Examples of Non-locality

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Abstract

We use κ-free but not Whitehead Abelian groups to construct Abstract Elementary Classes (AEC) which satisfy the amalgamation property but fail various conditions on the locality of Galois-types. We introduce the notion that an AEC admits closures (roughly closed under intersection). We conclude that for AEC which admit closures, the amalgamation property can have no positive effect on locality. There is a transformation of AEC's which preserves non-locality but takes any AEC which admits closures to one with amalgamation. More specifically we have: Theorem 5.3. There is an AEC with amalgamation which is not $(\aleph_0, \aleph_1)$-tame but is $(2^{\aleph_0}, \infty)$-tame; Theorem 3.1. It is consistent with ZFC that there is an AEC with amalgamation which is not $(\leq \aleph_2, \leq \aleph_2)$-compact.

A primary object of study in first order model theory is a *syntactic* type: the type of a over B in a model N is the collection of formulas $\phi(x, \mathbf{b})$ which are true of a in N. Usually the N is suppressed because a preliminary construction has established a universal domain for the investigation. In such a *homogeneous* universal domain one can identify the type of a over B as the orbit of a under automorphisms which fix B pointwise.

An abstract elementary class is a pair $(\boldsymbol{K}, \prec_{\boldsymbol{K}})$, a collection of structures of a fixed similarity type and a partial order on $\boldsymbol{K}$ which refines substructure and satisfies natural axioms which are enumerated in many places such as ([She87, Bal00, She99, Gro02, GVb]. In this case, 'Galois' types (introduced in [She87] and named in [Gro02]) are defined as equivalence classes of triples (M, a, N) where $a \in N - M$ under the equivalence relation generated by $(M_1, a_1, N_1) \sim$

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(M_2, a_2, N_2) if $M_1 = M_2$ and there is an amalgam of N_1 and N_2 over M_1 where a_1 and a_2 have the same image. If $\boldsymbol{K}$ has the amalgamation property the equivalence classes, i.e. the Galois types, of this equivalence relation can again be identified as orbits of automorphisms of a universal domain which fix the domain of the type. The notions and definitions which appear in this paper stem from a long series of papers by Shelah ([She87, She99, She01] etc.) They occur in the form used here in [Bal00]. Grossberg and Vandieren [GVb] isolated the notion of tame as a fruitful object of study. Recent work by such authors Grossberg, Kolesnikov, VanDieren, Villaveces [BKV00, GVb, GVa, GVc, GK] either assume or derive tameness. In particular, a number of results on the stablity spectrum and transfer of categoricity have been proved for tame AEC.

Unless we specifically add hypotheses, $\boldsymbol{K}$ denotes an arbitrary AEC. The Löwenheim-Skolem number of an AEC $\boldsymbol{K}$ is denoted $\mathrm{LS}(\boldsymbol{K})$. We introduce in this paper two new notions: admitting closures and model completeness. As explained in Section 1, in an AEC which admits closures the notion of Galois type is better behaved. Model completeness is the natural analog of the first order notion in this context.

Syntactic types have certain natural locality properties. Any increasing chain of types has at most one upper bound; two distinct types differ on a finite set; an increasing chain of types has a realization. The translations of these conditions to Galois types do not hold in general. But there have been few specific examples of their failure. In the first section, we first give precise means to these three notions (in order): locality, tameness, and compactness and attach certain cardinal parameters to them. Precise statements of the results depend on these definitions and occur with the proofs. But vaguely speaking, in Section 2 we show there is an AEC with the amalgamation property which is not $\aleph_0$-tame and does not attain tameness at any small cardinal. In Section 3 we find a $\boldsymbol{K}$ which is not compact at one of $\aleph_1, \aleph_2$. In Section 4 we introduce a general construction which shows that one can transform a failure of locality in an AEC which admits closures to a failure in an AEC with amalgamation. And in Section 5, we combine Sections 2 and 4 and answer a question of [GVb] by providing an example which is not $\aleph_0$-tame but is $2^{\aleph_0}$-tame.

In the presence of amalgamation, the subject of this paper can be considered as a study of the automorphism group of the monster model. For example, compactness is the assertion: if M_i is an increasing sequence of strong submodels of $\mathcal{M}$, $G_i = \mathrm{aut}_{\mathrm{M_i}}(\mathcal{M})$, and X_i is a decreasing sequence of orbits under G_i, then the intersection of the X_i is nonempty. The cardinal parameters of the formal definition fix the cardinality of the M_i and the length of the chain.

1 Some notions of locality

We define the notions below with two cardinal parameters: the first is the size of a certain submodel or the length of a sequence of types; the second is the size of the models under consideration,. In a rough sense, the first parameter is the important one; ideally the second can be replaced by ∞. But the only theorems which derive locality from categoricity (without identifying Galois types with syntactic types of some sort) do so only for models of fixed size. So we use the fastidious notation. Replacing (λ, κ) by e.g. $(< \lambda, \kappa)$ has the obvious meaning. The following property holds of all AEC considered in this paper.

Definition 1.1 *We say the AEC* $(\boldsymbol{K}, \prec_{\boldsymbol{K}})$ admits closures *if for every* $X \subseteq M \in \boldsymbol{K}$*, there is a minimal closure of* X *in* M*. That is,* $M\restriction \bigcap\{N : X \subseteq N \prec_{\boldsymbol{K}} M\} = \mathrm{cl}_M(X) \prec_{\boldsymbol{K}} M$*.*

Any relation defined by such an intersection will have the monotonicity, finite character, and transitivity properties of a closure relation. Note that this property is nontrivial even if one restricts to first order theories with elementary submodel. In that case it applies to strongly minimal or *o*-minimal theories; the first order case was characterized by Rabin [Rab62]. And of course the condition is satisfied when one has Skolem functions. But we work in a more general situation. If an AEC admits closures we have a natural way to check equality of Galois types.

Lemma 1.2 *Let* $(\boldsymbol{K}, \prec_{\boldsymbol{K}})$ *admit closures.*

1. *Suppose* $M_0 \prec_{\boldsymbol{K}} M_1, M_2$ *with* $a_i \in M_i$ *for* $i = 1, 2$*. Then* $\mathrm{tp}(a_1/M_0, M_1) = \mathrm{tp}(a_2/M_0, M_2)$ *if and only if there is an isomorphism over* M_0 *from* $M_1\restriction\mathrm{cl}_{M_1}(M_0a_1)$ *onto* $M_2\restriction\mathrm{cl}_{M_2}(M_0a_2)$ *which maps* a_1 *to* a_2*.*

2. (M_1, a_1, N_1) *and* (M_2, a_2, N_2) *represent the same Galois type over* M_1 *iff* $M_1 = M_2$ *and there is an amalgam of* N_1 *and* N_2 *over* M_1 *where* a_1 *and* a_2 *have the same image.*

Proof. Immediate. $\square_{1.2}$

That is, while in general Galois equivalence may result from a finite composition of maps, in this context only one step is required. If $M \prec_{\boldsymbol{K}} N$, $p \in \mathrm{ga-S(M)}$, $q \in \mathrm{ga-S(M)}$, the notion that q extends p is similarly complicated in an arbitrary AEC. Lemma 1.2 yields a simpler characterization.

Fact 1.3 *Suppose* $\boldsymbol{K}$ *admits closures or has the amalgamation property. If* $M \prec_{\boldsymbol{K}} N$*,* $p \in \mathrm{ga-S(M)}$*,* $q \in \mathrm{ga-S(M)}$*, then* q *extends* p *if and only if for each* (N, b, N') *realizing* p *there is a* $\boldsymbol{K}$*-map fixing* M *and taking* b *to an* (M, a, M') *realizing* q*.*

Here is a characterization of realizing the union of a chain of types in terms of maps. The straightforward justification is in [Bal00] in the chapter on locality and tameness.

