STRONGLY MINIMAL STEINER SYSTEMS I: EXISTENCE

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Abstract. A linear space is a system of points and lines such that any two distinct points determine a *unique* line; a Steiner k-system (for $k \geqslant 2$) is a linear space such that each line has size exactly k. Clearly, as a two-sorted structure, no linear space can be strongly minimal. We formulate linear spaces in a (bi-interpretable) vocabulary τ with a single ternary relation R. We prove that for every integer k there exist $2^{\aleph_0}$-many integer valued functions μ such that each μ determines a distinct strongly minimal Steiner k-system $\mathcal{G}_\mu$, whose algebraic closure geometry has all the properties of the *ab initio* Hrushovski construction. Thus each is a counterexample to the Zilber Trichotomy Conjecture.

1. Introduction

Zilber conjectured that every strongly minimal set was (essentially) bi-interpretable either with a strongly minimal set whose associated acl-geometry was trivial or locally modular, or with an algebraically closed field. Hrushovski [Hru93] refuted that conjecture by a seminal extension of the Fraïssé construction of $\aleph_0$-categorical theories as 'limits' of finite structures to construct strongly minimal (and so $\aleph_1$-categorical) theories. In this paper we modify Hrushovski's method to construct $2^{\aleph_0}$-many strongly minimal Steiner systems that also violate Zilber's conjecture. The examples arising from Hrushovski's construction have been seen as pathological, and there has been little work exploring the actual theories. The new examples that we construct here are infinite analogs of concepts that have been central to combinatorics for 150 years. But most of these investigations (e.g. [BB93, CR99, RR10]) focus on finite systems.

Our construction of strongly minimal linear spaces via a Hrushovski's construction might lead in two directions: (i) explore infinite Steiner systems investigating combinatorial notions appearing in such papers as [Cam94, CW12, GW75, Ste56]; (ii) search for further mathematically interesting strongly minimal sets with exotic geometries. This paper is an essential prerequisite for the sequel [Bal19], where we address both issues by showing the examples here are *essentially unary*[1], expand the techniques used here to construct strongly minimal quasigroups, and extend the combinatorial analysis of [CW12] to to those quasigroups.

Our construction combines methods from the theory of linear spaces/combinatorics and model theory. A linear space (Definition 2.4) is a system of points and lines such that any two points determine a *unique* line. A Steiner k-system is a

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[1]There is no parameter-free definable function depending on more than one variable.

linear space such that all lines have size k. We explain strong minimality below, and explore its connection with Steiner systems in Section 2.1.

The key ingredient of our construction is the development in [Pao] of a new model theoretic rank function inspired by Mason's α-function [Mas72], which arose in matroid theory. Using this new rank to produce a strongly minimal set requires a variant on the Hrushovski construction [Hru93] with several new features.

This is the first of a series of papers exploring these examples. Here are the main results of this paper; they depend on definitions explained below.

- Theorem 2.9: The one-sorted (Definition 2.1) and two-sorted (Definition 2.4) notions of linear space are bi-interpretable.
- Theorem 2.7(2): For each k, with $3 \leqslant k < \omega$, there are $2^{\aleph_0}$-many strongly minimal theories T_μ (depending[2] on an integer valued function μ) of infinite linear spaces in the one-sorted vocabulary τ that are Steiner k-systems.
- Corollary 6.7: Each theory T_μ admits weak[3] but not strong elimination of imaginaries, its geometry is not locally modular, but it is CM-trivial and so it does not interpret a field. Thus, it violates Zilber's conjecture.

The last two results make sense only in the one-sorted vocabulary τ (see below for a more detailed explanation of this). This phenomena is symptomatic of the interplay among model theory, finite geometries and matroid theory. Notions in these areas are 'almost' the same. Sometimes 'almost' is good enough and sometimes not. The same intuitive structures are formalized in different vocabularies and in different logics depending on the field. Thus, the first task of this paper is to explain this interaction. The first main result addresses this issue; further refinements on bi-interpretability appear in Section 2.3 and even more in [Bal19].

As this investigation is aimed at researchers from three areas we are careful to explicate notions which are basic in one area but not others. For example, the notion of bi-interpretation used here has a similar purpose to cryptomorphism in matroid theory or polynomial equivalence in universal algebra. But bi-interpretation is both more precisely defined than cryptomorphism and much coarser than polynomial equivalence. Similarly, we recount salient properties of strong minimality to emphasize to non-model theorists the significance of that notion and we ensure that the proof of the basic existence result here is largely self-contained.

A structure M is strongly minimal if it is infinite and in *any* elementary extension N of M every definable subset X of N (i.e. X is the set of solutions in N of a first-order formula $\varphi(x, \overline{a})$) is either finite or cofinite. We observe in Section 2.1 that a strongly minimal linear space *cannot* be affine or projective. Furthermore, as we note in Fact 2.3, an infinite strongly minimal linear space must have all lines of a bounded finite cardinality.

Strongly minimal sets are the building blocks of many structures analyzed by model theoretic methods (see [Bal18, Hod87] for non-technical overviews or any stability theory text for details). These blocks are simply structures such that the algebraic closure relation determines a matroid (combinatorial pre-geometry) and such that any bijection between two bases extends to an automorphism of the structure. Stability theory allows the decomposition of many natural algebraic structures into manageable combinations of strongly minimal sets. Pure sets, vector

[2]The theory of course depends on the line length k; k is coded by μ so we suppress the k.

[3]In view of Lemma 5.29 and Fact 6.5 our argument may, in very special cases, require naming finitely many constants to guarantee that $\mathrm{acl}(\emptyset)$ is infinite.

spaces, and algebraically closed fields are natural examples of these building blocks. Zilber expresses his goal this way:

> The initial hope of this author in [Zil84] that any uncountably categorical structure comes from a classical context (the trichotomy conjecture), was based on the belief that logically perfect structures could not be overlooked in the natural progression of mathematics. Allowing some philosophical license here, this was also a belief in a strong logical predetermination of basic mathematical structures. As a matter of fact, it turned out to be true in many cases.

We investigate here a new case where the structures have classical roots. Much of the current research on strongly minimal theories (as opposed for example to the strongly minimal sets discovered in differentially closed fields) focuses on classifying the attached acl-geometry. Work of Evans, Ferreira, Hasson, and Mermelstein [EF11, EF12, HM18, Mer18] suggests that up to arity or more precisely, purity, (and modulo some apparently natural conditions[4]) any two acl-geometries associated with strongly minimal Hrushovski constructions are locally isomorphic. This analysis is orthogonal to our program, which focuses on the particular strongly minimal theories constructed.

The naive observation that a plane has Morley rank 2 motivated the construction in [Bal94] of an $\aleph_1$-categorical non-Desarguesian *projective plane* of Morley rank exactly 2. The novelty of that result is the failure of the Desarguesian axiom; while the projective plane over $\mathbb{C}$ has Morley rank 2, it is 'field-like' and so Desarguesian. The result here complements that example, weakening 'projective plane' to 'plane' (a linear space which admits the structure of a simple rank 3 matroid) while strengthening Morley rank 2 to strongly minimal (i.e. Morley rank 1 and Morley degree 1). And the examples turn out to be Steiner systems.

