Riemann existence theorem.

Fuchsian groups

Definition: Fuchsian group

1. A subgroup Γ of $\mathrm{isom}(\mathbb{H})$ is discrete if it is discrete in the induced topology.

2. A *Fuchsian group* is a discrete subgroup of $\mathrm{PSL}_2(\Re)$.

Definition

An automorphic function for a Fuchsian group Γ is a function such that for all $\tau \in \mathbb{H}$ and $g \in \Gamma$: $f(g\tau) = f(\tau)$.

The collection of all automorphic functions for Γ is a field generated (over $\mathbb{C}$) by an automorphic function j_Γ, called an hauptmodul or uniformizer for Γ.

[Kat92]

ODE return

Classically, each uniformizer j_Γ satisfies a third order ODE of Schwarzian type. Denoting the independent variable by t:

$$(*) \quad S_{\frac{d}{dt}}(y) + \frac{1}{2}(y')^2 R_\Gamma(y) = 0,$$

where R_Γ is a rational function with coefficients in $\mathbb{C}$, $y' = \frac{dy}{dt}$, and $S_{\frac{d}{dt}}(y) = \frac{y'''}{y'} - \frac{3}{2}\left(\frac{y''}{y'}\right)$ denotes the Schwarzian derivative and R_Γ is a rational function over $\mathbb{C}$.

Theorem [FS18]

The Schwarzian ODE satisfied by the classical j-function is strongly minimal and geometrically trivial but not ω-categorical.

This proof applied results of Pila's proved using o-minimality. (Generalization and DCF_0-proof below.)

19th to the 21st century:

Lindemann-Weierstrass Theorem:

If $\alpha_1, \ldots \alpha_n$ are linearly independent over $\mathbb{Q}$, then $e^{\alpha_1}, \ldots, e^{\alpha_n}$ are algebraically independent over $\mathbb{Q}$.

Schanuel Conjecture:

If $\alpha_1, \ldots \alpha_n$ are linearly independent over $\mathbb{Q}$, then $\{\alpha_1, \ldots \alpha_n, e^{\alpha_1}, \ldots, e^{\alpha_n}\}$ has transcendence degree at least n over $\mathbb{Q}$.

Ax-Lindemann-Weierstrass Problem:

Let $t_1, \ldots t_n \in \mathbb{C}(X)$, X an algebraic variety. Classify when $(j_\Gamma(t_1), \ldots, j_\Gamma(t_n))$ algebraically dependent.

Ax-Lindemann-Weierstrass with derivatives:

Notation

Suppose $P(y, \delta y, \ldots, \delta^{(n)} y) = 0$, where $P \in k[X_0, ..., X_n]$ is irreducible.

For each $m \geq 1$, consider the condition:

(C_m): For any m distinct solutions $a_1, \ldots, a_m \notin k^{alg}$, the sequence of $n \times m$ functions
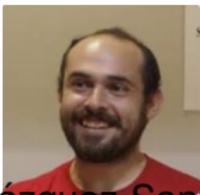
$\delta_j^i = \delta^i(a_j)$ for $i < n, j < m$
is algebraically independent over k.

Note C_{m+1} implies C_m.

ALW with derivatives I:

Casale   Blázquez-Sanz

Theorem: Casale, Freitag, Nagloo [CFN21]

Let Γ be a Fuchsian group of first kind and genus zero, and let X_Γ be the set defined by the Schwarzian equation (*). Then:

1. X_Γ is strongly minimal and (so) geometrically trivial.

2. X_Γ is ω-categorical if and only if Γ is nonarithmetic. Recognized algebraically by $[\mathrm{Comm}(\Gamma) : \Gamma]$ is infinite.

Theorem: Blázquez-Sanz, Casale, Freitag, Nagloo [BSCFN20]

This paper extends the above result by weakening the hypothesis on Γ.