Fact 1.4

1. *If* $p_i \in \mathrm{ga-S(M_i)}$ *for* $i < \delta$ *is a coherent chain of Galois types, there is a* $p_\delta \in \mathrm{ga-S(M_\delta)}$ *that extends each* p_i *so that* $\langle p_i : i \leq \delta\rangle$ *is a coherent sequence.*

2. *Conversely,* $p_\delta \in \mathrm{ga-S(M_\delta)}$ *extends each* p_i*, there is a choice of* $f_{i,j}$ *for* $i \leq j \leq \delta$ *that witness* $\langle p_i : i \leq \delta\rangle$ *is a coherent sequence.*

Definition 1.5 *Galois types are* (κ, λ)-compact *in* $\boldsymbol{K}$ *if for every continuous increasing chain* $M = \bigcup_{i<\kappa} M_i$ *of members of* $\boldsymbol{K}$ *with cardinality* λ *and every increasing chain* $\{p_i : i < \kappa\}$ *of members* $\mathrm{ga-S(M_i)}$ *there is a* $p \in \mathrm{ga-S(M)}$ *with* $p \upharpoonright M_i = p_i$ *for every* i*.*

Definition 1.6 $\boldsymbol{K}$ *has* (κ, λ)-local galois types *if for every continuous increasing chain* $M = \bigcup_{i<\kappa} M_i$ *of members of* $\boldsymbol{K}$ *with* $|M| = \lambda$ *and for any* $p, q \in \mathrm{ga-S(M)}$*: if* $p \upharpoonright M_i = q \upharpoonright M_i$ *for every* i *then* $p = q$*.*

The following results were stated by Shelah in e.g. [She99]; a full proof appears in [Bal00].

Lemma 1.7 *For any* λ*, if* $\boldsymbol{K}$ *has* $(< \kappa, \leq \lambda)$*-local Galois types. Then Galois types are* $(\leq \kappa, \leq \lambda)$-compact *in* $\boldsymbol{K}$*.*

Now we turn to the notion of tameness. The property was first isolated in [She99] in the midst of a proof. Grossberg and VanDieren [GVb] focused attention on the notion as a general property of AEC's. We introduce a parameterized version in hopes of deriving tameness from categoricity by an induction. And weakly tame is the version that can actually be proved. That is, the best result now known [She99, Bal00] is that if $\boldsymbol{K}$ is categorical in some regular λ greater than $H_1 = H(\boldsymbol{K})$ (the Hanf number for $\boldsymbol{K}$), then $\boldsymbol{K}$ is $(< \lambda, \chi)$-weakly tame for some $\chi < H_1$.

Definition 1.8

1. *We say* $\boldsymbol{K}$ *is* (χ, μ)-weakly tame *if for any* saturated $N \in \boldsymbol{K}$ *with* $|N| \leq \mu$ *if* $p, q, \in \mathrm{ga-S(N)}$ *and for every* $N_0 \leq N$ *with* $|N_0| \leq \chi$*,* $p \upharpoonright N_0 = q \upharpoonright N_0$ *then* $q = p$*.*
2. *We say* $\boldsymbol{K}$ *is* (χ, μ)-tame *if the previous condition holds for* all N *with cardinality* μ*.*
3. (χ, μ)*-weakly compact and* (χ, μ)*-weakly local are defined analogously.*

Thus the vague notion of κ-tame in the introduction is formally (κ, ∞)-tame. Finally, we say $\boldsymbol{K}$ is κ (weakly)-tame or (κ, ∞) (weakly)-tame if it (κ, λ)-(weakly)-tame for every λ greater than κ. There are a few relations between tameness and locality. The second was observed in conversation by Olivier Lessmann.

Lemma 1.9 *If* $\lambda \geq \kappa$ *and* $\mathrm{cf}(\kappa) > \chi$*, then* (χ, λ)*-tame implies* (κ, λ)*-local. If particular,* $(\aleph_0, \aleph_1)$*-tame implies* $(\aleph_1, \aleph_1)$*-local.*

Proof. Suppose $\langle M_i, p_i : i < \kappa \rangle$ is an increasing chain with $\bigcup_i M_i = M$ and $|M| \leq \lambda$. If both $p, q \in \mathrm{ga-S(M)}$ extend each p_i, by (χ, λ)-tameness, there is a model N of cardinality χ on which they differ. Since $\mathrm{cf}(\kappa) > \chi$, N is contained in some M_i. $\square_{1.9}$

Lemma 1.10 *If* $\boldsymbol{K}$ *is* $(< \mu, < \mu)$*-local and* $\mu \geq \mathrm{LS}(\boldsymbol{K})$ *then* M *is* $(\mathrm{LS}(\boldsymbol{K}), \mu)$*-tame.*

Proof. We prove the result by induction on μ and it is clear for $\mu = \mathrm{LS}(\boldsymbol{K})$. Suppose it holds for all $\kappa < \mu$. Let p, q be distinct types in $\mathrm{ga}-\mathrm{S(M)}$ where $|M| = \mu$ and write M as an increasing chain $\langle M_i : i < \mu \rangle$. with $|M_i| \leq |i| + \mathrm{LS}(\boldsymbol{K})$. Let p_i, respectively q_i denote the restriction to M_i. Since $p \neq q$, locality gives an M_j with $p_j \neq q_j$ and $|M_j| < \mu$. By induction there exists an $N \prec_{\boldsymbol{K}} M_j$ with $|N| = \mathrm{LS}(\boldsymbol{K})$ and $p_j|N \neq q_j|N$. But then, $p|N \neq q|N$ and we finish. $\square_{1.10}$

2 A Concrete Example of Non-tameness

In this section we find a concrete example of a class which is not $(\aleph_1, \aleph_1)$-local and so not $(\aleph_0, \aleph_1)$-tame. We encode some well-known 'incompactness' phenomena for Abelian groups.

Definition 2.1 *We say A is a* Whitehead group *if $Ext(A, \mathbb{Z}) = 0$. That is, every short exact sequence*

$$0 \to \mathbb{Z} \to H \to A \to 0,$$

splits or in still another formulation, H is the direct sum of A and $\mathbb{Z}$.

Every free group is Whitehead and a Whitehead group of power $\aleph_1$ is $\aleph_1$-free, i.e., every countable subgroup is free. Recall that J.H.C. Whitehead conjectured that every Whitehead group of cardinality $\aleph_1$ is free. We do not rely in this section on the Shelah's argument that the Whitehead conjecture is independent of ZFC. But we use some of the techniques of the argument and more appear in the next section. Now we contradict locality. And we rely on his construction, reported on page 228 of [EM90] of a group with the following properties.

Fact 2.2 *There is $\aleph_1$-free group G of cardinality $\aleph_1$ which is not Whitehead. Moreover, there is a countable subgroup R of G such that G/R is p-divisible for each prime p.*

Example 2.3 Let $\boldsymbol{K}$ be the class of structures $M = \langle G, Z, I, H \rangle$, where each of the listed sets is the solution set of one of the unary predicates $(\boldsymbol{G}, \boldsymbol{Z}, \boldsymbol{I}, \boldsymbol{H})$. G is a torsion-free Abelian Group. Z is a copy of $(\mathbb{Z}, +)$. I is an index set and H is a family of infinite Abelian groups. The vocabulary also includes function symbols $\boldsymbol{F}, \boldsymbol{k}$ and $\boldsymbol{\pi}$, naming functions F, k, and π. F maps H onto I and for $s \in I$, $+(\_, \_, s)$ is a group operation on $H_s = F^{-1}(s)$. Finally, π maps H onto G so that $\pi_s = \pi \restriction H_s$ is a projection from H_s onto G. The kernel of each π_s is isomorphic to Z via a map $\boldsymbol{k}(\_, s)$ where $k : Z \times I \mapsto H$.

Further, we write $M_0 \prec_{\boldsymbol{K}} M_1$ if M_0 is a substructure of M, but $\boldsymbol{Z}^{M_0} = \boldsymbol{Z}^M$ and $\boldsymbol{G}^{M_0}$ is a pure subgroup of $\boldsymbol{G}^{M_1}$.

Since each $\boldsymbol{G}^M$ is torsion-free, it follows that if $M_0 \prec_{\boldsymbol{K}} M_1$, for each $t \in \boldsymbol{I}^{M_0}$, $\boldsymbol{H}_t^{M_0}$ is pure in $\boldsymbol{H}_t^{M_1}$. The class $\boldsymbol{K}$ is almost first order definable; we require some infinitary logic to keep $\mathbb{Z}$ standard. But the notion of $\prec_{\boldsymbol{K}}$ is much weaker than elementary submodel. The models are essentially many exact sequences. They all have the same kernel $\mathbb{Z}$; and there may be many with the same image G, but the middle terms H are all disjoint. It is fruitful (see Section 5) to restrict the class of image groups. But it is delicate to do so while keeping $\boldsymbol{K}$ closed under unions and amalgamation.