A key difference from the finite situation is that k-Steiner systems of finite cardinality v occur only under strict number theoretic conditions on v and k. In contrast, for every k, we construct theories with countably many models in $\aleph_0$ and one in each uncountable power that are all Steiner k-systems. But the number theory reappears when we attempt to find algebraic structures associated with these geometries. One goal is to coordinatize the Steiner systems by nicely behaved algebras. A substantial literature [Ste57, Ste56, GW75, GW80] builds a correspondence between k-Steiner systems and certain varieties of universal algebras. But while this correspondence is a bi-interpretation for $k = 3$, it does not rise to that level in general. Indeed, for $k > 3$, we show [Bal19] that none of the strongly minimal Steiner systems constructed here interpret a quasi-group[5]. We also prove there that for q a prime power, and V an appropriate variety, for each of our theories T_μ there is a theory $T_{\mu,V}$ of a strongly minimal quasigroup in V that interprets a q-strongly minimal Steiner system.

As already mentioned, most of the literature on linear spaces focuses on finite structures, but Cameron [Cam94] asserts:

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[4] In [EF11, EF11], the class of finite structures is restricted only by the dimension function and properties of mu, that satisfy several technical conditions, which don't hold in some constructions in [Bal19], as opposed to such axioms as 'two points determine a line' here or the existence of a quasigroup structure in [Bal19].

[5] A quasigroup is a structure $(A, *)$ such that specification of any two of x, y, z in the equation $x * y = z$ determines the third uniquely.

> There is no theory of infinite linear spaces comparable to the enormous amount known about finite linear spaces. This is due to two contrasting factors. First, techniques which are crucial in the finite case (notably counting) are not available. Second, infinite linear spaces are too easy to construct; instead of having to force our configurations to close up, we just continue adding points and lines infinitely often! The result is a proliferation of examples without any set of tools to deal with them.

We import non-trivial constructions from model theory to build interesting linear spaces. Since we are interested in finding theories rather than structures, we construct families of similar Steiner systems that are similar both combinatorially and model theoretically. Perhaps this technique could become a tool in studying infinite linear spaces as stability has already influenced graph theory [MS14].

Studying the (a, b)-cycle graph [CW12] associated with Steiner triple systems (Definition 4.9), already yielded a perspicuous proof that we have constructed continuum many theories (Corollary 5.26). In [Bal19] we extend the notion of graph cycle from Steiner triple systems to Steiner q-systems, for q a prime power, and produce examples of T_μ that have only finite cycles (called paths in the more general situation) in the prime model but infinite cycles in all others. By cutting away from the class of finite models some that have low δ-rank, it is fairly easy to guarantee that all models of T_μ are 2-transitive. By making a relatively large such cut Hrushovski [Hru93, Example 5.2] produced an example, which as a side effect, is a Steiner triple system. But this construction does not generalize uniformly, as ours does, to get Steiner k-systems for larger k. With less extreme surgery we find in [Bal19] theories of q-Steiner system such that every model is 2-transitive and thus the path graph is uniform in a sense inspired by [CW12].

There have been a number of papers that use model theoretic techniques and, in at least one case, the Hrushovski construction, to investigate linear spaces and Steiner systems. Our approach differs by invoking a predimension function inspired by Mason's α-function, and focusing on the combinatorial *consequences of strong minimality* by investigating the family of similar (elementarily equivalent) structures of arbitrary cardinality arising from a particular strongly minimal k-Steiner system. In contrast, Evans [Eva04] constructs Steiner triple systems using a variant of the Hrushovski construction without discussing their stability class. At the opposite end of the stability spectra from our result, Barbina and Casanovas [BC1x] find existentially closed Steiner triple systems that are $TP2$ and $NSOP_1$ by a traditional Fraïssé construction. Remark 6.1 compares their example with ours in more detail. Between these extremes, Hytinnen and Paolini [HP] show that the Hall construction of free projective planes yields a strictly stable theory. Conant and Kruckman [CK16] find an existentially closed projective plane and prove it is $NSOP_1$ but not simple. Their construction involves a generalized Fraïssé construction for the existential completeness as well as the Hall construction.

Thus, there are four techniques that construct infinite linear spaces in a range of stability classes: taking all extensions in a given universal class but insisting on finite amalgamation in a standard Fraïssé construction [BC1x], building one chain of models carefully [HP], combining these two methods but allowing the amalgam of finite structures to be countable [CK16], and, as here, restricting the amalgamation class to guarantee a well-behaved acl-geometry.

Section 2.1 provides background on strong minimality, linear spaces, matroids, and proves the bi-interpretablity between the one and two-sorted approach. Sections 3 and 4 lay out the distinctions in the basic theory between the general Hrushovski approach and the specific dimension function for linear spaces studied here. In Section 5 we prove the main existence theorem for strongly minimal Steiner systems and in Section 6 we discuss the connection with recent work on the model theory of Steiner systems and expound the underlying properties which show that our examples have the usual 'geometric' properties of Hrushovski constructions.

2. Strong Minimality, Linear Spaces, Matroids and planes {smaetal}

The goals of this paper and the sequel are to construct strongly minimal linear spaces, in fact, Steiner systems and to investigate some of the relevant connections between model theory and combinatorics. In this section we describe strong minimality on the one hand, and the combinatorial notions of linear space, matroid, and some notions from design theory on the other. The sophisticated study of strongly minimal sets depends on the general framework of one-sorted first-order logic; linear systems are usually studied in a two-sorted first-order logic, while matroids are rarely formalized (See Section 2.4.). We explore here the role of and translations between these various 'formalisms'. Most of our work takes place in the following context: {taulin}

Definition 2.1 (Linear Spaces in τ)**.** *Let τ contain a single ternary relation symbol R which holds of sets of 3 distinct elements in any order. $\boldsymbol{K}^*$, the class of* linear spaces*, consists of the τ-structures that satisfy: any two distinct points determine a unique line when R is interpreted as collinearity. That is, $R(x,y,z) \wedge R(x,y,w) \rightarrow R(x,w,z)$. Each pair of elements is regarded as lying on a (trivial) line; each non-trivial line is a maximal R-clique.*
$\boldsymbol{K}^*_0$ denotes the collection of finite structures in $\boldsymbol{K}^*$.

The switch from a 2-sorted to a 1-sorted formalism leads to some peculiar notation. In the two-sorted world, a line in $(M; P^M, L^M)$ can gain points when M is extended. In the one-sorted context a line is a subset of the universe which is definable from any two points lying on it. But this definition is non-uniform. If the line is trivial (only two points) the definition is $x = a \vee x = b$; if the line is non-trivial the definition is $R(a, b, x)$. As a model M is extended, not only may a line gain points, but the correct such definition can change.

