Learning Seminar: Categoricity of Canonical Structures and Fucshian groups

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Sept 16, 2022

Formalization

Anachronistically, *full formalization* involves the following components.

1. Vocabulary τ: specification of primitive notions.
2. Logic $\mathcal{L}$:
   - a Specify a class $\mathcal{L}(\tau)$ of well formed formulas.
   - b Specify truth of a formula from this class in a structure.
   - c Specify the notion of a formal deduction for these sentences.
3. Axioms: specify the basic properties of the situation in question by sentences of the logic.

Three Logics

- The first order language ($\mathcal{L}_{\omega,\omega}(\tau)$) associated with τ is the least set of formulas containing the atomic L-formulas and closed under **finite** Boolean operations and quantification over finitely many individuals.
- The $\mathcal{L}_{\omega_1,\omega}(\tau)$ language associated with L is the least set of formulas containing the atomic τ-formulas and closed under **countable** Boolean operations and quantification over finitely many individuals.
- The second order $\mathcal{L}^2(\tau)$ language associated with τ is the least set of formulas extending $\mathcal{L}_{\omega,\omega}(\tau)$ by allowing quantification over sets and relations. Morally equivalent to set theory.

Structures and Definability

A vocabulary τ is collection of constant, relation, and function symbols.

A τ-structure is a set in which each τ-symbol is interpreted.

A subset A of a τ-structure M is $\mathcal{L}$-definable in M if there is $\mathbf{n} \in M$ and an $\mathcal{L}(\tau)$-formula $\phi(x, \mathbf{y})$ such that

$$A = \{m \in M : M \models \phi(m, \mathbf{n})\}.$$

$\mathrm{tp}_M(a/B)$ is the set of $L_{\omega_1,\omega}$ formulas with parameters from b that are satisfied in M (for $a, B \subseteq M$).

Note that if a property is defined without parameters in M, then it is uniformly defined in all models of $\mathrm{Th}(M)$.

Categoricity

Definition

An $\mathcal{L}$-theory T is categorical if there is a unique model of T.
An $\mathcal{L}$-theory T is categorical in power κ if there is a unique model of T with cardinality κ.

Context

1. Every structure is categorical in ZFC. (by definition)
2. Many natural structures are categorical in $\mathcal{L}^2$.
   $\Re(+,\times,<)$, $\mathcal{C}(+,\times,<)$, The Euclidean plane
3. In $L_{\omega,\omega}$ or $L_{\omega_1,\omega}$: Categoricity in power implies structural properties -specifically dimension; the structure is controlled by a geometry.
   $\mathcal{C}(+,\times,<)$ is categorical in all uncountable cardinals in $L_{\omega,\omega}$ and so in $L_{\omega_1,\omega}$.
   $\Re(+,\times,<)$ is not categorical in any uncountable cardinal by any sentence in $L_{\omega_1,\omega}$.

Zilber's program

Motto

Canonical structures should have informative descriptions
That is, an axiomatization that is more than a mere description.

> *The geometric value of the project is perhaps in the fact that the formulation of the categorical theory of the universal cover of a variety X (essentially the description of $\mathbb{U}$) is essentially a formulation of a complete formal invariant of X. [ZD21, 2]*

Combinatorial Geometries (matroids)

Definition. A pregeometry is a set G together with a 'dependence' relation

$$cl : \mathcal{P}(G) \to \mathcal{P}(G)$$

satisfying the following axioms.

A1. $cl(X) = \bigcup\{cl(X') : \; X' \subseteq_{fin} X\}$
A2. $X \subseteq cl(X)$
A3. If $a \in cl(Xb)$ and $a \notin cl(X)$, then $b \in cl(Xa)$.
A4. $cl(cl(X)) = cl(X)$
If points are closed the structure is called a geometry.

Quasiminimality

A class $(\boldsymbol{K}, \mathrm{cl})$ is *quasiminimal* if cl is a combinatorial geometry which satisfies on each $M \in \boldsymbol{K}$:

1. closure is determined by quantifier free types;
2. there is a unique type of a basis;
3. a technical homogeneity condition: $\aleph_0$-homogeneity over $\emptyset$ and over closed subsets.
4. Closure of countable sets is countable
5. ([BHH$^+$14]) Every model has infinite dimension.

Theorem

A quasiminimal class is $\aleph_1$-categorical.