ALW with derivatives II: Jaoui Moosa

When any 3 solutions are independent (Freitag, Jaoui, Moosa)

[FJM18] For an ODE of order $n > 1$, $C_3 \to C_m$ for all m.

3-independence implies independence. (A natural way to classify sm sets considered but discarded in the 80's.)

The results on the last two slides generalize [FS18]. They use the methods described here and no longer rely on o-minimality.

Generic ODE are strongly minimal Devilbiss

Theorem (Devilbiss-Freitag)

If X is the solution set of differential polynomial of arbitrary order and sufficiently large degree whose coefficients are differentially algebraically independent over K, X is strongly minimal.

For constant coefficients algebraic independence suffices.
[Jao21, DF21]

Definable Analysis

Definable Analysis

1. The domain of the functions is the universe of the model.
2. The functions being studied are the compositions of the functions named in the vocabulary; one cannot quantify over them.

The main model theoretic tool is *o*-minimality.
The study of linearly ordered structures such that **Every definable subset is a Boolean combination of intervals**.
Sample theorems involving *o*-minimality

1. $(\Re, +, \times, \exp)$ is o-minimal [Wil96]
2. Pila-Wilkie theorem [PW06]
3. Andrè-Oort [Pil11]
4. Ax-Lindemann-Weierstrass using o-minimality [PT14]

o-minimal approach to transcendence theory

$\Re_{an,exp}$ is the o-minimal theory of the reals expanded by analytic functions restricted to bounded domains and exponentiation.

The analytic function under consideration (restricted to an appropriate fundamental domain) is interpretable in $\Re_{an,exp}$. (See [CFN21].)

Key tool:

Pila-Wilkie theorem

There are few rational points on the transcendental part of a set definable in an o-minimal expansion of the real field.
The number of such points of height T grows more slowly that T^{ϵ}

(See [CFN21] for connections to Axiomatic analysis approach.)

Implicit Analysis

Implicit Analysis

Goal

Provide 'algebraic' characterizations of important mathematical structures by axiomatizations in infinitary logic that are categorical in power.

Example Zilber's $L_{\omega_1,\omega}(Q)$ theory of psuedo-exponentiation that is categorial in all uncountable powers and conjecturally an axiomatization $\mathrm{Th}(\mathbb{C}, +, \times, e^x)$.

Zilber's program

Zilber

The initial hope of this author in [Zil84] that any uncountably categorical structure comes from a classical context (the trichotomy conjecture), was based on the belief that logically perfect structures could not be overlooked in the natural progression of mathematics.
([Zil05a])

The geometric value of the project is perhaps in the fact that the formulation of the categorical theory of the universal cover of a variety X (essentially the description of $\mathbb{U}$) is essentially a formulation of a complete formal invariant of X. [ZD21, 2]

Example

The simplest example of such a cover is given by the short exact sequence:

$$0 \to \ker(\exp) \to (\mathbb{C}, +) \overset{\exp}{\to} (\mathbb{C}, \cdot) \to 1. \tag{1}$$

The logic $L_{\omega_1, \omega}$ is used to guarantee the kernel is 'standard' – isomorphic to $(\mathbb{Z}, +)$.

More recent works extends to algebraic geometry.

$$X^+ \overset{q}{\to} \mathbb{V} \to 1. \tag{2}$$

The map remains a projection but there is no longer a kernel when $\mathbb{V}$ is not a group. So the infinitary description needs at least a 'standard fibre' rather than a 'standard kernel'.

The two-sorted structure

We want to give an 'algebraic' axiomatization of a classical universal cover.

$$X^+ \xrightarrow{p} \mathbb{V} \to 1. \tag{3}$$

Classical Situation:

1. A variety $\mathbb{V}$ is given as the quotient of the action by a discrete group on $\mathbb{H}$ (hyperbolic space).
   (more generally a hermitian symmetric domain X^+ (Shimura varieties)

2. $T := \mathrm{Th}(\mathbb{V})$ in a large enough countable language that T has quantifier elimination.
   (Obtained because $\mathbb{V}$ is definable in $(\mathbb{C}, +, \times)$.)