2.1. **Strongly Minimal Theories** {sm}

A complete theory T is strongly minimal if every model of T is a strongly minimal structure. Prototypic examples of strongly minimal theories include completions of the pure theory of equality, vector spaces, and algebraically closed fields.

We define the *model theoretic algebraic closure* of a set $A \subseteq M$ to be[6]:

$$\mathrm{acl}_M(A) = \{b \in M : M \models \varphi(b, \overline{a}) \wedge \text{ for some } k\ (\exists^{k!} x)\varphi(x, \overline{a})\},$$

where the $\varphi(x, \overline{a})$ vary over all formulas with parameters from A. In any strongly minimal structure M, the operator acl induces a matroid (pre-geometry) on the subsets of M (see e.g. [BL71]). This pre-geometry is infinite dimensional if M is

[6] $(\exists^{k!} x)\varphi(x, \overline{a})$ means that there are exactly k-solutions of $\varphi(x, \overline{a})$; we similarly use $(\exists^{>k} x)$. These are abbreviations of first-order formulas.

saturated. If $\mathrm{acl}_M(a) = \{a\}$, for every $a \in M$, then (M, acl_M) is a simple matroid (a combinatorial geometry). Strong minimality imposes significant restrictions on the structure M due to the following:

{eliminf}

Fact 2.2. *If M is strongly minimal, then for every formula $\varphi(x, \overline{y})$, there is an integer $k = k_\varphi$ such that for any $\overline{a} \in M$, $(\exists^{>k_\varphi} x)\varphi(x, \overline{a})$ implies that there are infinitely many solutions of $\varphi(x, \overline{a})$, and thus finitely many solutions of $\neg\varphi(x, \overline{a})$.*

This is an easy consequence of the compactness theorem: if the conclusion fails the collection of sentences $\{(\exists^{>k} x)\varphi(x, \overline{y}) \wedge (\exists^{>k} x)\neg\varphi(x, \overline{y}) \colon k < \omega\}$ is finitely satisfiable and so realized by some $\overline{a}^*$ in an elementary extension N of M, which contradicts *strong minimality*. This result allows us, by suppressing the dependence of k on φ, to introduce the abbreviation $(\exists^{\infty} x)\varphi(x, \overline{a})$ for $(\exists^{>k_\varphi} x)\varphi(x, \overline{a})$. As, in our context, the second assertion implies the first, which is usually not first-order.

Fact 2.2 has an immediate consequence for any strongly minimal linear space, $(M, R) \in \boldsymbol{K}^*$ (cf. Definition 2.1), where all lines have at least 3 points: there can be no infinite lines. Suppose ℓ is an infinite line. Choose A not on ℓ. For each B_i, B_j on ℓ the lines AB_i and AB_j intersect only in A. But each line B_i has a point not on ℓ and not equal to A. Thus ℓ has an infinite definable complement, contradicting strong minimality. More strongly, we observe:

{bndlen}

Fact 2.3. *If (M, R) is a strongly minimal linear space, then there exists an integer k such that all lines have length at most k.*

As, $R(x, y, z)$ means[7] x, y, z are collinear, i.e. x is on the line determined by y, z, applying Fact 2.2 we see that there is $k = k_R$ such that $(\exists^{>k_R} x)R(x, a, b)$ implies the line through a, b is infinite, which contradicts the preceding paragraph. In particular, there can be no strongly minimal affine or projective plane, since in such planes the number points on a line must equal the number of lines through a point (+1 in the finite affine case).

{linsp}

2.2. (Families of) Linear Spaces

We begin with the notion of linear space as expounded in [BB93]. We formalize this notion in the usual first-order two-sorted way. In Definition 2.1 we provided a one-sorted formalization of linear spaces, and in Theorem 2.9 we will prove that the two definitions are bi-interpretable.

{linspace}

Definition 2.4 (Linear Spaces in τ^+)**.** A linear space *is a structure S for a vocabulary τ^+ with unary predicates P (points) and L (lines) and a binary relation I (incidence) satisfying the following properties:*

(A) any two distinct points lie on at exactly one line;
(B) each line contains at least two points.

$\boldsymbol{K}^+$ denotes the collection of τ^+-structures that are linear spaces.

Remark 2.5. *We omit in Definition 2.4 the usual non-trivality condition that there are at least three points not on a common line. It will of course be true of the infinite structures that we construct, but allowing even the empty structure is technically convenient.*

[7]We require any triple satisfying R to be of distinct points.

While [BB93] deals almost exclusively with *finite* linear spaces, the definition extends (as the authors noted) to allow infinite spaces. We pause to describe several different descriptions of linear spaces, most notably *pairwise balanced designs* (as defined in [Wil72]): {pbddef}

Definition 2.6 (PBD)**.** *A finite design is a pair $(X, \mathcal{L})$ where X is a finite set and $\mathcal{L}$ is a family $\{B_i : i \in I\}$ of (not necessarily distinct) subsets of X.*

(1) For $v \geqslant 0$ and $\lambda > 0$ integers, and K a set of positive integers, a design $(X, \mathcal{L})$ is a (v, K, λ)-PBD, Pairwise Balanced Design*, if and only if:*
   (a) $|X| = v$;
   (b) $|B_i| \in K$; {(iii)}
   (c) every two element subset of X is contained in exactly λ blocks B_i.

(2) A Pairwise Balanced Design is said to be a Steiner system if $\lambda = 1$ and $|K| = 1$ (i.e. all blocks have the same size).

An infinite PBD is obtained by relaxing the requirement that v is finite.

If $K = \{k\}$, we adopt the standard notation of Steiner k-system.

Condition (1)(c) is read as asserting that the design is *pairwise balanced* with index λ. Any finite linear space is a (v, K, λ)-PBD for some K and with $\lambda = 1$. Fact 2.3 gives (1) of the next theorem; (2) is a consequence of our main construction. {designcon}

Theorem 2.7. *(1) A strongly minimal infinite linear space in the vocabulary τ (cf. Definition 2.1) is a $(v, K, 1)$-PBD for some finite set of integers K.*

(2) For each $3 \leqslant k < \omega$, we construct continuum-many strongly minimal infinite linear spaces in the vocabulary τ that are Steiner k-systems.

2.3. One and Two-Sorted Formalization {12sort}

We explore the historical connections between the one and two-sorted approach to combinatorial geometry and indicate that while our formalizations are bi-interpretable in the usual sense of model theory they differ in important ways. In particular, as mentioned in Section 2.1, the one-sorted version can be strongly minimal while the two-sorted one cannot. Hilbert's axiomatization of geometry is naturally formulated as a first-order two-sorted incidence geometry[8] and this framework is developed in, e.g., Hall [HJ43]. This tradition is continued with Definition 2.4 of linear spaces as two-sorted structures in a vocabulary τ^+ for first-order logic. Tarski aimed for a first-order foundation for Euclidean geometry and pioneered a single-sorted approach to geometry summarised in [GT99]. Here the fundamental relation is a ternary predicate interpreted as 'betweenness' or more generally as 'collinearity'. In order to apply standard model theoretic tools, we provide a first-order single-sorted framework in a vocabulary τ that is equivalent (for our purposes; recall, however, that Morley rank is not preserved) to the study of linear spaces.