It is easy to check that under these definitions

Lemma 2.4 *The class $(\boldsymbol{K}, \prec_{\boldsymbol{K}})$ defined in Example 2.3 is an abstract elementary class.*

We defined the notion of an AEC admitting closures in Definition 1.1

Lemma 2.5 *The class $(\boldsymbol{K}, \prec_{\boldsymbol{K}})$ defined in Definition 2.3 admits closures.*

Proof. To find the the required closure of a subset A of M, first close A under the functions of the language: $\boldsymbol{F}^M, \boldsymbol{\pi}^M$ and the group operation in G and in each fiber of $\boldsymbol{H}^M$ to form a set X'. Then take the pure closure of $X' \cap \boldsymbol{G}^M$ in $\boldsymbol{G}^M$ as X''. Finally add those $y \in \boldsymbol{H}^M$ which satisfy both $\boldsymbol{\pi}^M(y) \in X''$ and $\boldsymbol{F}^M(y) \in X''$. If N is the substructure of M with this universe it is easy to check that $N \prec_{\boldsymbol{K}} M$ and N is contained in any N' with $A \subset N' \prec_{\boldsymbol{K}} M$. $\square_{2.5}$

The next easy lemma provides a nice characterization for this example of equality for certain Galois types.

Lemma 2.6 *Suppose $M_0 \prec_{\boldsymbol{K}} M_1, M_2$ and the group $G = \boldsymbol{G}^{M_0}$ is the same in each of the three structures. Let t_1, t_2 be in $\boldsymbol{I}^{M_1} - \boldsymbol{I}^{M_0}, \boldsymbol{I}^{M_2} - \boldsymbol{I}^{M_0}$ respectively. The following are equivalent.*

1. $\mathrm{tp}(t_1/M_0, M_1) = \mathrm{tp}(t_2/M_0, M_2)$.
2. *There is an isomorphism h from $\boldsymbol{H}_{t_1}^{M_1}$ onto $\boldsymbol{H}_{t_2}^{M_2}$ such that*
   (a) *For $z \in \boldsymbol{Z}^{M_0}$, $h(\boldsymbol{k}^{M_1}(z, t_1)) = \boldsymbol{k}^{M_2}(z, t_2)$ and*
   (b) *for $y \in \boldsymbol{H}_{t_1}^{M_1}$, $h(\pi^{M_1}(y, t_1)) = \pi^{M_1}(h(y), t_2)$.*

Recall that the class of torsion-free abelian groups has the amalgamation property for pure embeddings. Specifically, to amalgamate G_1 and G_2 over G_0 just form $G_0 \times G_1$ and factor out the subgroup of elements $\{(x, -x) : x \in G_0\}$. The purity of G_0 in G_1 and G_2 guarantees the amalgam is torsion free. Note that if $H_0 \subset H_1, H_2$ and we have maps from H_1 onto G_1 and from H_2 onto G_2 with common kernel contained in H_0, these maps extend coordinate-wise to maps from the amalgam of the H's to the amalgam of the G's.

Lemma 2.7 $(\boldsymbol{K}, \prec_{\boldsymbol{K}})$ *has the amalgamation property.*

Proof. We want to amalgamate M_1 and M_2 over M_0 to construct M_3. Without loss of generality we can assume M_1 and M_2 intersect in M_0. So just take disjoint union on $\boldsymbol{I}$, the group amalgams on $\boldsymbol{G}$ and also the group amalgam of each $\boldsymbol{H}_t^1$ and $\boldsymbol{H}_t^2$ if $t \in \boldsymbol{I}^{M_0}$ and extend the functions naturally. If $t \in \boldsymbol{I}^1 - \boldsymbol{I}^2$, $\boldsymbol{H}_t^1 = \boldsymbol{H}_t^3$. (The case $\boldsymbol{I}^1 - \boldsymbol{I}^2 \neq \emptyset$ is similar.) In particular, $\boldsymbol{\pi}^{M_3}(y_1, y_2)$ is only defined if $F^{M_1}(y_1) = F^{M_2}(y_2)$ and in that case, the value is $((\boldsymbol{\pi}^{M_1}(y_1), (\boldsymbol{\pi}^{M_2}(y_2))/\boldsymbol{G}^{M_0}$. There is no interaction between the problems for H_s and H_t if $s \neq t \in \boldsymbol{I}^{M_1} \cup \boldsymbol{I}^{M_2}$.

$\square_{2.7}$

Lemma 2.8 $(\boldsymbol{K}, \prec_{\boldsymbol{K}})$ *is not* $(\aleph_1, \aleph_1)$*-local. That is, there is an* $M^0 \in \boldsymbol{K}$ *of cardinality* $\aleph_1$*, a continuous increasing chain of models* M_i^0 *for* $i < \aleph_1$*, and two distinct types* $p, q \in \mathrm{ga-S(M^0)}$ *with* $p \restriction M_i^0 = q \restriction M_i$ *for each* i*.*

Proof. We define p and q, show they are distinct, and then show their restrictions are the same.

Let G be the Abelian group from Fact 2.2 of cardinality $\aleph_1$ which is $\aleph_1$-free but not a Whitehead group. Then, there is an H such that,

$$0 \to \mathbb{Z} \to H \to G \to 0$$

is exact but does not split. Say $g_1 : H \to G$. But, we can write G as a continuous increasing chain $\cup_i G_i$ of countable free groups such that each exact sequence:

$$0 \to \mathbb{Z} \to H_i \to G_i \to 0,$$

splits, where $H_i = g_1^{-1}(G_i)$, because G_i is free.

Let M^0 have $\boldsymbol{G}^{M^0} = G$, $\boldsymbol{Z}^{M^0} = \mathbb{Z}$, $\boldsymbol{I}^{M^0} = \{a\}$, and $\boldsymbol{H}^{M^0}$ a trivial group. Now define M^1, M^2 which have one additional point $t_i \in \boldsymbol{I}^{M_i}$. The key point is that $\boldsymbol{H}^{M^1}$ is H and $\boldsymbol{H}^{M^2}$ is $G \oplus \mathbb{Z}$. Let $\boldsymbol{\pi}^{M1}(\_, t_1)$ be g_1 and $\boldsymbol{\pi}^{M_2}(\_, t_2)$ be the projection map g_2 from $G \oplus \mathbb{Z}$ onto G. Let $p = \mathrm{tp}(t_1/M^0, M^1)$ and $q = \mathrm{tp}(t_2/M^0, M^2)$. Since the exact sequence for $\boldsymbol{H}^{M^2}$ splits and that for $\boldsymbol{H}^{M^1}$ does not, it is immediate from Lemma 2.6 that $p \neq q$.

Let $G = \bigcup_{i<\aleph_1} G_i$. Now we define the models M_i^ℓ for $i < \aleph_1$ and $\ell = 0, 1, 2$. Then M_i^0 is naturally obtained by letting $\boldsymbol{G}^{M_i^0} = G_i$ and leaving the other components as in M^0. For $\ell = 1, 2$, let M_i^ℓ be the restriction of M^ℓ to $\langle G_i, \mathbb{Z}, \{t_\ell\}, \{y \in \boldsymbol{H}^{M^\ell} : g_\ell(y) \in G_i\}\rangle$.

By Lemma 1.2, $\mathrm{tp}(t_\ell/M_i^\ell, M^\ell) = \mathrm{tp}(t_\ell/M_i^\ell, M^\ell)$ for each i and $\ell = 1, 2$. By the choice of H, for $\ell = 1$ and by the restriction of g_2 for $\ell = 2$ each of the exact sequences:

$$0 \to \mathbb{Z} \to H_i^\ell \to G_i^\ell \to 0$$

splits. This implies there is an isomorphism h from M_i^1 onto M_i^2 over M_i^0 mapping t_1 to t_2. That is, $\mathrm{tp}(t_1/M_i^0, M_i^1) = \mathrm{tp}(t_2/M_i^0, M_i^2)$. Thus, $p \restriction M_i^0 = q \restriction M_i^0$. We have the required counterexample. $\square_{2.8}$

Remark 2.9 While the existence of an $\aleph_1$-free group which is not Whitehead can be done in ZFC, κ-free but not Whitehead groups for larger regular κ become sensitive to set theory. But if, for example $V = L$ (much weaker conditions suffice), the class $\boldsymbol{K}$ is not (κ, κ)-local for arbitrary regular κ.

3 Incompactness

In this section we construct an increasing sequence of Galois types which has no upper bound. The model theoretic example is the same as Section 2 but the choice of groups for the counterexample is different. In contrast to nonlocality we obtain only a consistency result.

Theorem 3.1 *Assume* $2^{\aleph_0} = \aleph_1$, *and* $\diamondsuit_{\aleph_1}, \diamondsuit_{S^2_1}$ *where*

$$S^2_1 = \{\delta < \aleph_2 : \mathrm{cf}(\delta) = \aleph_1\}.$$

Then, the $\boldsymbol{K}$ *from Section 2 fails either* $(\aleph_1, \aleph_1)$ *or* $(\aleph_2, \aleph_2)$*-compactness.*

Note that this is a more precise version of the statement in the abstract that $\boldsymbol{K}$ is not $(\leq \aleph_1, \leq \aleph_1)$-compact.

