Learning Seminar: Categoricity of Canonical Structures and Fucshian groups

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Zilber's program: The role of model theory

Formalization

Anachronistically, *full formalization* involves the following components. [Bal18, Chapter 1]

1. ***Choose* Vocabulary τ: specification of primitive notions.**
2. Choose a Logic $\mathcal{L}$:
   a. Specify a class $\mathcal{L}(\tau)$ of well formed formulas.
   b. Specify truth of a formula from this class in a structure.
   c. Specify the notion of a formal deduction for these sentences.
3. **Axioms: specify the basic properties of the situation in question by sentences of the logic.**

A model theoretic description of $\mathcal{H}$

Use Hilbert's two sorted axiomatization of neutral geometry (e.g. in modern form see [Bal18, §9]). The vocabulary η includes predicates for betweenness, congruence (of segment, triangles ...), etc. Add Hilbert's axiom L [Har00, p 374],which implies the geometry is hyperbolic and limiting parallels exist to get a theory [Har00, p 374] that is bi-interpretable with the theory of ordered fields [Har00, Thm 43.1].

Each model of $T_{\mathcal{H}}$ defines a coordinatizing field.

Adding axioms specifying the field is real closed yields a complete first order η-theory . As an η-structure $\mathcal{H} \models T_{\mathcal{H}}$ and its automorphism group in the vocabulary η is $PSL_2(\Re)$.
But can be no theory in $L_{\omega_1,\omega}$ is categorical in (all) uncountable powers and that this structure satisfies. There is an order.

Find the vocabulary and axioms for categoricity.

key references: [Har14] [DH17]

Group Action

For the hyperbolic space $\mathcal{H}$, $G = PSL_2(\Re)$.
G acts on $\mathcal{H}$ by fractional linear transformations.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \tau = \frac{a\tau + b}{c\tau + d},$$

Vocabulary

Let τ_{cov} contain two sorts H and F, a projection q from H onto F and function symbols f_g for g in a group G, field structure on F, and constants.

Special points

Just as the covering space in the $\mathcal{C}$ case had only addition, the group action is now encoded using only **unary** function symbols.
Consider the upper half plane $H = \{\gamma \in \mathcal{C} : im(\gamma) > 0\}$ with the group $G = GL^+(\mathbb{Q})/Z(GL^+(\mathbb{Q}))$ (+ means positive determinant) acting on it via fractional linear translations.
Why not $PSL_2(\mathbb{Z})$?

Fact: Each $a \in H$ is either special or non-special.

where

1. non-special: No $g \in G$ fixes a.
2. special: There is a $g_a \in G$ with exactly one fixed point a in H. (The quadratic equation derived from $gz = z$ has two complex conjugate roots.)
   So there are countably many special points and they are each in quadratic extensions of $\mathbb{Q}$.
   Note that for every $g \in G$ either $\forall x f_g(x) \neq x$ or $\exists x f_g(x) = x$ in the theory of $\langle H, f_g : g \in G$.

The 'covering sort' $\langle H, \{f_g; g \in G\}\rangle$

The universe is partitioned into *countable* orbits under G.

1. non-special orbit O : Every pair of points in $a, b \in O$ satisfy $f_{a,b}(a) = b$ and $f^{-1}_{a,b}(b) = a$
   G acts strictly 2-transitively on non-special orbits.
2. special orbit O of special point a: Like a non-special orbit but each element of the orbit is fixed by a conjugate of g_a.
   Note that if g fixes a, the conjugate of g by $f_{a,b}$ fixes b.

Thus the structure $\langle H, \{f_g; g \in G\}\rangle$ is *strongly minimal* with trivial geometry.
i.e. $\mathrm{cl}(X) = \bigcup_{x \in X} \mathrm{cl}(x)$.

Congruence Subgroups, and Shimura Varieties

Fuchsian groups, Congruence Subgroups, and Shimura Varieties

Definition

A Fuchsian group is a discrete subgroup of $PSL_2(\Re)$.