In the next definitions, we regard a linear space in the vocabulary τ^+ (cf. Definition 2.4) as a τ-structure (cf. Definition 2.1); this is easily done. Given a τ^+-structure B as in Definition 2.4, define a τ-structure A by letting A be the points of B and defining $R(a, b, c)$ if and only there is line ℓ in B such that each of a, b, c is on ℓ. {def_plane}

Remark 2.8. We now show that the class $\boldsymbol{K}^*$ (Definition 2.1) of single-sorted linear spaces is bi-interpretable with the class $\boldsymbol{K}^+$ of linear spaces in the two-sorted vocabulary τ^+ (cf. Definition 2.4). Notice that conditions (A) and (B) of

[8]Although he includes two non-first-order axioms; all the properly geometric work is first-order axiomatized [Bal17a, Bal17b].

Definition 2.4 imply that every pair of distinct lines intersects in at most one point. Also, recall that we allow models with no points or lines.

We define a pair of mutually inverse bijections from the models of a class of τ-structures to a class of τ^+-structures and back that are uniformly definable, respect isomorphism, and preserve substructure. The notion that 'bi-interpretability' means 'same' requires some clarification. On the one hand, we have already mentioned that the transformation here does not preserve Morley rank/degree. This is because the lines of the τ^+ structure are interpreted as imaginary elements (equivalence classes) of the associated τ-structure (More concretely; this is a 2-dimensional interpretation [Hod93, 212].). On the other hand, such properties as decidability, $\aleph_1$-categoricity, and λ-stability are preserved by first-order bi-interpretability.

While the next theorem explicitly gives an isomorphism of categories (with embeddings as morphisms), by changing notation we could construct a bi-interpretation in the classical sense of [Hod93, Section 5.3] between $\boldsymbol{K}^+$ and $\boldsymbol{K}^*$. For example, the domain of the interpretation of $\boldsymbol{K}^+$ into $\boldsymbol{K}^*$ in part (1), which Hodges would label ∂_F, is: $\Delta(A^2) \cup (A^2 - \Delta(A^2))/E$. Our formulation is awkward for the usual applications to decidability but natural for our 'equivalence' between structures. Such a reformulation is a real strengthening since bi-interpretability of A and B is equivalent to their endomorphism rings being continuously isomorphic [AZ86] while mere isomorphism of those monoids gives equivalent categories of models as in [Las82]. But [BEKP16] shows that there are $\aleph_0$-categorical structures which have isomorphic but not continuously isomorphic endomorphism monoids.

{bint}

Theorem 2.9. *(1) There is an interpretation F of $\mathbf{K}^+$ into $\mathbf{K}^*$. That is, for every $A \in \mathbf{K}^*$ there is a τ^+-structure $F(A) \in \boldsymbol{K}^+$ definable without parameters in A.*

(2) There is an interpretation G of $\mathbf{K}^*$ into $\mathbf{K}^+$. That is, for every $B \in \mathbf{K}^+$ there is a τ-structure $G(B) \in \boldsymbol{K}^*$ definable without parameters in B.

(3) For any $A \in \boldsymbol{K}^*$, $G(F(A))$ is definably isomorphic to A and for any $B \in \boldsymbol{K}^+$, $F(G(B))$ is definably isomorphic to B. Thus we have a bi-interpretation.

Proof. We prove (1). Let $A \in \mathbf{K}^*$. Set $P = \{(a, a) \colon a \in A\}$ as the set of points of the τ^+-structure $F(A)$. Towards describing the lines, define the following equivalence relation E on $A^2 - P$ by declaring $(a, b)E(c, d)$ if and only if the following condition is met:

$(\star)$ $\quad |\{a, b, c, d\}| = 1$ or $\{a, b\} = \{c, d\}$ or $\{a, b\} \cup \{c, d\}$ is an R-clique.

We verify that E is transitive. To this end, suppose that $(a, b)E(c, d)$ and $(c, d)E(e, f)$, $e \neq f$, $\{a, b\} \neq \{c, d\}$ and $\{c, d\} \neq \{e, f\}$. Since each pair is of distinct elements both $\{a, b, c, d\}$ and $\{c, d, e, f\}$ are R-cliques and since two points determine a line $\{a, b, c, d, e, f\}$ is an R-clique and transitivity is established. Now, let

$$L = \{[(a, b)]_E : (a, b) \in A^2 \text{ such that } a \neq b\}$$

be the set of lines of $F(A)$. For $(p, p) \in P$ and $[(a, b)]_E \in L$ define the following point-line incidence relation:

$$(p, p)I[(a, b)]_E \Leftrightarrow \exists (c, d) \in [(a, b)]_E \text{ such that } p \in \{c, d\}.$$

Clearly, $F(A)$ is definable in the τ-structure (A, R). We show that $F(A) \in \boldsymbol{K}^+$, i.e. Definition 2.4 is satisfied. Obviously, Axiom (B) is satisfied. We prove axiom (A). Towards this goal, let ℓ_1 and ℓ_2 be two distinct lines of $F(A)$ that intersect (via the definition of I) in two distinct points (b_1, b_1) and (b_2, b_2). By hypothesis

$\ell_1 \neq \ell_2$ and so, we can assume $\ell_1 = [(b_1, b_2)]_E$ and there is $(c, d) \in A^2$ such that $c \neq d$, $\neg E((b_1, b_2), (c, d))$ and $(c, d) \in \ell_2$. Note that any E-equivalence class of element with more than 3 elements consists of an R-clique and distinct R-cliques can intersect in only one point; so, we finish.

We prove (2). Let $B \in \boldsymbol{K}^+$. Define the τ-structure $G(B) = (A, R)$ by letting A be the points of B and defining $R(a, b, c)$ if and only if a, b, c are distinct and there is a line ℓ in B such that each of a, b, c is on ℓ. Since B is a linear space the axioms of $\boldsymbol{K}^*$ are immediate.