Excellence

Let M be a quasiminimal pregeometry structure and $B \subseteq M$ be a countable independent subset of M.
Write $M_B = \mathrm{cl}(B)$, and $\mathbf{b} = \langle b_0, \ldots b_{n-1} \rangle$.
$\partial_i(B) = \mathrm{cl}(B - b_i)$
the n-crown: $\partial M_B = \bigcup_{i<n} \partial_i(M_B)$

Definition

A quasiminimal pregeometry structure $<$ is *excellent* if for every $n \in \mathbb{N}$ with $n > 2$ and every n-crown $\partial M_B \subset M$, and every finite $\boldsymbol{a} \in M_B$, there is finite $X \subset \partial(M_B)$ such that $\mathrm{tp}(\boldsymbol{a}/X) \models \mathrm{tp}(\boldsymbol{a}/\partial(M_B))$.

The importance of excellence

1. [Zil04] Every quasiminimal class is $\aleph_1$-categorical.
2. [Zil04] Every excellent quasiminimal class is categorical in all uncountable cardinalities.
3. [BHH+14] Every quasiminimal class is excellent.

Covers by $\mathcal{C}$

$\mathbb{Z}$-Covers of Algebraic Groups

Definition A $\mathbb{Z}$-cover of a commutative algebraic group $\mathbb{A}(\mathcal{C})$ is a short exact sequence

$$0 \to Z^N \to V \stackrel{\exp}{\to} \mathbb{A}(\mathcal{C}) \to 1. \quad (1)$$

where V is a $\mathbb{Q}$ vector space and $\mathbb{A}$ is an algebraic group, defined over k_0 with the full structure imposed by $(\mathcal{C}, +, \cdot)$ and so interdefinable with the field.

Axiomatizing $\mathbb{Z}$-Covers: first order

Let $\mathbb{A}$ be a commutative algebraic group over an algebraically closed field F.
Let T_A be the first order theory asserting:

1. $(V, +, f_q)_{q \in \mathbb{Q}}$ is a $\mathbb{Q}$-vector space.
2. The complete first order theory of $\mathbb{A}(F)$ in a language with a symbol for each k_0-definable variety (where k_0 is the field of definition of $\mathbb{A}$).
3. $\exp$ is a group homomorphism from $(V, +)$ to $(\mathbb{A}(F), \cdot)$.

Axiomatizing Covers: $L_{\omega_1,\omega}$

Let $\Lambda(x)$ be first order formula defining the kernel of $\exp$. Then $T_A + \Lambda = \mathbb{Z}^n$ asserts the kernel of $\exp$ is standard. Namely,

$$(\exists \mathbf{x} \in (\exp^{-1}(1))^n)(\forall y)[\exp(y) = 1 \rightarrow \bigvee_{\mathbf{m} \in \mathbb{Z}^n} \Sigma_{i<n} m_i x_i = y]$$

Definition of closure

For $X \subset V \models T_A + \Lambda = \mathbb{Z}^n$

$$\mathrm{cl}(X) = \exp^{-1}(acl(\exp(X))).$$

The categoricity result is extended to covers of arbitrary semi-abelian varieties using Kummer theory in [BHP20].

Categoricity Problem

Is $T_A + \Lambda = \mathbb{Z}^N$ categorical in uncountable powers?

paraphrasing Zilber:

> *Categoricity would mean the short exact sequence is a reasonable 'algebraic' substitute for the classical complex universal cover.*

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Theorem

$T_A + \Lambda = \mathbb{Z}^N$ is quasiminimal and so categorical in all uncountable powers.
[Zil04, BHH$^+$14]

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Hyperbolic space and Fuchsian groups