Central Example: Congruence Subgroups

For $\Gamma \subseteq SL_2(Z)$, let
Fix $N > 0$. Let

$$\Gamma(N) = \Gamma_N = \left\{ \gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(\mathbb{Z}) \mid \gamma \equiv \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \mathrm{mod} N \right\}.$$

Shimura Curves I

Notation

1. Let B be a quaternion algebra over a totally real number field F such that $\Re \otimes_F B$ is isomorphic to $M_2(\Re)$ for exactly one embedding of F into $\Re$.
2. let $G = G(R)$ be the algebraic group over Q whose R-points for any Q-algebra R are the elements of $\Re \otimes B$ of norm 1; when $R = \Re$, $G = GL_2(\Re)$ [DH17, p 2].
3. there is a surjective homomorphism $\phi : G(\Re) \to \mathrm{aut}(\mathcal{H})$ with compact kernel.
4. A Shimura curve is the quotient, $S(\mathcal{C})$, of $\mathcal{H}$ by (the image in $\mathrm{aut}(\mathcal{H})$ of a congruence subgroup of $G(\mathbb{Q})$), say $p : \mathcal{H} \to S(\mathcal{C})$.

[Mil12] is a splendid summary which includes higher dimensional Shimura varieties.

Shimura Curves II

[DH17] give a more abstract definition replacing $\mathcal{H}$ by a Hermitian symmetric domain X^+ which has a canonical model over a finite abelian extension E, E^{ab} is a maximal abelian extension of E, and $E^{ab}(\Sigma)$ is the field obtained by adjoining Σ, the images in $S(\mathcal{C})$ of the special points.
They replace $GL_2(\Re)$ by the *countable* group:

Definition– Lemma

1. In general G^{ad} denotes $G/Z(G)$. G^+ means elements of G with positive determinant.
2. $G^{ad}(\mathbb{Q}^+) = G^{ad}(\mathbb{Q}) \cap G^{ad}(\Re)^+$
   For $\mathbf{g} \in G^{ad}(\mathbb{Q}^+)$, $Z_{\mathbf{g}}$ is the image of the map $f : X^+ \to S(\mathcal{C})^n$ by $x \mapsto \langle p(g_1 x), \ldots p(g_n x)\rangle$.

Lemma: $Z_{\mathbf{g}}$ is an algebraic variety defined on E^{ab}.

Returning to the axiomatization

General Scheme

1. The domain and the range are both algebraically (first order) as given next.
2. Prove that the theory T' (with infinitary axioms) of the two sorted structure is quasiminimal and so categorical in power.
3. Replace the domain of the cover sort by an 'analytically given object'
   Examples:
   1. In the covers of multiplicative group the universal covering space with all its analytic structure
   2. j function and Shimura case: certain inverse limits.

   But with the structure on the domain still given by the $\{f_g; g \in G\}$.
4. Prove the new 'cover' with a appropriate projection still satisfies T'.

The image (field) sort

Definition

1. the structure $\langle \mathcal{C}, +, \times, ; \mathbb{Q}(j(S)) \rangle$.
   $\mathbb{Q}(j(S))$ means a constant for each element of this field.
2. the first order theory

$$\mathrm{Th}(\langle H, \{f_g; g \in G\}\rangle) \cup \mathrm{Th}(\langle \mathcal{C}, +, \times; \mathbb{Q}(j(S))\rangle)$$

   By $\mathrm{Th}(\langle \mathcal{C}, +, \times; \mathbb{Q}(j(S))\rangle)$ is meant in a vocabulary containing a relation for each Zariski closed set of $\mathcal{C}$ defined over $\mathbb{Q}(j(S))$.

In the standard model, the projection map q (e.g. j) takes (finite sequences of) elements G to a variety $Z_{\mathbf{g}}$ subset of $\mathcal{C}$ that is definable over the constants. [DH17, p 14] $Z_{\mathbf{g}}$ is biholomorphic with $\Gamma_{\mathbf{g}}\mathcal{H}$, where $\Gamma_{\mathbf{g}} = \Gamma \cap \Gamma^{g_1} \cap \ldots \Gamma^{g_n}$.