We will use several times the following fact which is one of the equivalent conditions in Lemma IV.2.3 of [EM90].

Fact 3.2 (Pontryagin's criterion) *G is an $\aleph_1$-free Abelian group if and only G is torsion free and every finite subset of G is contained in a finitely generated pure subgroup of G.*

We now introduce some set-theoretic notation and construct a family of groups to show Theorem 3.1.

Notation 3.3

1. *Let $\langle C_\alpha : \alpha < \omega_2, \alpha \text{ limit}\rangle$ be a square (i.e., C_α is a club of α with $|C_\alpha| = \aleph_0$ when $\mathrm{cf}(\alpha) = \aleph_0$, and if $\beta \in C_\alpha$ is a limit ordinal, $C_\beta = C_\alpha \cap \beta$).*

2. *For each α, let*
$$C'_\alpha = \{\beta \in C_\alpha : \beta = \sup(C_\alpha \cap \beta)\}$$
and
$$S = \{\alpha \in \omega_2 : \alpha = \sup(C'_\alpha)\}.$$

3. *S_0 is the elements of S with cofinality ω and S_1 is the elements of S with cofinality ω_1. (Note that, in fact, S_1 contains all ordinals less than $\aleph_2$ of cofinality ω_1.)*

4. *Choose a ladder system $\{\eta_\delta : \delta \in S_0\}$ from the C'_δ. That is, each η_δ is an increasing ω sequence with limit δ* of elements of C'_δ.

Definition 3.4 *For $\alpha \leq \aleph_2$, let G_α be the Abelian group generated by*

$$\{x_\beta : \beta < \alpha\} \cup \{y_{\delta,n} : \delta \in S_0 \cap \alpha, n < \omega\}$$

subject only to the relations $n! y_{\delta,n+1} = y_{\delta,n} - x_{\eta_\delta(n)}$.

We use interval notation in the ordinals, writing $\{\gamma : \alpha + 1 \leq \gamma < \beta\}$ as $[\alpha + 1, \beta]$.

Lemma 3.5 *With the notation above,*

1. *The G_α form an increasing continuous sequence of $\aleph_1$-free abelian groups.*
2. *For $\alpha < \beta \leq \aleph_2$, $G_{\alpha+1}$ is a direct summand of G_β.*

Proof. Check 1) using Pontryagin's criteria. We now prove 2. For $\delta \in S_0 \cap [\alpha + 1, \beta]$, choose $b(\delta) = b < \omega$ so that $\eta_\delta(b) > \alpha + 1$. Let $G_{\alpha+1,\beta}$ be the group generated by

$$\{x_\gamma : \alpha + 1 \leq \gamma < \beta\} \cup \{y_{\delta,m} : \delta \in S_0 \cap [\alpha + 1, \beta], b(\delta) \leq m\}.$$

Then $G_\beta = G_{\alpha+1} \oplus G_{\alpha+1,\beta}$ as required, since there are no relations between the generators of $G_{\alpha+1}$ and $G_{\alpha+1,\beta}$ and each $y_{\delta,n}$ with $\eta_\delta(n) < b(\delta)$ can be written as sum of elements from $G_{\alpha+1}$ and $G_{\alpha+1,\beta}$. $\square_{3.5}$

Notation 3.6 *Let $\langle F_\delta : \delta \in S_1 \rangle$ be a diamond sequence, i.e.,*

1. *F_δ is a two-place function with domain $\delta, F_\delta(\gamma_1, \gamma_2)$ is a permutation of some $\beta_{\delta,\gamma_1,\gamma_2} < \delta$ for $\gamma_1 < \gamma_2 < \delta$.*
2. *If $\overline{f} = \langle f_{\gamma_1,\gamma_2} : \gamma_1 < \gamma_2 < \aleph_2 \rangle, f_{\gamma_1,\gamma_2}$ is a permutation of some $\beta_{\gamma_1,\gamma_2} < \aleph_2$ then $\{\delta \in S_1 : (\forall \gamma_1 < \gamma_2 < \delta) f_{\gamma_1,\gamma_2} = F_\delta(\gamma_1, \gamma_2)\}$ is stationary.*

We can assume the universe of G_α is an ordinal $\delta_\alpha < \aleph_2$, since the G_α are a continuous increasing sequence.

Now we construct by induction on α an $(\aleph_2, \aleph_2)$-array of Abelian groups: $\langle < H_{\beta,\alpha}, g_{\beta,\alpha} >: \beta \leq \alpha < \aleph_2 \rangle$ and $\langle \pi_{\beta,\alpha} : \beta \leq \alpha < \aleph_2 \rangle$ which satisfy the following pair of conditions:

- A_α:
    1. $H_{\alpha,\alpha}$ is an abelian group with universe δ_α.
    2. $g_{\alpha,\alpha} = g_\alpha$ is a homomorphism from $H_{\alpha,\alpha}$ onto G_α with kernel $\boldsymbol{Z}$.
    3. For $\beta < \alpha$, $H_{\beta,\alpha} = H_{\alpha,\alpha} \restriction \{x \in H_{\alpha,\alpha} : g_{\alpha,\alpha}(x) \in G_\beta\}$; H_β denotes $H_{\beta,\beta}$.
    4. $g_{\beta,\alpha} = g_{\alpha,\alpha} \restriction H_{\beta,\alpha}$.
    5. If $\beta \leq \alpha < \aleph_1$, $\pi_{\beta,\alpha}$ is an isomorphism from H_β onto $H_{\beta,\alpha}$ such that $g_{\beta,\beta} = g_{\beta,\alpha} \circ \pi_{\beta,\alpha}$. (Note each $\pi_{\beta,\beta}$ is an identity map).

- B_α: If $\alpha \in S_1$ and for $\beta < \alpha$, $F_\alpha(\beta)$ is an automorphism of H_β, the permutation $F_\alpha(\alpha)$ of H_α given by the diamond is not an automorphism such that $\langle H_\beta; \pi_{\beta_2,\beta_1} \circ F_\alpha(\beta_i) : \beta \leq \alpha, \beta_1 \leq \beta_2 \leq \alpha\rangle$ form a sequence of commuting maps. That is, for $\beta_1 \leq \beta_2 \leq \beta_3 \leq \alpha$ and any $x \in H_{\beta_1}$:

$$(\pi_{\beta_3,\beta_1} \circ F_\alpha(\beta_1))(x) = (\pi_{\beta_3,\beta_2} \circ F_\alpha(\beta_2)((\pi_{\beta_2,\beta_1} \circ F_\alpha(\beta_1))(x)).$$

These automorphisms and projections are a slightly different formalism for describing the realization of unions of types than that described in Fact 1.4. Also, we working directly with the groups; they are represented by single elements by the map $\boldsymbol{\pi}$ as in Section 2.

We need one more lemma concerning the structure of the G_α.

Lemma 3.7 *If $\alpha \in S_1$ then G_α can be decomposed as $G_\alpha = G'_\alpha \oplus G''_\alpha$ where G'_α is countable and free.*

Proof. Let

$$G'_\alpha =: \langle \{x_\beta : \beta \in C'_\alpha\} \cup \langle y_{\beta,n} : \beta \in S_0 \cap C'_\alpha, n < \omega\}\rangle_{G_\alpha}$$

and

$$G''_\alpha = \langle \{x_\beta : \beta \in \alpha \backslash C'_\alpha\} \cup \{y_{\delta,n} : \delta \in \alpha \cap S_0 \backslash C'_\alpha \text{ and } \eta_\delta(n) > \sup(C'_\alpha \cap C'_\delta)\}\rangle_{G_\alpha}.$$

Now if $\delta \in \alpha \cap S_0 \backslash C'_\alpha$ then $\{n : \eta_\delta(n) \in C'_\alpha\}$ is finite (otherwise $\delta = \sup(C'_\delta \cap C'_\alpha)$ hence $\delta \in C'_\alpha$). And the $y_{\delta,n}$ with $\eta_\delta(n) \leq \sup(C'_\alpha \cap C'_\delta)$ are represented as sums of elements in G'_α and G''_α. Since C'_α is countable, G'_α is countable and free since G_α is $\aleph_1$-free. $\square_{3.7}$

Construction 3.8 We construct groups $H_{\beta,\alpha}$ and functions $g_{\alpha,\beta}$, $\pi_{\alpha,\beta}$ for $\beta \leq \alpha \leq \aleph_2$ by induction to satisfy conditions A_α and B_α.