The upper half plane

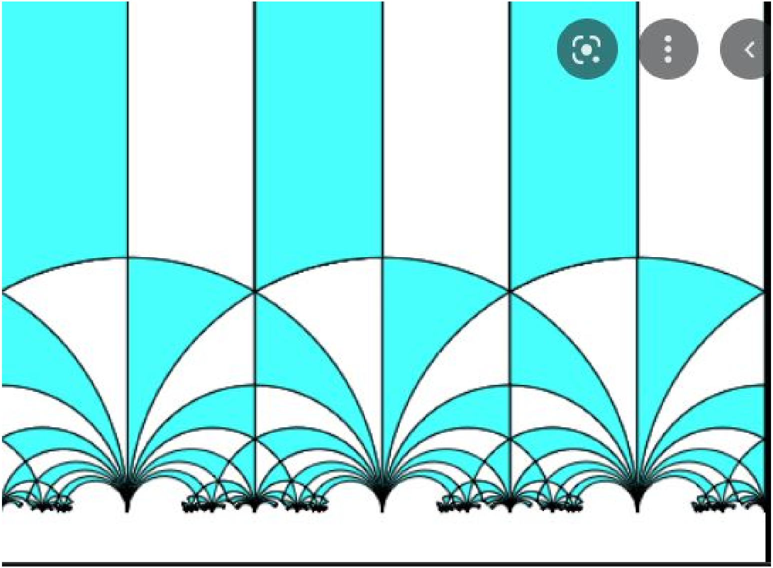

Figure: upper half plane

What is the hyperbolic plane?

Define ‘automorphism

What is the hyperbolic plane?

Define ‘automorphism

1. model theory: permutations of a structure which preserve the relations/functions in the vocabulary.
   e.g. The automorphisms of a geometry are the collineations.
2. category theory: invertible morphism
   in algebraic geometry a morphism is (cf [Poi01]) a constructible (generically quasi-rational) bijection.

Automorphisms of the hyperbolic plane

1. The hyperbolic $\mathcal{H}$.
   1. [Kat92, §1.1, 1.3] and [Miy89] define $\mathcal{H}$ as the metric space with domain the (open) upper half plane under the 'hyperbolic metric'.
   2. This induces a geometry by taking geodesics as lines.
2. $\mathrm{aut}(\mathcal{H})$
   1. [Miy89] denotes by $\mathrm{aut}(\mathcal{H})$ the group of all complex analytic 'automorphisms' of $\mathcal{H}$. This might be taken as distance preserving permutations. But

      $$GL_2^+(\Re)/\Re^\times \approx SL_2(\Re)/\{\pm 1\} \approx \mathrm{aut}(\mathcal{H}) \approx \mathrm{PSL}_2(\Re).$$

      Such maps are conformal so orientation preserving. [CK17, 188 bottom].
   2. [Kat92, §1.3] The group of all isometries of $\mathcal{H}$ is generated by the fractional linear transformation plus $z \rightarrow -\overline{z}$.
      $[PSL_2(\Re) : ISOM(\mathcal{H})] = 2$

Define a 'correct' model theoretic structure on $\mathcal{H}$

Problem

How does one describe 'orientation' in a model theoretic way without introducing an order?

Solution: Follow Klein

Let τ_{cov} contain two sorts H and F, a projection ρ from H onto F and functions symbols f_g for g in a group G.

For $\mathcal{H}$, $G = PSL_2(\Re)$.
G acts on $\mathcal{H}$ by fractional linear transformations.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \tau = \frac{a\tau + b}{c\tau + d},$$

Thus the structure on H is the quotient with domain the orbits under $PSL_2(\Re)$. [Har14, 58]

Fuchsian groups and Congruence Subgroups

Definition

A Fuchsian group is a discrete subgroup of $PSL_2(\Re)$.

Central Example: Congruence Subgroups

For $\Gamma \subseteq SL_2(Z)$, let

$$\Gamma_N = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \;,\; b \equiv c \equiv 0,\; a \equiv d \mod N \right\}.$$

[Har14, 2.2.5] Likely, it is intended that $a \equiv d \equiv 1 \mod N$ ([Miy89, ZD21]).

The j function

The j function as a cover of $\mathbb{A}^1(\mathcal{C})$ by $\mathcal{H}$

j is *precisely* the universal cover of $(\mathcal{H} \cup P^1(\mathcal{C})) \setminus \{0, 1728\}$'.

moduli space of elliptic curves

$PSL_2(Z) \setminus \mathcal{H}$ is the *moduli space* of elliptic curves: j enumerates the isomorphism classes of elliptic curves.
For $z, z' \in \mathcal{C}$, the following are equivalent:

1. $j(z) = j(z')$;
2. The lattices Λ, Λ' generated by $\langle 1, z\rangle$ and $\langle 1, z'\rangle$ are isomorphic (and so the elliptic curves when factored by Λ, Λ' are isomorphic.
3. z and z' are in the same orbit under $PSL_2(\mathbb{Z})$.