We prove (3) by showing that up to definable isomorphism G is F^{-1}. Fix A and $F(A)$ from (1). We analyze the composition $G(F(A))$ and show the image is definably isomorphic to A. The set of points, $P^{F(A)}$, is the diagonal $\Delta(A^2)$ of A^2. Map (a, a) to a. The set of lines of $F(A)$ is $L^{F(A)} = (A^2 - \Delta(A^2))/E$. Let $m \in L^{F(A)}$ and suppose $(a_0, a_0), (a_1, a_1), (a_2, a_2)$ are on m, where the a_i are distinct. By the definition of I in $F(A)$, for each $i < 3$ there exists an a_i' such that for $i \neq j$, $[(a_i, a_i')]_E = [(a_j, a_j')]_E$. By $(*)$ this implies the a_i, a_i' for $i < 3$ (some may be repeated) form an R-clique in A. Thus $G(F(A))$ is definably isomorphic to A. Now we reverse the procedure and show that for $B \in \boldsymbol{K}^+$, $F(G(B))$ is definably isomorphic to B. This is even easier. If a, b, c are collinear in B, then $G(B) \models R(a, b, c)$ (Note P^B is the domain of $G(B)$). For this, recall the argument in part (1) showing $F(A) \in \boldsymbol{K}^*$ takes collinear points of A into a clique composed of elements of the diagonal of $G(B)$, which correspond to a clique in B. Applying this argument to $G(B)$ completes the proof.
Finally, this shows, in the case at hand, the essential point of [Mak18], that F is onto from $\boldsymbol{K}^*$ to $\boldsymbol{K}^+$. ■

2.4. Connections with Matroids {sec_matroids}

The convention in matroid theory is to regard the rank as (normal geometrical dimension) + 1. For example, a 'plane' is a rank 3 matroid. By a plane we here mean a model of a first-order single-sorted representation of the class of simple matroids of rank 3. In describing this representation we lay out a formal correspondence (i.e. a bi-interpretation) between the matroidal and axiomatic approaches (as incidence structures) to geometry. The functorial correspondence between matroids and certain incidence structures is well-known to experts, but, at the best of our knowledge, the formal correspondence by model-theoretic means in Lemma 2.12 has, like that in Theorem 2.9, not been made explicit in the literature.

As is well-known, see e.g. [WN85, Chapter 2], matroids can be defined using many different notions as primitive. Among them are the notions of dependent set, circuit, independent set, basis, etc. In this work we will assume as primary the notion of *dependent set*. In e.g. [Oxl92, WN85], a collection $\mathcal{D}$ of dependent sets is any collection of non-empty finite sets, closed under superset, and satisfying the well-known Exchange Axiom of Definition 2.10(2). The matroid theorist writes axioms in the fashion of Euclid, Hilbert in 1899 [Hil71], or Bourbaki; there is no formal language. In fact, no standard logic can directly express these axioms, since the collection of dependent sets contains finite set of various cardinalities. Notionally, his arguments and definitions can be formalized in ZFC, but this is not an issue to him. It is however crucial to our enterprize to describe our structures in first-order single-sorted logic.

For this, as in [Bal84], we first work in a relational vocabulary $\check{\tau} = \{R_n : 1 \leqslant n < \omega\}$, where R_n is an n-ary relation symbol. Our axioms on $\check{\tau}$-structures *first require* that each R_k is a *uniform-k-hypergraph*, that is, $M \models R_n(a_1, ..., a_n)$ implies:
(1) $a_i \neq a_j$ for every $1 \leqslant i < j \leqslant n$;
(2) $M \models R_n(a_{\sigma(1)}, ..., a_{\sigma(n)})$, for each $\sigma \in Sym(\{1, ..., n\})$.

Consequently, if $X \subseteq M$, $(x_1, ..., x_n)$ is an injective enumeration of X and $M \models R_n(x_1, ..., x_n)$, then we can write $M \models R_n(X)$. Given a $\check{\tau}$-structure M and $D \subseteq_\omega M$ we say that a set D is dependent if $M \models R_{|D|}(D)$. The *further axioms* in Definition 2.10 require the R_n to code in this way dependent sets of size n, for $1 \leqslant n < \omega$. For emphasis, we write the first and third axioms as $\check{\tau}$ sentences but we use the abbreviations introduced above to make the exchange axiom easier to read.

{preaxiomatization}

Definition 2.10 (Planes in $\check{\tau}$)**.** *Following [WN85] (see, in particular, [WN85, Proposition 2.2.3 and Theorem 2.2.6]) the class* $\mathbf{K}^{\check{\tau}}$ *of simple matroids of rank* $\leqslant 3$ *can be defined as the class of* $\check{\tau}$*-structures* M *such that each* R_k *is a uniform-k-hypergraph and satisfy the following further axioms:*

{small}
(1) $(\forall x)\neg R_1(x)$, $(\forall x, y)[x \neq y \to \neg R_2(x, y)]$*;*

{exchange}
(2) if $D_1, D_2 \subseteq_\omega M$ *are dependent and* $D_1 \cap D_2$ *is not dependent, then for every* $a \in M$ *we have that* $D_1 \cup D_2 - \{a\}$ *is dependent;*
(3) for all $n \geqslant 4$, $\forall x_1, \dots x_n R_n(x_1, \dots x_n)$*.*

We call $\boldsymbol{K}^{\check{\tau}}$ *the class of planes.*

In matroid parlance, condition (1) asserts that we consider only *simple* matroids; in the language of combinatorial geometry it asserts that the structure is a geometry, not merely a pre-geometry. The more usual requirement for a matroid that a superset of a dependent set is dependent follows immediately from (1) and (3). When dealing with simple matroids of rank 3 we can replace $\check{\tau}$ by the vocabulary with a single ternary relation symbol $R = R_3$ (see Definition 2.1 and Theorem 2.9).

The formulation in the last paragraphs deliberately smudges the transition between the informal and the formal first-order viewpoint. In the further development we will have to remind ourselves of the conditions we put on the R_n for $n > 3$.

Table 1 may help in navigating among the various choices of vocabulary (language) for first-order axiomatizations of linear spaces and matroids.

Language	Class	Context
τ	$\mathbf{K}^*$	One-sorted linear spaces (cf. Definition 2.1)
τ^+	$\mathbf{K}^+$	Two-sorted linear spaces (cf. Definition 2.4)
$\check{\tau}$	$\mathbf{K}^{\check{\tau}}$	Matroids of rank 3 (cf. Definition 2.10)
τ	$\check{\mathbf{K}}$	Matroids of rank 3 as τ-structures (cf. Definition 2.11)

TABLE 1. The various contexts/languages of Section 2. tab2.4

{axiomatization}

Definition 2.11 (Planes in τ)**.** *Let* τ *contain a single ternary relation symbol* R*. And, let* $\psi_n(x_1, \dots x_n)$ *assert that the* x_i *are distinct.* $\check{\boldsymbol{K}}$ *is the class of* τ*-structures that satisfy Definition 2.10 when* $x \neq x$ *is substituted for* $R_1(x)$*,* $x = y$ *for* $R_2(x, y)$*,* $R(x, y, z)$ *for* $R_3(x, y, z)$*, and* ψ_n *for* R_n *when* $n \geqslant 4$*.*
$\check{\boldsymbol{K}}_0$ *denotes the collection of finite structures in* $\check{\boldsymbol{K}}$*.*

Definition 2.11 is motivated by the following result, showing that the objects we create are planes in the matroid sense (cf. Definition 2.10).

{keyobs}

Lemma 2.12. *The axiom schema of Definition 2.11 determines a rank 3 matroid structure on each member of* $\check{\boldsymbol{K}}$.