Let H_0 be $\mathbb{Z} \oplus G_0$; g_0 is the projection g_0; $\pi_{0,0}$ is the identity.

To satisfy A_α in limit stages of cofinality ω: (One is tempted to just take unions; the $\pi_{\beta,\alpha}$ have not been constructed to commute so this may fail.) For $\delta \notin S_0$, choose a sequence $\eta_\delta(n)$ with limit δ; for $\delta \in S_0$, we already have one. For the moment, we consider only the structures $H_{\eta_\delta(m),\eta_\delta(k)}$ with $m, k < \omega$. We form new maps $\pi^*_{\eta_\delta(m),\eta_\delta(k)}$ by the composition of $\pi_{\eta_\delta(r),\eta_\delta(r+1)}$ for $m \leq r \leq k$. Now for each $k < \omega$ the $\langle H_{\eta_\delta(s)}, \pi^*_{\eta_\delta(s),\eta_\delta(t)} : s \leq t < \omega\rangle$ form a direct system and we can choose H_δ as the direct limit of this system with limit maps $\pi^*_{\eta_\delta(s),\delta}$ from $H_{\eta_\delta(s)}$ into H_δ. Denote the range of $\pi^*_{\eta_\delta(s),\delta}$ as $H_{\eta_\delta(s),\delta}$. Now define $g_{\eta_\delta(r),\delta}$ from $H_{\eta_\delta(r),\delta}$ onto $G_{\eta_\delta(r)}$ as $g_{\eta_\delta(r),\eta_\delta(r)} \circ (\pi^*)^{-1}_{\eta_\delta(r),\delta}$. This gives 1) through 3) of A_δ. Now we satisfy 4) and 5) by defining $H_{\beta,\delta}$ and $g_{\beta,\delta}$ in the natural manner. It remains to define $\pi^*_{\delta,\gamma}$ when γ is not in the range of η_δ. Choose m such that $\eta_\delta(m) > \gamma$ and let

$$\pi^*_{\delta,\gamma} = \pi^*_{\delta,\eta_\delta(m)} \circ \pi_{\eta_\delta(m),\gamma}.$$

To satisfy $A_{\alpha+1}$ in a successor stage: given $0 \to \mathbb{Z} \to H_\alpha \to G_\alpha \to 0$, we proceed as follows. Let $\alpha = \beta+1$. If β is not in S_0, $G_\alpha = G_\beta \oplus \langle x_\beta \rangle$ and we just extend H_β freely by a single generator. If $\beta \in S_0$, choose elements $x'_{\beta,n} \in H_\beta$ with $g_\beta(x'_{\beta,n}) = x_{\eta_\beta(n)}$. Now form H_α by adding to H_β elements x'_β and $y'_{\beta,n}$ subject only to the relations:

$$n! y'_{\beta,n} = y'_{\beta,n-1} - x'_{\beta,n}.$$

Now if we map H_α to G_α by g_β on H_β and just dropping the primes on the generators of H_β over G_β, we have the required homomorphism.

We now consider α of cofinality $\aleph_1$. Let $C'_\alpha = \{\gamma_\varepsilon : \varepsilon < \aleph_1\}, \gamma_\varepsilon$ increasing continuous with ε. We choose by induction objects $\langle H^0_{\beta,\alpha}, g^0_{\beta,\alpha}, \pi^0_{\beta,\alpha} : \beta \leq \gamma_\varepsilon \rangle$ to satisfy the relevant parts of A_α. Let $H^0_{\alpha,\alpha} = \cup\{H^0_{\beta,\alpha} : \beta < \alpha\}, \pi^0_{\alpha,\alpha} = \mathrm{id}_{H^0_{\alpha,\alpha}}, g^0_{\beta,\alpha} = \cup\{g^0_{\beta,\alpha} : \beta < \alpha\}$. Consider an $M \in \boldsymbol{K}$ with $\boldsymbol{G}^M = G_\alpha$ and for some $t \in I$, $\boldsymbol{H}^M_t = H^0_{\alpha,\alpha}$. If Condition A_α fails then the example is not $(\aleph_1, \aleph_1)$-compact. If B_α holds for this choice, i.e. $F_\alpha(\alpha) \neq \mathrm{id}_{H^0_{\alpha,\alpha}}$, we are done. If not, recall the decomposition of G_α in Lemma 3.7.

Let

$$H'_\alpha = \{x \in H^0_{\alpha,\alpha} : g^0_{\alpha,\alpha}(x) \in G'_\alpha\}$$

$$H''_\alpha = \{x \in H^0_{\alpha,\alpha} : g^0_{\alpha,\alpha}(x) \in G''_\alpha\}$$

(so their intersection is the copy of $\mathbb{Z}$).

Since we have assumed $\diamond$, every Whitehead group of power $\aleph_1$ is free. So, $\mathrm{Ext}(G'_\alpha, \mathbb{Z}) \neq 0$ as G'_α is not free. Hence, we can find (H^*_α, g^*_α) such that

- (a) H^*_α is an abelian group;
- (b) g^*_α is a homomorphism from H^*_α onto G'_α, which does not split;
- (c) $\mathrm{Ker}(g^*_\alpha) = \mathrm{Ker}(g^0_{\alpha,\alpha}) \subseteq H_{\alpha,\alpha}$;
- (d) $H^*_\alpha \cap H''_\alpha = \mathrm{Ker}(g^*_\alpha)$;

Now we have a new candidate for $H_{\alpha,\alpha}$:

$$H^1_{\alpha,\alpha} = H^*_\alpha \bigoplus_{\mathrm{Ker}(g^0_{\alpha,\alpha})} H''_\alpha$$

where we define $g^1_{\alpha,\alpha} \in \mathrm{Hom}(H^1_{\alpha,\alpha}, G_\alpha)$ by extending $g^0_{\alpha,\alpha}$ on H''_α and g^*_α on H^*_α.

It remains to construct $\pi^1_{\beta,\alpha}$ for $\beta < \alpha$. Note that

$$G_\beta = (G'_\alpha \cap G_\beta) \oplus (G''_\alpha \cap G_\beta).$$

On those $x \in H_\beta$ with $g_\beta(x) \in G''_\alpha \cap G_\beta$, let $\pi^1_{\beta,\alpha}(x) = \pi^0_{\beta,\alpha}(x)$. We need to define $\pi^1_{\beta,\alpha}(x)$ on $\hat{H}_\beta = \{x \in H_\beta : g_\beta(x) \in \hat{G} = G''_\alpha \cap G_\beta\}$. Let $H^*_\beta = \{x \in H^*_\alpha :$

$g^1_\alpha(x) \in \hat{G}\}$. Since $\hat{G}$ is free, $0 \to \mathbb{Z} \to H^*_\beta \to \hat{G} \to 0$ splits; g^*_α has an inverse f_β. Let $\pi^1_{\beta,\alpha} = f_\beta \circ g_\beta$.

This shows B_α is satisfied since $F_\alpha(\alpha)$ cannot commute with both the $\pi^0_{\beta,\alpha}$ and $\pi^1_{\beta,\alpha}$. Thus we choose $(H_{\alpha,\alpha}, g_{\beta,\alpha}, \pi_{\beta,\alpha})$ from $(H^0_{\alpha,\alpha}, g^0_{\beta,\alpha}, \pi^0_{\beta,\alpha})$ and $(H^1_{\alpha,\alpha}, g^1_{\beta,\alpha}, \pi^1_{\beta,\alpha})$ as the sequence that satisfies B_α.