Abstract formulation

A structure will be of the form $\langle D, R, q \rangle$ where

1. **covering sort** The structure on the domain D will include function symbols f_g for $g \in G$, and perhaps more complicated relations $\mathcal{R}_D$.

2. **field sort** The range R is the realizations of $\mathbb{V}$ (i.e. the Shimura variety) over a field F with a specified named set of constants (of geometric significance).
   The relations $\mathcal{R}_S$ will be imposed from an underlying field.

3. $q : D \rightarrow R$ is a function.

An arbitrary structure of this vocabulary will be denoted $\langle \tilde{M}, M, q \rangle$

A Classical Example

$$\langle\langle\mathbb{H}, \{f_g : g \in \mathrm{PSL}_2(\mathbb{Z})\}\langle, \langle\mathbb{C}, +, \times\rangle j\rangle$$

$\mathbb{H}$ is D, $\mathbb{C}$ is R, j is q.

The classical j function gives a complex analytic isomorphism

$$j : (\mathrm{PSL}_2(\mathbb{Z}) \setminus \mathbb{H}) \to \mathbb{C}$$

realizing $\mathbb{A}^1(\mathbb{C})$ as the moduli space of elliptic curves. For $\tau, \tau' \in \mathbb{H}$.

1. $j(\tau) = j(\tau')$
2. $\tau = g\tau'$ for some $g \in \Gamma$
3. τ and τ' generate isomorphic elliptic curves.

The 'etale' cover

Let $\Gamma = \mathrm{PSL}_2(\mathbb{Z})$, and

$$\Gamma_N = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma : \begin{bmatrix} a & b \\ c & d \end{bmatrix} \equiv_N \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\}.$$

Note that each Γ_N has finite index in Γ and if $N|M$ then $\Gamma_M \subseteq \Gamma_N$.
Let Z_N be subvariety of $\mathrm{PSL}_2(\mathbb{Z}) \setminus \mathbb{H}$ given by $\Gamma_N \setminus \mathbb{H}$.

A non-classical cover

Central to each proof is the construction of an 'etale' ('profinite') cover $\tilde{M}$.

$$\hat{M} = \varprojlim Z_N$$

with maps $\rho_{M,N} : Z_N \twoheadrightarrow Z_M$ if $M|N$ and $\rho : \tilde{M} \twoheadrightarrow M$

$\hat{M}$ is D, M is R, ρ is p.

Proof Strategy

General strategy

1. Axiomatize $L_{\omega,\omega}$-theory of $(\hat{M}, M, q)$ by a superstable, quantifier-eliminable theory $\hat{T}$.

2. Add $L_{\omega_1,\omega}$-axioms to obtain $\hat{T}_\infty$ and gain ω-stability, control the fibers, insist certain subgroups have finite index.

3. Show the classical cover $(X^+, \mathbb{V}, p)$ satisfies $\hat{T}_\infty$.

4. Show for any $(\tilde{M}, M, q) \models \hat{T}_\infty$, the isomorphism type of the triple is determined by the isomorphism type (cardinality) of M and of the kernel:

A fork in the road

- **A** $\mathbb{V}$ is a (modular or Shimura) curve.
- **B** $\mathbb{V}$ is a Shimura variety
- **C** Abelian varieties can be handled by either method.

A closer look at the forks: Shelah's analysis returns

A Method : quasiminimal excellence:
The setting is an abstract elementary class with a global combinatorial geometry and n-dimensional amalgamation for all *n* (excellence).
The result is categoricity in all uncountable powers.