Proof. Let the τ-structure A satisfy the axioms from Definition 2.11 concerning R_3, in particular, the exchange axiom for R_3. The only obstruction now is checking the exchange axiom under the hypothesis that every four or more element set is declared dependent in A. Suppose D_1 and D_2 are arbitrary sets with at least four elements. By the substitutions for R_1 and R_2, $|D_1 \cap D_2| \geqslant 3$ and so $D_1 \cup D_2$ has at least seven points and so is dependent. Thus A is a simple rank 3 matroid. ∎

{distinctions}

Remark 2.13. Lemma 2.12 could be generalized to any k using only R_i for $3 \leqslant i \leqslant k < \omega$. Of course, if the space arises as e.g. F_q^n, the n-space over a q-element field, and $k < n$ the matroid dependence by membership in $\check{\boldsymbol{K}}$ will be stronger than the dependence relation arising from the native linear space.

We distinguish among $\boldsymbol{K}^*$, $\boldsymbol{K}^{\check{\tau}}$ and $\check{\boldsymbol{K}}$ in order to be able to treat axiomatize certain notions in first-order logic. We often say a structure 'is' a matroid, meaning a matroid structure can be imposed. The notion of a matroid is a property expressed in ZFC. But if we formalize the matroid or linear space notions in one-sorted first-order logic we must be more careful. A $\check{\tau}$-structure which belongs to $\boldsymbol{K}^{\check{\tau}}$ is a matroid if it is a model of the axioms in Definition 2.10. A τ-structure in $\boldsymbol{K}^*$ may admit matroid structures of any finite rank. But a τ-structure in $\check{\boldsymbol{K}}$ is a matroid of rank 3 because it satisfies the sentences ψ_n from Definition 2.11. We are pedantic about the ψ_n in order to ensure that the structures at the end of our complicated construction are rank 3 matroids and so 'planes'. In view of Lemma 2.12, we can regard any linear system as a rank 3 matroid (and the limit structures to have any finite rank we please). That is, any such linear system admits the structure of a rank k matroid for any finite k, and so there was no need to restrict to the rank 3 case.

3. The Specific Context

{context}

In this section we introduce the specific context in which we will work for the rest of the paper. The main component of this section is the introduction of a new predimension function δ (cf. Definition 3.4), which will be the essential ingredient in the construction of our strongly minimal Steiner systems. This predimension function δ was introduced in [Pao] and it is inspired by Mason's α-function, a well-known measure of complexity for matroids introduced by Mason in [Mas72]. We will give an explicit definition of our function δ without introducing the matroid theoretic machinery needed to define the α-function. For the reader interested in this connection we refer to [Pao, Section 3], where this is carefully explained.

{hascnot}

Notation 3.1. *(1) For any class* $\boldsymbol{K}_0$ *of finite structures for a vocabulary* σ *that is closed under substructure,* $\hat{\boldsymbol{K}}_0$ *denotes the class of all* σ*-structures* M *such that every finite substructure of* M *is in* $\boldsymbol{K}_0$. {K_0}

(2) Given an arbitrary class of structures $\boldsymbol{L}$ *for a vocabulary* σ *we denote by* $\boldsymbol{L}_0$ *the class of finite structures in* $\boldsymbol{L}$*. (For convenience, we allow the empty structure.)*

(3) We write $\cong$ *for isomorphism,* $X \subseteq_\omega Y$ *for finite subset, and if* $B \subsetneq C$, $\hat{C}$ *for* $C - B$.

We will define below several classes of structures; in particular $\boldsymbol{K}_0$, $\boldsymbol{K}_\mu$, and $\boldsymbol{K}_d^\mu$ (see Table 3 for references). Furthermore, there will be five (closure operators)/(dependence relations) on structures in each class. Since all five of them could naturally be called 'geometric', we avoid this term and give them each a different tag (see also Table 2):

{indnot}

Notation 3.2 (Notions of Dependence)**.** *Let $M \in \mathbf{K}^*$ (cf. Def. 2.1) and $A \subseteq M$.*

(1) The dependence relation witnessing that M is a rank 3 matroid (cf. Definition 2.11) is denoted by $\text{m-cl}(A)$ and called m-dependence.
(2) The intrinsic closure operator (cf. Definition 3.7) is denoted by $\text{icl}(A)$.
(3) The d-closure operator (cf. Definition 5.4) is denoted by $\text{cl}^{\text{d}}(A)$.
(4) The algebraic closure operator is denoted by $\text{acl}(A)$. (We use the standard model theoretic notion for algebraic closure, i.e. $a \in \text{acl}(B)$ means that a is in a finite set definable with parameters from B.)
(5) The subspace closure $\text{cl}_R(X)$ in A, the smallest subset B of A containing X such that if $a \in A$ satsfies $R(b_1, b_2, a)$ with the $b_i \in B$, then $a \in B$.

A key fact, Lemma 5.27, asserts that on a d-closed structure M (cf. Definition 5.4) in the class $\boldsymbol{K}_\mu$, notions (3) and (4) are equivalent; this is central for proving strong minimality.

Tables 2 and 3 fix the notation introduced in Definition 3.2 and the classes of models discussed at various places in the text.

Notation	Name
$\text{m-cl}(A)$	m-dependence
$\text{icl}(A)$	intrinsic closure
$\text{acl}(A)$	algebraic closure
$\text{cl}^{\text{d}}(A)$	d-closure
$\text{cl}_R(X)$	subspace closure

TABLE 2. Notions of dependence. tab_dep

Notation	References
$\mathbf{K}^*$	Definition 2.1
$\mathbf{K}_0^*$	Definitions 2.1 and 3.1(2)
$\mathbf{K}_0$	Definition 3.7
$\hat{\boldsymbol{K}}_0$	Definitions 3.7 and 3.1(1)
$\boldsymbol{K}_\mu$	Definition 5.2(3)
$\boldsymbol{K}_d^\mu$	Definition 5.4(4)

TABLE 3. The classes of structures relevant to our construction. tab_classes

The following notation will clarify the distinction between 2-element lines (a.k.a. trivial lines) which are understood to hold of arbitrary pairs of elements from models in $\boldsymbol{K}^*$ and lines where the relation symbol R is explicit (cf. Definition 2.1).

{def_rank}

Definition 3.3. *Let $A \in \mathbf{K}^*$ and $A \subseteq B$ with $B \in \mathbf{K}^*$ (cf. Definition 2.1).*

(1) *The m-rank of $X \subseteq A$, denoted as* $\text{m-rk}(X)$*, is the cardinality of the largest m-independent subset of X. Notice that in this paper* $\text{m-rk}(X)$ *is usually at most 3 and that if $|X| < 3$, then* $\text{m-rk}(X) = |X|$.
(2) *A* line *of A is an R-closed subset of A of m-rank 2. In particular, if two points $a \neq b \in A$ and there is no $c \in A$ with $R(a, b, c)$, then $\{a, b\}$ is a line. We call such lines 'trivial'.*
(3) *We denote the cardinality of a line $\ell \subseteq A$ by $|\ell|$, and, for $B \subseteq A$, we denote by $|\ell|_B$ the cardinality of $\ell \cap B$.*
(4) *We say that a line ℓ contained in A is* based *in $B \subseteq A$ if $|\ell \cap B| \geqslant 2$, in this case we write $\ell \in L(B)$.* {nullity}
(5) *The* nullity of a line ℓ *contained in a structure $A \in \mathbf{K}^*$ is:*

$$\mathbf{n}_A(\ell) = |\ell| - 2.$$

Note that if $B \subseteq A$ are both in $\mathbf{K}^*$, and $\ell \subseteq A$ is a line then $\ell \cap B$ may be in $L(B)$ (if it has at least two points) but may not be R-closed in A (i.e. if $\ell - B \neq \emptyset$).

