Dividing line strategies for Classification

John T. Baldwin
University of Illinois at Chicago
2021 Fudan Model Theory and Philosophy of Mathematics Conference

November 26, 2022

Overview

Thanks to Chris Laskowski

Setting the Stage

I am grateful for this great honour. While it is great to find full understanding of that for which we have considerable knowledge, I have been attracted to trying to find some order in the darkness, more specifically,

finding meaningful (successful) dividing lines among general families of structures.

This means that there are

meaningful things to be said on both sides of the divide: characteristically, understanding the tame ones and giving evidence of being complicated for the chaotic ones. [She13]

Good test problems help to find the right dividing lines. [She20]

The dividing line strategy

Based on [She20, Bal21]; updating [Bal18, §13]

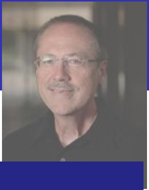
Goals

1. Explore the evolution (at least my understanding) of this notion.
2. Describe the success of the ur-example
3. Examine desirable properties of dividing lines for several examples.

Interlocking notions

- virtuous property
- classification
- dividing line/quasi-order
- role of test question

Detlefsen asked:



Question A

Which view is the more plausible—that theories are the better the more nearly they are categorical, or that theories are the better the more they give rise to significant non-isomorphic interpretations?

Question B

Is there a single answer to the preceding question? Or is it rather the case that categoricity is a virtue in some theories but not in others?

Midwest PhilMath Workshop

What is virtue?

Pragmatic Criterion: [Bal18]

A property of a theory T is virtuous if it has significant mathematical consequences for T or its models.

A property P is a dividing line if both P and $\neg P$ are virtuous.

Successful vs bi-virtuous

A Contrast

1. Successful depends on the test question.
2. P and $\neg P$ each virtuous does not.

Classification

Definition (Button and Walsh [BW17])

Given: A class C of mathematical objects, an equivalence relation E on them, and a set of invariants *Inv*,
a classification is an *easily calculable* function $\iota : C \rightarrow \mathrm{Inv}$ from a canonical presentation of $X \in C$ that

1. maps all elements of an equivalence class to the same invariant;
2. it is also easy to determine for $X, Y \in C$ whether $\iota(X) = \iota(Y)$.

Shelah's classification program

1. The objects of the classification are complete first order theories, not models.
2. There are many different classifications of these theories, e.g.
   1. Keisler order (obtaining saturation)
   2. Stability hierarchy: counting models
      1. By isomorphism
      2. By isomorphic embedding
   3. order by the spectra of λ where T has a universal model ([She20, Bal21])
   4. exact saturation: spectrum of λ with λ but not λ^+ saturated model
3. each classification is stimulated by a test question and usually involves a quasi-order on theories.
4. The strategy extends to other kinds of classes: univeral classes, AEC, etc.

The ur-example- the solution to Morley's conjecture

Morley's conjecture and Shelah's reformulation

The number of non-isomorphic models M of T with $|M| = \kappa$ is $I(T, \kappa)$ – the spectrum function of T.

Test Question: Morley's conjecture

The spectrum function of a countable first order theory is increasing on uncountable cardinals.

Shelah's reformulation

The possible spectrum function of a countable first order theory can be listed; all are increasing.

Stronger version

The spectrum function of a countable first order theory is an invariant; all are increasing.

How the reformulation works

Shelah's strategy

There is a finite list of dividing lines $\{P_i : i \leq n\}$ such that:

1. For any i, $\neg P_i(T)$ implies the spectrum of T is 2^κ and P_i extends control over the number of models of T in each κ. **The question matters**
2. For every T
   1. $P_k(T) \rightarrow P_{k-1}(T)$.
   2. $P_n(T)$ implies there is a ZFC definable function $g(\aleph_\alpha, \gamma)$ and a $\gamma_T = \mathrm{dp}(T) < \omega_1$ such that:

$$g(\aleph_\alpha, \gamma_T) = I(T, \aleph_\alpha) \leq \beth_{\omega_1}(|\alpha| + \omega).$$

   3. Every theory satisfies some P_i.

The choice of the P_i is encapsulated in the dividing line strategy.