B Method: first order excellence: $\hat{T}$ is superstable. There are excellent (n-dimensional amalgamation) I-independent systems of 'constructible' models.
Each model in $\aleph_1$ can be 'characterized'.
Are the characterizations 'fully formalized'?
Categoricity holds in the continuum. [BHP20, BGH14]

Categoricity of Covering Spaces

topic	paper	method
Complex exponentiation	[Zil05b]	quasiminimality
cover of mult group	[Zil06]	quasiminimality
	[BZ11]	quasiminimality
j-function	[Har14]	quasiminimality
Modular/Shimura Curves	[DH17]	quasiminimality
	[ZD21]	quasiminimality
Abelian Varieties & Kummer	[BGH14]	finite Morley rank groups
	[BHP20]	fmr & notop
Shimura *varieties*	[Ete22]	notop

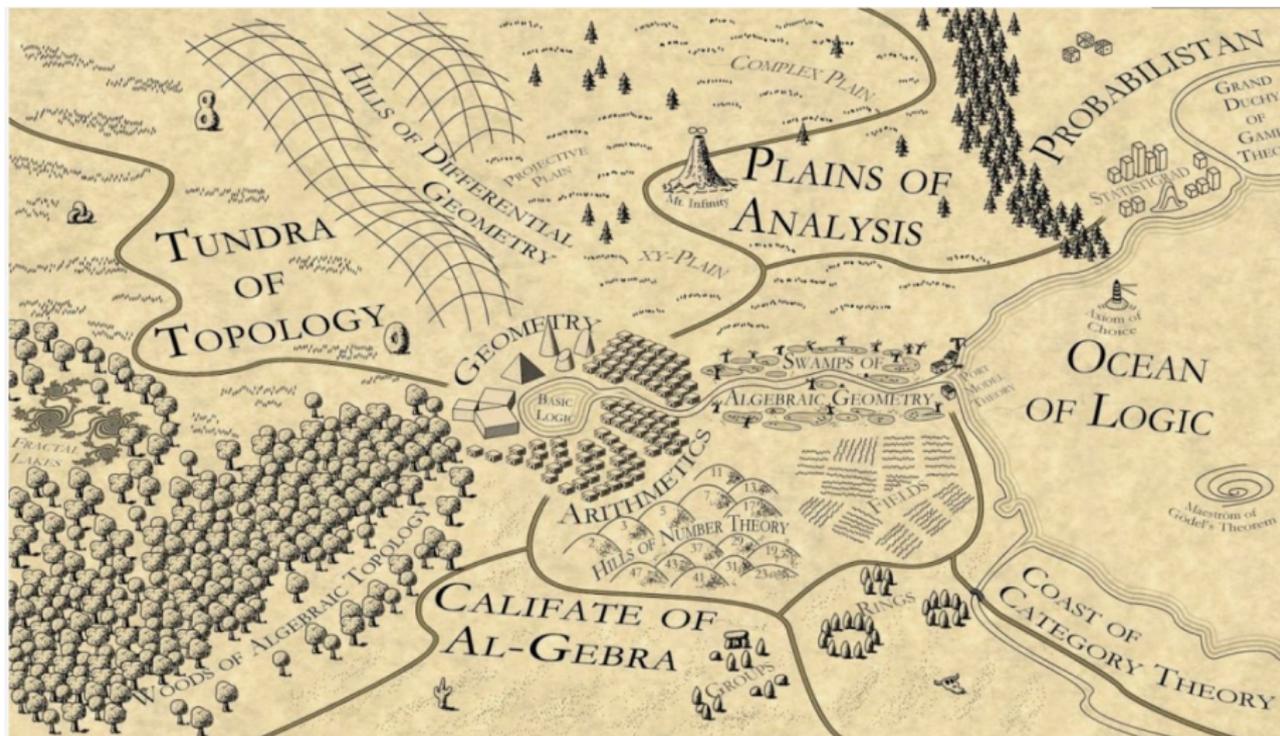# Authors

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missing pictures: Gavrilovich Harris

Geography of Mathematics: Deep not Basic Logic

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