With these notions in hand, using m-rank as just defined, we introduce the new rank δ that is central to this paper[9]. It has two key features: (i) it is based on the notion of 'dimension' of a line; (ii) the associated geometry is flat, and so we get counterexamples to Zilber's conjecture (see Section 5 for details.).

{defdelrank}

Definition 3.4. *For $A \in \mathbf{K}^*_0$ (recall Definitions 2.1 and 3.1(2)), let:*

$$\delta(A) = |A| - \sum_{\ell \in L(A)} \mathbf{n}_A(\ell).$$

We rely on Lemma 19 of [Pao], which asserts:

{comp_prop}

Proposition 3.5. *Let A and B disjoint subsets of a structure $C \in \mathbf{K}^*_0$. Then:*
(1) *if $\ell \in L(AB)$ and $\ell \in L(B)$, then $\mathbf{n}_{AB}(\ell) - \mathbf{n}_B(\ell) = |\ell|_A$;*
(2) *$\delta(A/B) := \delta(AB) - \delta(B)$ is equal to:*

$$|A| - \sum_{\substack{\ell \in L(AB) \\ \ell \in L(A) \\ \ell \notin L(B)}} \mathbf{n}_{AB}(\ell) - \sum_{\substack{\ell \in L(AB) \\ \ell \in L(A) \\ \ell \in L(B)}} |\ell|_A - \sum_{\substack{\ell \in L(AB) \\ \ell \notin L(A) \\ \ell \in L(B)}} |\ell|_A.$$

Reference [BS96] provides a set of axioms for *strong substructure*; we give the instance of strong substructure relevant for this paper and recall some further terminology for classes with such a relation. These axioms can be seen to hold in our situation by modifying the argument for [BS96, Theorem 3.12], and using Proposition 3.5 and Axiom A6; on this see also [Pao, Section 4].

{conclusion_ax}

Fact 3.6. $(\boldsymbol{K}^*_0, \leqslant)$ *satisfies Axiom A1-A6 from [BS96, Axioms Group A], i.e.:*
(1) *if $A \in \boldsymbol{K}^*_0$, then $A \leqslant A$;*
(2) *if $A \leqslant B \in \boldsymbol{K}^*_0$, then A is a substructure of B;*
(3) *if $A, B, C \in \boldsymbol{K}^*_0$ and $A \leqslant B \leqslant C$, then $A \leqslant C$;*
(4) *if $A, B, C \in \boldsymbol{K}^*_0$, $A \leqslant C$, B is a substructure of C, and A is a substructure of B, then $A \leqslant B$;*
(5) *$\emptyset \in \boldsymbol{K}^*_0$ and $\emptyset \leqslant A$, for all $A \in \boldsymbol{K}^*_0$;* {item6}
(6) *if $A, B, C \in \boldsymbol{K}^*_0$, $A \leqslant B$, and C is a substructure of B, then $A \cap C \leqslant C$.*

[9] Mermelstein [Mer18] has independently studied variants on this rank, but only in the infinite rank case so the intricate analyis of primitives in this paper does not arise.

{K0def}

Definition 3.7. *(1) Let:*

$$\boldsymbol{K}_0 = \{A \in \mathbf{K}_0^* \text{ such that for any } A' \subseteq A, \delta(A') \geqslant 0\},$$

and $(\boldsymbol{K}_0, \leqslant)$ *be as in [BS96, Definition 3.11], i.e. we let* $A \leqslant B$ *if and only if:*

$$A \subseteq B \wedge \forall X(A \subseteq X \subseteq B \Rightarrow \delta(X) \geqslant \delta(A)).$$

(2) We write $A < B$ *to mean that* $A \leqslant B$ *and* A *is a proper subset of* B.
(3) For any X*, the least subset of* A *containing* X *that is strong in* A *is called the* intrinsic closure *of* X *in* A *and denoted by* $\mathrm{icl}_A(X)$ *or* $\overline{X}$ *(cf. Remark 3.9).*

Remark 3.8. Note that $\boldsymbol{K}_0$ has many fewer structures that $\boldsymbol{K}_0^*$. In particular, no projective plane (except the Fano plane, Example 4.3)or space A over a finite field is in $\boldsymbol{K}_0$; as, for each such A, $\delta(A) < 0$.

{remark_uni}

Remark 3.9. The existence and uniqueness of finite intrinsic closures of finite sets as defined in Definition 3.7 follows as in [BS96, Theorem 2.23].

Note that the verification of Axiom (6) from Fact 3.6 depends on the existence of a dimension function as in Proposition 3.10 of [BS96].

{various_def}

Definition 3.10. *Let* $A, B \in \mathbf{K}_0$*. We say that* A *is a* primitive extension *of* B *if* $B \leqslant A$ *and there is no* A_0 *with* $B \subsetneq A_0 \subsetneq A$ *such that* $B \leqslant A_0 \leqslant A$.

The Hrushovski construction is an outgrowth of the Fraïssé [Fra54] construction of 'homogeneous-universal' countable models; unlike Fraïssé the resulting theory need not be $\aleph_0$-categorical and the stability class of the resulting *generic* model can be controlled. We use the following notion of genericity:

{defgen}

Definition 3.11. *The countable model* $M \in \hat{\boldsymbol{K}}_0$ *is* $(\boldsymbol{K}_0, \leqslant)$-generic *when:*

(1) if $A \leqslant M, A \leqslant B \in \boldsymbol{K}_0$*, then there exists* $B' \leqslant M$ *such that* $B \cong_A B'$*;*
(2) for every finite $A \subseteq M$*,* $\mathrm{icl}_M(A)$ *is finite.*

The amalgamation property is a fundamental hypothesis for Fraïssé's work. We now define the appropriate notion in our context of a free amalgam $A \oplus_C B$ in $\boldsymbol{K}_0$.

{defcanam}

Definition 3.12. *Let* $A \cap B = C$ *with* $A, B, C \in \boldsymbol{K}_0$*. We define* $D := A \oplus_C B$ *as follows:*

(1) the domain of D *is* $A \cup B$*;*
(2) a pair of points $a \in A - C$ *and* $b \in B - C$ *are on a non-trivial line* ℓ' *in* D *if and only if there is a line* ℓ *based in* C *such that* $a \in \ell$ *(in* A*) and* $b \in \ell$ *(in* B*). Thus, in this case,* $\ell' = \ell$ *(in* D*).*

{remark_canonical_amalgam}

Remark 3.13. Notice that the amalgam $D := A \oplus_C B$ of Definition 3.12 can be characterized in more lattice theoretic terms as the following τ-structure:

(1) the domain of D is $A \cup B$;
(2) $R^D = R^A \cup R^B \cup \{\{a, b, c\} : a \vee b \vee c = a' \vee b' \text{ and } \{a', b'\} \subseteq C\}$.