Dividing lines for Morley's conjecture

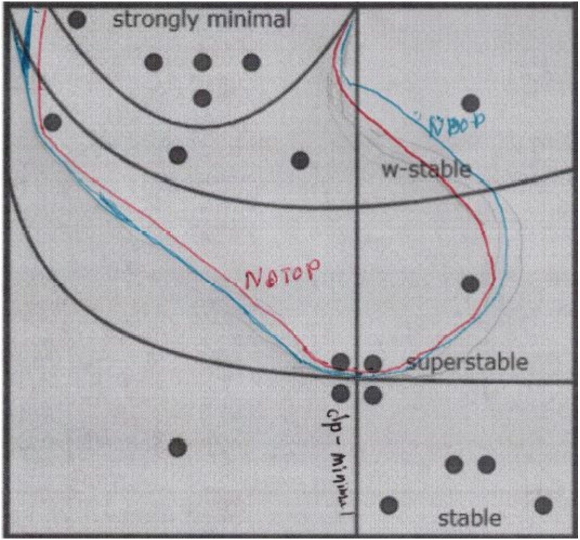
The Stability Hierarchy

Every complete first order theory falls T into one of the following classes.

1. stable
2. superstable
   1. and ndop
   2. and notop
      1. There is a tree associated with T. If it is not well-founded or has infinite depth the spectrum function is an invariant.
      2. For finite depth [HHL00] show that specifying for each theory T further cardinal parameters (each small) and a further model theoretic condition determines the spectrum function of T.

This classification is set theoretically absolute

Refining the map: Missing Dividing Lines



Classifiable theories

External (semantic?) Definition: T is **HHL-classifiable** if

1. Every model N of T is prime and minimal over an independent tree of countable, elementary submodels.
2. If the tree is always well-founded the theory is **shallow** and the maximum depth of such a tree is the $\mathrm{dp}(T)$. Otherwise the theory is **deep**.

Theorem: internal (syntactic) definition

T is **classifiable** iff it is (in order) stable, superstable, ndop, notop and **shallow**.

Corollary

If T is classifiable $I(T, \aleph_\alpha) \leq \beth_{\omega_1}(|\omega + \alpha|)$; otherwise $I(T, \aleph_\alpha) = 2^{\aleph_\alpha}$. Always increasing.

Increasing vs Invariants

1. The positive side of a dividing line implies more structure; the negative side implies $I(T, \kappa) = 2^{\kappa}$, solving the conjecture.
2. Shelah's proof only established the spectrum function as an invariant of T if $\mathrm{dp}(T) \geq \omega$. However, he established each 'potential' spectrum function is increasing.
3. For finite depth [HHL00] show that specifying for a theory two further cardinal parameters (each small) and a further model theoretic condition determines the spectrum function.

What properties must invariants for models have?

1. There must be a proper class of invariants.
2. We require a set of invariants for each κ.
3. But, there should be a uniform method for assigning the invariants for each κ.
4. **easily calculable** The 'form' of the decomposition tree is determined by examining 'small models'; this yields the formula for the spectrum function which is a definable function in ZFC.

What are plausible invariants for models?

Definition

1. cardinal-like invariants [She09, She85]
2. infinitary sentences
   1. $L_{\infty,\lambda}$ [She90, She87]
   2. $L_{\infty,\aleph_\epsilon}(d.q)$ quantifies over (enumerated) algebraic closures of finite sets and
   3. $L_{\infty,\omega_1}(d.q)$ allows quantification over arbitrary (enumerated) countable sets.