Structures for the vocabulary $\mathcal{L}_{cov}$

We say $s \in \mathcal{H}$ (and also $j(s)$) is special if s is fixed by member of G.

the setup

$\tau_{cov} = \mathcal{L}_{cov}$ has unary predicates H and F. A structure for $\mathcal{L}_{cov}$ has the form

$$\mathcal{M} = \mathcal{H} \xrightarrow{j} \langle F, +, \times, C \rangle$$

where $\mathbb{H} = \langle H, \{f_g : g \in G\} \rangle$ with the f_g are unary functions acting on H, G is $GL_2^+(\mathbb{Q})$ mod its center, F is a field, and C is a set of constants.

Th(*j*)

$Th(j)$ is the first order theory of the 'standard model'

$$\mathcal{C}_j = \mathcal{H} \xrightarrow{j} \langle C, +, \times, j(S) \rangle$$

S is the set of special points. We say $s \in \mathcal{H}$ (and also $j(s)$) is special if s is fixed by a member of G.

Harris's results on the *j*-function

Harris summary

[Har14] gives a first-order axiomatisation of the theory of the j-function and shown to be complete with quantifier elimination. An object similar to a pro-ètale cover is defined and shown to be a model of the same first-order theory.
He shows categoricity can be seen as the statement that this pro-ètale cover contains the same types of independent tuples as the standard model. Categoricity then becomes equivalent to certain statements about Galois representations in the geometric ètale fundamental group. In particular, it is close to an instance of the adelic Mumford-Tate conjecture regarding the images of Galois representations in the Tate-modules of products of elliptic curves.

Harris – more detail

Main idea

Using the pro-étale cover of C satisfies $\mathrm{Th}(j)$.

pro-étale cover $\hat{\mathcal{C}}$

$G = GL^+(\mathbb{Q})/Z(GL^+(\mathbb{Q}))$.

$$\hat{\mathcal{C}} = \varprojlim_{g \subset G} Z_g$$

The Z_g are defined in [Har14, 2.2.12].

Let $\mathrm{Th}(j)$ denote the first order theory above plus the standard fiber axiom:

$$\forall x \forall y (j(x) = j(y) \to \bigvee_{\gamma \in SL_2(\mathbb{Z})} x = f_\gamma(y))$$

Harris – main theorem

Theorem

Define a set $\hat{U}$ to be the union of the special G-orbits of H, and the non-special G-orbits of $\hat{C}$. Now define the (model-theoretic) pro-étale cover to be the two-sorted structure

$$\hat{\mathcal{U}} := \langle \hat{U}, G \rangle :\xrightarrow{\hat{j}} \langle \mathcal{C}, +, \times; \mathbb{Q}(j(S)) \rangle$$

where $\hat{j}$ is defined to be j on $\mathcal{H}$.
Then $\hat{\mathcal{U}} \models \mathrm{Th}(j)$.

Crucially, $\mathbb{Q}(j(S))$ means constants for each member of that field.
The main tool of the proof is proving the theory is quasiminimal.

Daw-Harris [DH17] extend these results to covers of Shimura varieties.

Covers by $\tilde{\mathcal{H}}$

Shimura Curves

Notation

1. Let B be a quaternion algebra over a totally real number field F such that $\Re \otimes_F B$ is isomorphic to $M_2(\Re)$ for exactly one embedding of F into $\Re$;
2. let G be the algebraic group over Q whose R-points for any Q-algebra R are the elements of $\Re \otimes B$ of norm 1;
3. there is a surjective homomorphism $\phi : G(\Re)\mathrm{aut}(\mathcal{H})$ with compact kernel.
4. A Shimura curve is the quotient of $\mathcal{H}$ by the image in $\mathrm{aut}(\mathcal{H})$ of a congruence subgroup of $G(\mathbb{Q})$.

For the higher dimensional Shimura varieties see [Mil12].

Modular Curves and the pseudo-analytic cover

A major shift in [ZD21] is the choice of the universal covering space.

$$\tilde{\mathcal{H}} = \varprojlim(\Gamma_N \setminus \mathcal{H})$$

Differences from Daw-Harris

1. different inverse limit for the covering space
2. The elements of the field generated by the special points are not named. But the substitute is unclear.

Transcendence properties

Casales, Freitag, Nagloo, Scanlon etc.

consider transcendence properties of solutions of Schwartzian equations (including *j*) and Fuchsian groups.

The key insight is that the solution set of certain Schwartzian differential equations are strongly minimal subsets of models of DCF_0 so have a natural geometry.

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