Connecting the sorts

The connection axioms

1. first order axioms: recall $q : H \to F$.
   1. modularity axioms:
      1. $(\forall x \in H)(q(g_1 x), \ldots q(g_n(x)) \in Z_{\mathbf{g}}$
      2. $(\forall z \in Z_{\mathbf{g}}(\exists x \in H)q((g_1 x), \ldots q(g_n(x)) = z$
   2. A first order scheme called 'special points axiom'. In [DH17]

$$SP_x := (\forall y \in H)[g_x y = y \Rightarrow q(y) = p(y)]$$

   Clearly a typo; perhaps $q(x) = q(y)$ as this connects the sorts. But unnecessay since hypothesis implies $x = y$, as x is the unique fixed point of g_x.

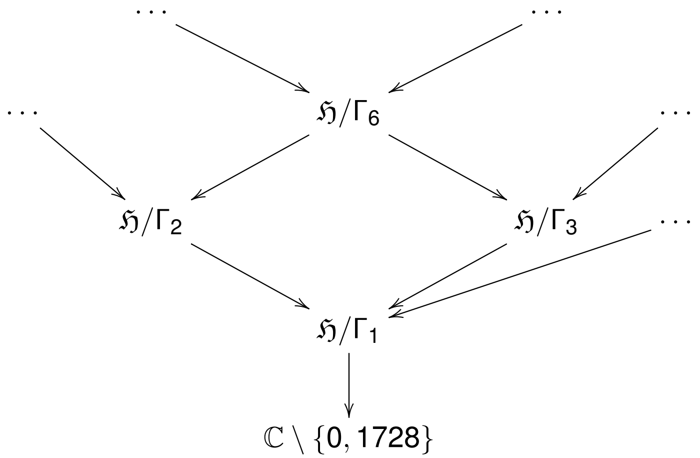
2. infinitary standard fiber axiom:

$$\forall x \forall y [j(x) = j(y) \to \bigvee_{g \in G} x = f_g(y)]$$

Fuchsian Groups and DCF_0

The Modular ‘tower’

Note that $N|M$ implies Γ_M projects to Γ_N. Yielding:



Getting back to $\mathcal{C}$

Fix $\Gamma = \Gamma_N$ or some subgroup of $SL_2(\mathbb{Q})$ contained in *some* Γ_N. Then $\mathfrak{H}/\Gamma$ is a Riemann surface. We may embed $\mathfrak{H}/\Gamma$ (as an affine variety over $\mathbb{C}$) into $\mathcal{H} \cup \mathbb{Q} \cup \{\infty\} /\Gamma$ (as a projective variety over $\mathbb{C}$). Fix the notation $Z_N := \mathcal{H}/\Gamma_N$.

Transcendence properties

Casales, Freitag, Nagloo, Sanz, Scanlon etc.

consider transcendence properties of solutions of Schwartzian equations (including j) and Fuchsian groups.

The key insight is that the solution set of certain Schwartzian differential equations are strongly minimal subsets of models of DCF_0 so have a natural geometry.
Important and old mathematical questions are resolved by using when the combinatorial geometry is trivial and $\aleph_0$-categorical.

Strongly minimal means every first order definable set is finite or cofinite; acl is a combinatorial geometry on any strongly minimal set.

Questions

1. Is the strategy described above used by Daw-Harris [DH17], Daw-Zilber [ZD21]?
2. Do the three papers Harris thesis [Har14] Daw-Harris [DH17], and Daw-Zilber [ZD21] have the same notion of analytic cover?
3. The word ‘étale is central in [ZD21] and [Har14]; I can’t find it in [DH17].
4. How does one distinguish the aims and applications of the infinitary and DCF approach to facilitate interaction?

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