What classifications of models are possible or not

Results

1. cardinal-like invariants [She09, She85, Bal88] code a canonical presentation? all the possible decomposition trees for the model.
2. infinitary sentences
   1. Shelah: if T is a classifiable theory,
      1. then the isomorphism type of any model M of T is determined by the theory T_M of that model in $L_{\infty,|M|}$. If not classifiable $I(T_M, \kappa) = 2^\kappa$.
      2. the isomorphism type of any model M of T is determined by $Th_{L_{\infty,\omega_1}(d.q)}(M)$.
   2. [BH06]
      1. For ω-stable theories of depth at most 2, $L_{\infty,\aleph_\epsilon}(d.q)$ does determine the isomorphism type.
      2. But this result fails for ω-stable theories in general and even for a superstable theory of depth 1.

A different notion of classification

Gowers [Gow08, §I.2.1] suggests a second notion of classification:
Identify a class of 'basic' structures from which each member of the target class can be build in a simple way.
This is precisely what Shelah does. Indeed, this is the notion of classification used in my exposition of the main gap. [Bal18, §5.5].

Classification project exposes algebraic relations

John T. Baldwin University of Illinois at Chicago 2021 Fudan Model Theory and Philosophy of Mathematics Conference

An algebraic component is inevitable:
[HHL00], 'mention for instance that any model of a complete theory whose uncountable spectrum is

$$\min(2^{\aleph_\alpha}, \beth_{d-1}(|\alpha+\omega| + \beth_2))$$

for some finite $d > 1$ interprets an infinite group.'

Section III: Dividing Lines

Shelah's Properties of Dividing Lines I

Properties of dividing lines

A property is:

1. robust:
   1. **internally** if it has an *internal* definition, i.e. definable by first order formulas with parameters) in $M \in \boldsymbol{K}$,
   2. and **externally** if there is an equivalent such as having few models up to isomorphism, or that the ultra-powers of any $M \in K$ are 'easily saturated', etc.
2. successful,
   1. **downward** if there is a serious structure theory on the positive side. E.g. we have a general definition of non-forking, or of dimension;
   2. **upward** when it helps to prove complicated models exist for T

Shelah uses internal/external for both 1) and 2).

Shelah's Properties of Dividing Lines II

Further properties of dividing lines

A candidate for being a dividing line is

1. **fruitful**, when the positive theory has applications in parts of mathematics outside model theory.
2. **versatile**, if also for contexts not falling in our framework the machinery developed is helpful.

The stability taxonomy [Bal18, §13.4]

Shelah's program for the Morley conjecture gave

1. a finite hierarchy of successful dividing lines
2. The dividing lines were
    1. **fruitful**: The structure theory given by non-forking and orthogonality has proved its worth across mathematics.
    2. **versatile**: The machinery developed is helpful for logical contexts not falling in our framework
        1. Trivial Strongly Minimal Sets are model complete after naming constants. The spectrum of computable models of any trivial, strongly minimal theory is Σ^0_5. [GHL+03]
        2. resplendency and recursive saturation [Poi91].

Thus, the dividing line strategy was a great success. However, the other classifications do not seem to satisfy all these goals.

Other classification schemes

A map of complete theories

https://www.forkinganddividing.com/

The meaning of the map

Do the lines represent

1. Various overlapping taxonomies
2. individual dividing lines
3. random virtuous properties

Virtuous Properties/Successful Dividing Lines

Virtuous Properties

1. strongly minimality and o-minimality
2. ω-stability and $\aleph_1$-categoricity
3. simplicity
4. n-dependence (Chernikov: n-ary vs binary relations on tuples)

Dividing Lines

1. stable, superstable, ndop, notop for spectrum problem
2. NIP

Are these dividing lines?

Candidate Dividing Lines

1. NSOP –upward successful for MC but not needed
2. $NSOP_2$ upward successful for Keisler order
3. $NSOP_1$ downward successful [KR20]
4. Monadic NIP (argued below)

A general test question

A quasi-order is a transitive reflexive binary relation.