We finish as follows:

Lemma 3.9 *Suppose we carry out Construction 3.8 for $\aleph_2$-steps. It is not possible to define $\pi_{\beta,\aleph_2} : \beta \leq \aleph_2$ so that:*

- *The sequence $\langle H_{\beta,\aleph_2}, g_{\aleph_2,\beta} \circ \pi_{\beta,\aleph_2} : \beta \leq \aleph_2\rangle$ satisfies:*
  - *(a) $H_{\aleph_2,\aleph_2}$ an abelian group;*
  - *(b) $g_{\aleph_2,\aleph_2}$ maps $(H_{\aleph_2,\aleph_2}$ onto $G_{\aleph_2})$;*
  - *(c) $H_{\beta,\aleph_2} = \{x \in H_{\aleph_2,\aleph_2} : g_{\aleph_2,\aleph_2}(x) \in G_\beta\}$;*
  - *(d) $g_{\beta,\aleph_2} = g_{\aleph_2,\aleph_2} \restriction H_{\beta,\aleph_2}$;*
  - *(e) $\pi_{\beta,\aleph_2}$ is an isomorphism from $H_{\beta,\aleph_2}$ onto $H_{\beta,\beta}$;*
  - *(f) if $x \in H_{\beta,\aleph_2}$ then $g_{\beta,\aleph_2}(x) = g_{\beta,\beta}(\pi_{\aleph_2,\beta}(x)) \in G_\beta$.*

Proof. Suppose for contradiction that we have constructed such a sequence. Then letting,

$$\rho_{\beta_2,\beta_1} = \pi^{-1}_{\beta_2,\beta_1} \circ \pi^{-1}_{\aleph_2,\beta_2} \circ \pi_{\aleph_2,\beta_1}$$

we have a system $\langle H_{\beta,\beta}, (\pi_{\beta_2,\beta_1} \circ \rho_{\beta_2,\beta_1}) : \beta < \aleph_2, \beta_1 \leq \beta_2 < \aleph_2\rangle$ of commuting maps. But by the choice of $\langle F_\delta : \delta \in S_1\rangle$ and since S_1 is stationary in $\aleph_2$ for some δ^* we have:

$$(\forall \gamma_1 \leq \gamma_2 < \delta^*) F_{\delta^*}(\gamma_1, \gamma_2) = \rho_{\gamma_2,\gamma_2}.$$

This contradicts B_{δ^*} in the construction and we finish. $\square_{3.9}$

We are able to carry out the construction unless $\boldsymbol{K}$ is not $(\aleph_1, \aleph_1)$ compact. But then we obtain a counterexample to $(\aleph_2, \aleph_2)$-compactness where the $H_{\beta,\aleph_2}$ from Lemma 3.9 give rise to a sequence of Galois types (of singletons via the coding spelled out in Section 2) over the G_β which have no common extension over $G_{\aleph_2}$. $\square_{3.1}$

Fact 1.7 implies that if $\boldsymbol{K}$ were $(\aleph_0, \aleph_1)$-local then it would be $(\aleph_1, \aleph_1)$-compact and we could remove the awkard disjunction from Theorem 3.1. However, $\boldsymbol{K}$ even fails $(\aleph_0, \aleph_0)$-locality (and failure of $(\aleph_0, \aleph_1)$-locality is an easy consequence in this case).

Lemma 3.10 *$\boldsymbol{K}$ is not $(\aleph_0, \aleph_0)$-local.*

Proof. We construct a sequence of pairs of Abelian group (H_α, G_α) such that for $\alpha < \omega$, $H_\alpha = Z \oplus G_\alpha$, but H_ω is not a split extension of G_ω. Since $Z \oplus G_\omega$ is another limit of this chain, we contradict locality.

Let H^+ be the $\mathbb{Q}$ vector space generated by elements x, z, y_n for $n < \omega$. Fix distinct odd primes p_n and $q_{n,k}$ for $n, k < \omega$. We denote $(p_n - 1)/2$ by r_n.

For each $n < \omega$, H_n is the subgroup of H^+ generated by x, z, y_n for $n < \omega$ and the elements $\frac{p_n y_n + x - r_n z}{q_{n,k}}$ for $k < \omega$.

Clearly H_n is a pure subgroup of H_{n+1}; let $H_\omega = \cup_n H_n$

Claim 3.11 $\mathbb{Z}z = \{nz : n \in \mathbb{Z}\}$ *is a direct summand of* H_n.

Proof. Since $\mathbb{Z}$ is free every projection onto $\mathbb{Z}z$ splits. So we need only construct a homomorphism h_n from H_n onto $\mathbb{Z}z$. Choose, by the Chinese remainder theorem r'_n such that $r'_n \equiv r_\ell \mod p_\ell$ for all $\ell < n$. Now let $h_n(z) = z$, $h_n(x) = r'_n z$, $h_n(y_\ell) = -\frac{(r'_n - r_n)z}{p_\ell}$, and $h_n(\frac{p_n y_n + x - r_n z}{q_{n,k}}) = 0$.

The choice of r'_n guarantees that each $h_n(y_\ell) \in \mathbb{Z}z$ (the coefficient is an integer). Clearly h_n maps onto $\mathbb{Z}z$; the danger is that it is not well defined. It suffices to show that from the values of h_n on z, x and the y_l, $h_n(p_n y_n + x - r_n z) = 0$ since that makes our definition consistent. For this, we compute:

$$h_n(p_\ell y_\ell + x - r_\ell z) = -p_\ell \frac{(r'_n - r_n)z}{p_\ell} + r'_n z - r_\ell z = 0.$$

$\square_{3.11}$

Claim 3.12 $\mathbb{Z}z = \{nz : n \in \mathbb{Z}\}$ *is not a direct summand of* H_ω.

Proof: Suppose for contradiction that h retracts H_ω onto Zz (i.e. $h(z) = z$). Now, for any n, $h(p_n y_n + x - r_n z) \in Zz$ is divisible by $q_{k,n}$ for all k. This implies that

$$h(p_n y_n + x - r_n z) = 0.$$

That is, $h(x - r_n z) \in p_n \mathbb{Z}z$. Since $h(x) = rz$ some $z \in \mathbb{Z}$, this implies $r \equiv r_n \equiv (p_n - 1)/2 \mod p_n$. But it impossible for this to happen for infinitely many n so $\mathbb{Z}z$ is not a direct summand of H_ω. $\square_{3.12}$

With these two claims we complete the proof of Lemma 3.10. $\square_{3.10}$

4 A General Construction for Amalgamation

Let $(\boldsymbol{K}, \prec_{\boldsymbol{K}})$ be an aec in *a relational language* τ which admits closures and is model complete. In this section we construct from $(\boldsymbol{K}, \prec_{\boldsymbol{K}})$ an AEC $(\boldsymbol{K}', \prec_{\boldsymbol{K}'})$ which satisfies the amalgamation property and has the same non-locality properties as $\boldsymbol{K}$. The construction will apply to all AEC which admit closures. We proceed in three steps; we first make a cosmetic change in $\boldsymbol{K}$ to guarantee that it has quantifier free closures (Definition 4.1). Then we establish some important properties of AEC with quantifier free closures and finally make the main construction.

Throughout this section we assume that $\boldsymbol{K}$ admits closures (Definition 1.1); this simplifies the notions of Galois type and extension of Galois type (Lemma 1.2).

We require some preliminary definitions and a lemma for our main construction. Note that throughout we write boldface $\boldsymbol{a}$ for a finite sequence of elements of a model and a for a single element.

Definition 4.1 *Suppose $(\boldsymbol{K}, \prec_{\boldsymbol{K}})$ admits closures.*

1. *$\boldsymbol{K}$ is said to have* quantifier-free closure *if the satisfaction of '$b \in \mathrm{cl}_M(\mathbf{b})$' depends only on the quantifier free (syntactic) type of $\mathbf{b}b$.*

2. *$\boldsymbol{K}$ is* model complete *if $N \subset M$ and $N \in \boldsymbol{K}$, implies $N \prec_{\boldsymbol{K}} M$.*

Lemma 4.2 *For any AEC $(\boldsymbol{K}, \prec_{\boldsymbol{K}})$ which admits closures there is an associated AEC $(\boldsymbol{K}', \prec_{\boldsymbol{K}'})$ with exactly the same spectrum of models which has quantifier-free closure.*

Proof. Add to the language τ of $\boldsymbol{K}$, $n+1$-ary relation symbols for each n and expand $M \in \boldsymbol{K}$ to $M' \in \boldsymbol{K}'$ by making $R_n(\boldsymbol{a}, a)$ hold just if $a \in \mathrm{cl}_M(\boldsymbol{a})$; $\boldsymbol{a}$ has length n. Let $\boldsymbol{K}'$ be exactly the models of this form and define $M' \prec_{\boldsymbol{K}'} N'$ if and only if $M' \restriction \tau \prec_{\boldsymbol{K}} N' \restriction \tau$. The isomorphism of $\boldsymbol{K}$ and $\boldsymbol{K}'$ is immediate and we have introduced quantifier-free closure by fiat. $\square_{4.2}$

We now introduce a property that will be key in establishing amalgamation and show that it follows from either model completeness or having quantifier-free closure.