Two examples of quasi-orders on theories

1. Keisler order: $T_1 \leq T_2$ iff every regular ultrafilter that saturates models of T_2 saturates models of T_1.
2. counting models: $T_1 \leq T_2$ iff $I(T_2, \kappa)$ eventually dominates $I(T_1, \kappa)$

What is the structure of the quasi-order?

The Keisler order

Definition: Keisler order

For complete countable first order theories T_1, T_2, we write $T_1 \trianglelefteq T_2$

1. $T_1 \trianglelefteq T_2$ if for any set I, $A_1 \models T_1, A_2 \models T_2$, and regular ultrafilter D on I, if A_2^I/D is I^+-saturated then A_1^I/D is I^+-saturated.
2. $T_1 \trianglelefteq^* T_2$ is a variant.

Test Questions for the Keisler order

1. Understand the order!
2. Is it finite? linear?
3. Are maximal/minimal classes specifiable?

Understanding the order

1. There are two stable classes
   1. without fcp – minimal
   2. with fcp – second lowest
2. SOP_2 implies maximal and is equivalent to $\trianglelefteq^*$ maximal. Robust yes! Successful ???

But, there are infinite descending $\trianglelefteq^*$ chains and $2^{\aleph_0}$ incomparable simple unstable theories. [MS21]

Evaluating the Keisler order I

Classifying first order theories by their place in the Keisler order

1. The order is robust:
   1. **internally** Malliaris [Mal09] gives a syntactic characterization
   2. and **externally** The ultrafilter definition

The individual dividing lines are more tenuous. But SOP_2 almost defines the class of maximal theories in the Keisler order.

Investigating the order has led to much better of understanding of simple theories.

Evaluating the Keisler order II

Further properties of dividing lines

1. **fruitful**: applications to study of the Szemerédi's Regularity Lemma in combinatorics [MS14, MP16]
2. **versatile**, the machinery developed is helpful for logical contexts not falling in our framework.
   1. deep connections with set theory, especially
   2. the $\mathfrak{p} = \mathfrak{t}$ problem

Evaluating the Monadic NIP

1. robust:
   1. **internally** Monadic formula witnesses independence
   2. and **externally** Every theory failing monadic NIP interprets arbitrary structures and has 2^κ models.
2. successful,
   1. **downward** Powerful new notion of independence.
   2. **upward** Every theory failing MNIP (monadically) interprets arbitrary structures and has 2^κ models.
3. **fruitful**: calculating the growth rate of finite structures
4. **versatile** ?

Classifying Strongly minimal sets

Strongly minimal theories with non-locally modular algebraic closure [BV21]

1. Diversity
   1. $2^{\aleph_0}$ theories of strongly minimal Steiner systems (M, R) with no $\emptyset$-definable binary function
   2. $2^{\aleph_0}$ theories of strongly minimal quasigroups $(M, R, *)$ + an example of Hrushovski
   3. Non-Desarguesian projective planes definably coordinatized by ternary fields [Bal95]
   4. strongly minimal eliminates imaginaries (flat) INFINITE vocabulary) (Verbovskiy)

Classifying Strongly minimal sets

Strongly minimal theories with non-locally modular algebraic closure [BV21]

1. Diversity
    1. $2^{\aleph_0}$ theories of strongly minimal Steiner systems (M, R) with no $\emptyset$-definable binary function
    2. $2^{\aleph_0}$ theories of strongly minimal quasigroups $(M, R, *)$ + an example of Hrushovski
    3. Non-Desarguesian projective planes definably coordinatized by ternary fields [Bal95]
    4. strongly minimal eliminates imaginaries (flat) INFINITE vocabulary) (Verbovskiy)
2. Classifying
    1. discrete
    2. non-trivial but no binary function
    3. non-trivial but no commutative binary function
    4. Non-Desarguesian projective planes definably coordinatized by ternary fields [Bal95]

Summary

1. Shelah proposes various classifications of theories for various purposes (title of the bible).
2. Finding dividing lines successfully resolved Morley's conjecture and the search for structure has mathematical consequences.
3. Keisler order is much more complex than hoped. But the minimum was found early and the maximum is being resolved. The search generated specific problems and opened up areas.
4. The search for dividing lines is a useful heuristic.