Definition 4.3 *A class $(\boldsymbol{K}, \prec_{\boldsymbol{K}})$, which admits closures, is said to be* nice for unions *if whenever $\langle M_i : i \leq \delta \rangle$ is a continuous increasing chain of $\boldsymbol{K}$-extensions and A is a finite subset of $N \subseteq M_\delta$ with $N \in \boldsymbol{K}$, there is an N' and an $i < \delta$ such that $A \subseteq N' \prec_{\boldsymbol{K}} N$ and $N' \prec_{\boldsymbol{K}} M_i$.*

Note that for $A \subset M_1 \prec_{\boldsymbol{K}} M_2$, $\mathrm{cl}_{M_1}(A) = \mathrm{cl}_{M_2}(A)$. Moreover, if $\boldsymbol{K}$ admits closures for any $A \subseteq N \in \boldsymbol{K}$, $\mathrm{cl}_N(A)$ is defined: take direct limits of the closures of finite sets.

Lemma 4.4 *Let $(\boldsymbol{K}, \prec_{\boldsymbol{K}})$ be an AEC which admits closures. If $\boldsymbol{K}$*

1. *is model complete or*

2. *has quantifier free closure*

then it is nice for unions.

Proof. As A is finite $\mathrm{cl}_{M_\delta}(A) = \mathrm{cl}_{M_i}(A)$ when $i = \min\{j < \delta : A \subseteq M_j\}$. Thus, $\mathrm{cl}_{M_\delta}(A) \prec_{\boldsymbol{K}} M_i$.

Case 1: model complete

So as $N, M_\delta \in \boldsymbol{K}, N \subseteq M_\delta$ we have $N \prec_{\boldsymbol{K}} M_\delta$ so

$$\mathrm{cl}_N(A) = \mathrm{cl}_{M_\delta}(A) \prec_{\boldsymbol{K}} M_i.$$

Case 2: has quantifier free closure

Clearly $\mathrm{cl}_N(A) = \mathrm{cl}_{M_\delta}(A)$ and as we observed to start the proof $\mathrm{cl}_{M_\delta}(A) \prec_{\boldsymbol{K}} M_i$.

$\square_{4.4}$

Now we pass to the main construction.

Definition 4.5 *Let $\boldsymbol{K}$ be an AEC with a relational vocabulary τ. The vocabulary τ' of $\boldsymbol{K}'$ is obtained by adding one additional binary relation R. We say the domain of a τ'-structure A is an R-set if R induces a complete graph on A.*

1. *The class $\boldsymbol{K}'$ is those τ'-structures M such that:*
   (a) *If the finite subset A of M is an R-set there is a τ'-structure M_A such that $A \subseteq M_A \subseteq M$ with $|M_A| \leq \mathrm{LS}(\boldsymbol{K})$, M_A is an R-set, and $M_A\restriction\tau \in \boldsymbol{K}$.*
   (b) *If $N \subset M$ satisfies the conditions on M_A in requirement 1), then $M_A\restriction\tau \prec_{\boldsymbol{K}} N\restriction\tau$.*
   (c) *For each M and A, we denote M_A by $\mathrm{cl}'_M(A)$.*
2. *If $M_1 \subseteq M_2$ are each in $\boldsymbol{K}'$, then $M_1 \prec_{\boldsymbol{K}'} M_2$ if for each finite R-set A in M_1, $\mathrm{cl}'_{M_1}(A) = \mathrm{cl}'_{M_2}(A)$.*

Note that if $M \subset N$ are τ-structures in $\boldsymbol{K}'$ and $M\restriction\tau \prec_{\boldsymbol{K}} N\restriction\tau$ then for any finite $A \subset M$, $\mathrm{cl}'_M(A) = \mathrm{cl}'_N(A)$.

Lemma 4.6 *Let $(\boldsymbol{K}, \prec_{\boldsymbol{K}})$ be an aec in* a relational language *which admits closures and is nice for unions. Then, $(\boldsymbol{K}', \prec_{\boldsymbol{K}'})$ is an AEC with amalgamation.*

Proof. The axioms for an AEC which do not involve unions are easy. For example, we show 'coherence' in $\boldsymbol{K}'$. Suppose $M \subset N \prec_{\boldsymbol{K}'} N'$ are τ'-structure and also $M \prec_{\boldsymbol{K}'} N'$. Fix any finite $A \subset M$. Since $N \prec_{\boldsymbol{K}'} N'$, $\mathrm{cl}'_N(A) = \mathrm{cl}'_{N'}(A)$. Since $M \prec_{\boldsymbol{K}'} N'$, $\mathrm{cl}'_M(A) = \mathrm{cl}'_{N'}(A)$. Thus, $\mathrm{cl}'_M(A) = \mathrm{cl}'_N(A)$ and $M \prec_{\boldsymbol{K}'} N$.

Suppose $\langle M_i : i \leq \delta\rangle$ is a continuous increasing chain of $\boldsymbol{K}'$-extensions. Let A be a finite R-set contained in M_δ. Fix the least $j < \delta$ with $A \subset M_j$. Then, each $i \geq j$, $\mathrm{cl}_{M_i}(A) = \mathrm{cl}_{M_j}(A)$; call this set M_A; it satisfies the conditions of Definition 4.5 1) a). Consider any other $N \subset M_\delta$ which satisfies the conditions of Definition 4.5 1) a); that is, N is an R-set containing A and $N\restriction\tau \in \boldsymbol{K}$. To show $M_\delta \in \boldsymbol{K}'$, we must show $M_A\restriction\tau \prec_{\boldsymbol{K}} N\restriction\tau$. Since $\boldsymbol{K}$ is nice for unions there is a $k < \delta$ and a τ-structure N' with $N' \prec_{\boldsymbol{K}} M_k\restriction\tau$, $A \subset N'$ and $N' \prec_{\boldsymbol{K}} N\restriction\tau$. We don't know whether the τ' structure with universe N' is in $\boldsymbol{K}'$. But M_A is an R-set; $M_A\restriction\tau \prec_{\boldsymbol{K}} M_k$, $M_A \subseteq N'$ and $N' \prec_{\boldsymbol{K}} M_k\restriction\tau$. So, by coherence in $\boldsymbol{K}$, $M_A\restriction\tau \prec_{\boldsymbol{K}} N'$ and we know $N' \prec_{\boldsymbol{K}} N\restriction\tau$. By transitivity of $\prec_{\boldsymbol{K}}$, this is exactly what is needed.

Now we show the second union axiom. Suppose $\langle M_i : i < \delta \rangle$ is a continuous increasing $\boldsymbol{K'}$-chain with each $M_i \prec_{\boldsymbol{K'}} M$. Then $M_\delta \prec_{\boldsymbol{K'}} M$, since we have shown for any finite $A \subset M_\delta$ there is an i with $\mathrm{cl}'_{M_\delta}(A) = \mathrm{cl}'_{M_i}(A)$.

To show amalgamation suppose $M_0 \prec_{\boldsymbol{K'}} M_1, M_2$. Without loss of generality $M_1 \cap M_2 = M_1$. Now, form the no-edges amalgamation of the underlying graphs of M_1 and M_2 over M_0. The structure with this domain is in $\boldsymbol{K'}$, as each finite R-subset A of it is in either M_1 or M_2; the closure of each such A to satisfy Definition 4.5 is easily found.

$\square_{4.6}$

Since that it is easy to obtain the hypothesis in Lemma 4.6 that $\boldsymbol{K}$ is nice for unions by the transformation in Lemma 4.2, we have shown:

Corollary 4.7 *To any AEC $(\boldsymbol{K}, \prec_{\boldsymbol{K}})$ which admits closures, we can assign $(\boldsymbol{K'}, \prec_{\boldsymbol{K'}})$ which has the amalgamation property.*

Now we note that any example of nonlocality in $\boldsymbol{K}$ remains such an example in $\boldsymbol{K'}$. As opposed to Lemma 4.2, this second transformation plays havoc with the spectrum function. We spell out the result for locality but the same holds for tameness.

Lemma 4.8 *If $\boldsymbol{K}$ is not (δ, λ)-local then $\boldsymbol{K'}$ is not (δ, λ)-local.*

Proof. Let $|M| = \lambda$ and suppose $p, q \in \mathrm{ga}-\mathrm{S(M)}$, $M = \bigcup_{i<\delta} M_i$, $p \neq q$ and $M, M_i \in \boldsymbol{K}$. Let M^1, M^2 be $\boldsymbol{K}$-extensions of M and let $a \in M_1$ realize p and $b \in M^2$ realize q. Now consider the models $\overline{M}, \overline{M}_1, \overline{M}_2$ in $\boldsymbol{K'}$ obtained by adding a complete R-graph to M, M_1, M_2. Then for each $i < \delta$, $\mathrm{tp}(a/\overline{M}_i, \overline{M}^1) = \mathrm{tp}(b/\overline{M}_i, \overline{M}^2)$. (Witness with the same maps just adding a complete R-graph.) But $\mathrm{tp}(a/\overline{M}, \overline{M}^1) \neq \mathrm{tp}(b/\overline{M}, \overline{M}^2)$ since any τ'-isomorphism taking a and b to the same point would restrict to a similar τ-isomorphism. $\square_{4.8}$

We now show that at least for regular λ, if $\boldsymbol{K'}$ is not (λ, λ)-local then it is not weakly (λ, λ)-local.