John T. Baldwin.
Classification theory:1985.
In John T. Baldwin, editor, *Classification Theory: Chicago, 1985 Proceedings of the U.S.-Israel Binational Workshop on Model Theory in Mathematical Logic*, Heidelberg, 1988. Springer Verlag.

John T. Baldwin.
Some projective planes of Lenz Barlotti class I.
Proceedings of the A.M.S., 123:251–256, 1995.

John T. Baldwin.
Model Theory and the Philosophy of Mathematical Practice: Formalization without Foundationalism.
Cambridge University Press, 2018.

John T. Baldwin.
The dividing line methodology: Model theory motivating set theory.
Theoria, 87:361–393, 2021.
theoria: https://doi.org/10.1111/theo.12297
http://homepages.math.uic.edu/~jbaldwin/pub/atheoriabibjan619.pdf.

E. Bouscaren and E. Hrushovski.
Classifiable theories without finitary invariants.
Annals of Pure and Applied Logic, 142:296–320, 2006.

John T. Baldwin and V. Verbovskiy.
Towards a finer classification of strongly minimal sets.
preprint: Math Arxiv:2106.15567, 2021.

T. Button and S. Walsh.
Philosophy and Model Theory.
Oxford University Press, 2017.

Goncharov, Harizanov, Laskowski, Lempp, and McCoy.
Trivial strongly minimal sets are model complete after naming constants.
Proceedings of the American Mathematical Society, 131:3901–3912, 2003.

T. Gowers.
Introduction.
In T. Gowers, editor, *The Princeton Companion to Mathematics*, pages 1–76. Princeton University Press, 2008.

B. Hart, E. Hrushovski, and C. Laskowski.
The uncountable spectra of countable theories.
Annals of Mathematics, 152:207–257, 2000.

I. Kaplan and N. Ramsey.
On Kim-independence.
J. Eur. Math. Soc. (JEMS)., 22:1423–1474, 2020.
arXiv:1702.03894v2.

M. Malliaris.
Realization of ϕ-types and Keisler's order.
Ann. Pure Appl. Logic, 157:220–224, 2009.

M. Malliaris and A. Pillay.
The stable regularity lemma revisited.
Proc. Amer. Math. Soc., 144:1761–1765, 2016.

M. Malliaris and S. Shelah.
Regularity lemmas for stable graphs.
Trans. AMS, 366:1551–1585, 2014.

M. Malliaris and S. Shelah.
Keisler's order is not simple (and simple theories may not be either).
math arxiv: `arXiv:1906.10241`, 2021.

Bruno Poizat.
Review of Baldwin, John T. the spectrum of resplendency. Journal of Symbolic Logic 55 (1990).
Math Reviews, 1991.
MR1056376.

S. Shelah.
Classification of first order theories which have a structure theory.
Bulletin of A.M.S., 12:227–232, 1985.
sh index 200.

S. Shelah.
Existence of many $L_{\infty,\lambda}$-equivalent non-isomorphic models of T of power λ.
Annals of Pure and Applied Logic, 34:291–310, 1987.

S. Shelah.
Classification Theory and the Number of Nonisomorphic Models.
North-Holland, 1990.
second edition.

S. Shelah.
Classification Theory for Abstract Elementary Classes.
Studies in Logic. College Publications, 2009.

S. Shelah.
Response to the award of the 2013 Steele prize for seminal research.
A.M.S. Prize Booklet, 2013, page 50, 2013.
http://www.ams.org/profession/prizebooklet-2013.pdf.

S. Shelah.
Divide and conquer: Dividing lines and universality.
Theoria, 2020.