Definition 4.9 *For any property P which can hold of models, we say that* (κ, λ)-almost all models *of $\boldsymbol{K}$ satisfy P if Player II has a winning strategy for the following game. The game lasts κ moves. At each stage each player must choose a model of cardinality λ extending all the preceding models in the chain. Player II wins if the union satisfies P.*

Note that if two properties are satisfied by (κ, λ)-almost all M, then (κ, λ)-almost all M satisfy both of them. Also when λ is regular, if there is a saturated model in power λ, (λ, λ)-almost all M are saturated.

We say a model is, e.g. (κ, λ)-compact if every union of types over a decomposition of the model has a limit.

Lemma 4.10 *Let $\boldsymbol{K}$ and $\boldsymbol{K'}$ be as in Definition 4.5.*

1. *If $\boldsymbol{K}$ is not (κ, λ)-compact then (κ, λ)-almost all models of $\boldsymbol{K}'$ are not (κ, λ)-compact.*
2. *If $\boldsymbol{K}$ is not (κ, λ)-local then (κ, λ)-almost all models of $\boldsymbol{K}'$ are not (κ, λ)-local.*

Proof. First we consider compactness. Let $\langle M_i, p_i\rangle$ for $i < \kappa$ be a continuous increasing sequence of τ-structures that witnesses the incompactness. Expand M_κ to a τ' structure by making it an R-set. Let $\langle N_i\rangle$ for $i < \kappa$, be player I's moves. At stage α, let player II choose for his move M'_α so that $N_\alpha \prec_{\boldsymbol{K}'} M'_\alpha$ and $M_\alpha \prec_{\boldsymbol{K}} M'_\alpha \restriction \tau$. Let $p_\alpha = \mathrm{tp}(a_\alpha/M_\alpha, L_\alpha)$. Let L'_α be an amalgam of the expansion of L_α to an R-set with M'_α (over $M'_{\alpha-1}$ at successors; take unions at limits.); let $p'_\alpha = \mathrm{tp}(a_\alpha/M'_\alpha, L'_\alpha)$. Now there is no realization of $\bigcup p'_\alpha$ since it would reduct to a realization of $\bigcup p_\alpha$. So Player II wins the game that asks each model to be extended to one which witnesses incompactness. For locality, do the same argument but choose L so that there are $a, b \in L$ such that $\mathrm{tp}(a/M_i, L) = tp(b/M_i, L)$ for each $i < \kappa$ but $\mathrm{tp}(a/M, L) \neq tp(b/M, L)$ and finish as in Lemma 4.8. $\square_{4.10}$

It is immediate from Lemma 4.10 and the remark before it that

Corollary 4.11 *If $\boldsymbol{K}$ is not (κ, λ)-compact then $\boldsymbol{K}'$ is weakly-(κ, λ)-compact.*

5 Gaining tameness

We gave in Section 2 an example of an AEC with the amalgamation property and Lowenheim-Skolem number $\aleph_0$ which is not $\aleph_0$-tame. But at least consistently there are arbitrarily large κ for which it is not (κ, ∞)-tame. Here, we respond to a question of Grossberg and VanDieren [GVb] and provide an example of an AEC with the amalgamation property and Lowenheim-Skolem number $\aleph_0$ which is not $\aleph_0$-tame but is $(2^{\aleph_0}, \infty)$-tame. The example is very close to that in Section 2 but we bound the size of the image group G.

We require one other property of the crucial group G from Section 2. There is a countable subgroup R of G such that every element of G/R is divisible by every prime. See [EM90].

Example 5.1 Let $\boldsymbol{K}^s$ be the class of structures $M = \langle G, Z, I, H, R\rangle$, where each of the listed sets is the solution set of one of the unary predicates $(\boldsymbol{G}, \boldsymbol{Z}, \boldsymbol{I}, \boldsymbol{H}, \boldsymbol{R})$. The first four predicates are interpreted exactly as in Example 2.3 but $\boldsymbol{R}$ is interpreted as the subgroup R described above. Crucially, we require that the group G be not merely torsion-free but $\aleph_1$-free. The notion of strong submodel is as before except in addition $M \prec_{\boldsymbol{K}} N$ implies $\boldsymbol{R}^M = \boldsymbol{R}^N$.

Lemma 5.2 *The class $(\boldsymbol{K}, \prec_{\boldsymbol{K}})$ defined in Definition 5.1 is an AEC which admits closures.*

Proof. To show that $\boldsymbol{K}^s$ is closed under unions of chains (the interesting case is countable unions) apply Pontryagin's criterion. We can construct closures exactly as in Lemma 2.5. $\square_{5.2}$

Since we have required that G is $\aleph_1$-free, the amalgamation property is no longer trivial. But we are saved by Section 4. (The amalgamation of torsion-free groups is torsion-free but we don't have an amalgamation of $\aleph_1$-free groups into $\aleph_1$-free groups.)

Theorem 5.3 *There is an AEC with the amalgamation property in a countable language with Lowenheim-Skolem number $\aleph_0$ which is not $(\aleph_0, \aleph_1)$-tame but is $(2^{\aleph_0}, \infty)$-tame.*

Proof. Since $\boldsymbol{K}^s$ (Example 5.1) admits closures we can get the desired example with the amalgamation property from Corollary 4.7, provided we show Example 5.1 is $(2^{\aleph_0}, \infty)$-tame. Since the source of non-tameness is types over the target group G, it suffices to show the cardinality of $\boldsymbol{G}^M$ is at most the continuum for any $M \in \boldsymbol{K}^s$. But if G is torsion free and for some countable subgroup R, G/R is p-divisible for all p, then $|G| \leq 2^{\aleph_0}$. For, if G is larger there exist $x, y \in G$ which realize the same first order type over R. Thus for each p there is an x_1 such that $px_1 - x = r \in R$ and a y_1 such that $py_1 - y = r \in R$. But then $x - y$ is a non-zero element of G which is divisible by every prime. By Pontryagin's criterion, there is no such element in an $\aleph_1$-free group. $\square_{5.3}$

6 Conclusion

This paper has several messages. The notion of an AEC admitting closures is rather natural; it has come up without being named in investigations of the Hrushovski construction. It simplifies the treatment of Galois-types while being much weaker than the amalgamation property; we think it deserves further investigation.

We have shown that 'locality' has several facets. There has been considerable work on categoricity transfer for tame AEC [GVa, GVc, Les]; under further locality assumptions [HV] begins a 'geometric stability' theory. This paper shows that in general these are real assumptions. But could $(\mathrm{LS}(\boldsymbol{K}), \infty)$-tameness be a consequence of categoricity? Let κ be the Löwenheim-Skolem number of $\boldsymbol{K}$ and let H_1 denote $\beth_{(2^{\kappa}))^+}$. Analysis of the Hart-Shelah examples should give examples of an $\boldsymbol{K}$ which is categorical in small cardinals and fails tameness in a cardinality where it has the amalgamation property. In contrast, the next question should be quite hard. Shelah proved (see [Bal00] for a short account):

Theorem 6.1 *[She99] Suppose $\boldsymbol{K}$ has the amalgamation property and arbitrarily large models. Suppose $\boldsymbol{K}$ is λ^+-categorical with $\lambda > H_1$. $\boldsymbol{K}$ is (κ, χ)-weakly tame for some $\chi < H_1$.*

Question 6.2 *Suppose $\boldsymbol{K}$ has the amalgamation property and arbitrarily large models. Suppose $\boldsymbol{K}$ is λ^+ categorical with $\lambda > H_1$.*

Is there any way to reduce the upper bound on χ in Theorem 6.1

Here are questions which naturally arise in extending this work.

Problem 6.3 *Find an example of non-compactness in ZFC.*

Question 6.4 *Is the example in Section 3 (Section 5),* $(\aleph_1, \aleph_1)$*-compact.*

Question 6.5 *Find examples of* $\boldsymbol{K}$ *with the amalgamation property which gain tameness as in Section 5 but for inherent algebraic reasons rather than through the transformation of Section 4.*

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