Here $\vee$ refers to the join in the canonically associated geometric lattice $G(D)$ (see [HP18] and [Pao] for details). Apart from these additional tuples the amalgam D is simply the free relational amalgam of A and B over C, i.e. the structures on $A \cup B$ whose R-tuples are the union of the R-tuples in A and the R-tuples in B.

This notion was treated in more generality in [HP18] and in [Pao, Lemma 26]. In particular, Example 4.1 of [HP18] provides a counterexample to amalgamation in $\boldsymbol{K}_0$ when neither extension is strong. The amalgamation arguments given in [HP18] and [Pao] generalize to the case where only one extension is strong, but we choose here to give a direct proof using only the tools that appear in this paper.

{canext}

Lemma 3.14. *If $E \cap F = D$, $D \leqslant E$ and $E, F, D \in \boldsymbol{K}_0$ then $G = F \oplus_D E$ is in $\boldsymbol{K}_0$. Moreover, $F \leqslant G$.*

Proof. We need to check that each pair of points a_0, a_1 determine a unique line in G. Without loss of generality, one is in $F - D$ and the other in E. Suppose for contradiction there are two distinct lines on which both a_0, a_1 are incident. If both lines are contained in F, the claim is obvious. But, if not, Definition 3.12 guarantees there are $d_1, d_2 \in D$ on which one of these lines, say ℓ_0, is based. Now any $a \in G$ that purports not to be on ℓ_0 is either in $F - D$ or in $E - D$ and since both F and E are in $\boldsymbol{K}_0$, we must have $R(d_1, d_2, a)$, and so a is on ℓ_0.

We now show that $F \leqslant G$. Without loss of generality, we can assume that E is primitive over D. Compare $\delta(\hat{G}/D)$ with $\delta(\hat{G}/F)$. Note that $G - F = E - D = \hat{E}$. So $E = D\hat{E}$ but we use the second notation in the equations below to avoid excessive uses of $(E - D)$. Recall that the lines in these equations are all subsets of G and, for example, $\ell \in L(\hat{E})$ means $|(\ell \cap \hat{E})| \geqslant 2$. By Proposition 3.5 we have that:

$$(1)\quad \texttt{\{eq3\}}\ \delta(\hat{E}/D) = |\hat{E}| - \sum_{\substack{\ell \in L(D\hat{E}) \\ \ell \in L(\hat{E}) \\ \ell \notin L(D)}} \mathbf{n}_{D\hat{E}}(\ell) - \sum_{\substack{\ell \in L(D\hat{E}) \\ \ell \in L(\hat{E}) \\ \ell \in L(D)}} |\ell|_{\hat{E}} - \sum_{\substack{\ell \in L(D\hat{E}) \\ \ell \notin L(\hat{E}) \\ \ell \in L(D)}} |\ell|_{\hat{E}},$$

and

$$(2)\quad \texttt{\{eq4\}}\delta(\hat{E}/F) = |\hat{E}| - \sum_{\substack{\ell \in L(F\hat{E}) \\ \ell \in L(\hat{E}) \\ \ell \notin L(F)}} \mathbf{n}_{F\hat{E}}(\ell) - \sum_{\substack{\ell \in L(F\hat{E}) \\ \ell \in L(\hat{E}) \\ \ell \in L(F)}} |\ell|_{\hat{E}} - \sum_{\substack{\ell \in L(F\hat{E}) \\ \ell \notin L(\hat{E}) \\ \ell \in L(F)}} |\ell|_{\hat{E}}.$$

We compare each of the four terms on the right hand sides of Equations (1) and (2). The first terms are obviously equal.

For the second terms, suppose $\ell \in L(\hat{E})$ as in Equation 2. We next show:

(i) ℓ is one of the lines considered in the second term of Equation 1 if and only if it is one of the lines considered in the second term of Equation 2;
(ii) $\ell \subseteq DE$, and so $n_{FE}(\ell) = n_{DE}(\ell)$.

Suppose for contradiction that $\ell \cap (F - D) \neq \emptyset$; then, by Definition 3.12, ℓ is based in D. But also from the second term of Equation (2), $\ell \notin L(F)$. Thus, any line ℓ counted in the second term in either equation is contained in DE and $n_{FE}(\ell) = n_{DE}(\ell)$. So both equations give the same value to the second term.

For the third and fourth terms, note that again by Definition 3.12 any line in $L(F)$ that intersects $\hat{E}$ must already be based in D. Since for any ℓ, $|\ell|_{\hat{E}} = |\ell \cap \hat{E}|$ does not depend on the ambient model, the two terms considered have the same value. While the line may gain points between DE and FE, those points do not affect the value of the formula. ∎

We cannot claim that $E \leqslant G$ in Lemma 3.14 as it may very well be that $D \not\leqslant F$.

4. Primitive Extensions and Good Pairs

{primgood}

Using only the δ function one can build up models in $\boldsymbol{K}_0$ from well-defined building blocks: primitive extensions (Definition 3.10) and good pairs (Definition 4.1). This section is an analysis of these foundations. In the next section we use them to study the complete theories we are constructing.

{preal gebraic}

Definition 4.1. *Let $A, B \in \boldsymbol{K}_0$ with $A \cap B = \emptyset$ and $A \neq \emptyset$.*

(1) When we have a pair B, A with $B \leqslant A$ we often write $\hat{A}$ for $A - B$ to simplify notation (cf. Notation 3.1). Equivalently, we describe a primitive pair as (B, A) where B and A are disjoint (and so BA is the set in the initial description).

(2) We say that the pair A/B is primitive *or A (AB) is primitive over B if $BA \in \boldsymbol{K}_0$, $B \leqslant BA$ is primitive (cf. Definition 3.10). If $\delta(A/B) = 0$, we write 0-primitive. We stress that in this definition while B may be empty, A cannot be.*

(3) We say that the 0-primitive pair A/B is good *if for there is no $B' \subsetneq B$ such that (A/B') is 0-primitive. When discussing good pairs, usually A and B are disjoint; for ease of notation, sometimes A is confused with $A \cup B$.*

(4) If A is 0-primitive over B and $B' \subseteq B$ is such that we have that A/B' is good, then we say that B' is a base *for A (or sometimes for AB).*

(5) If the pair A/B is good, then we also write (B, A) is a good pair*.*

{rmk_gp}

Remark 4.2. Note that if C is primitive over the empty set then the unique base for C is $\emptyset$. For, if there is $B \neq \emptyset$ with $B \subsetneq C$ with C based on B, then $\emptyset \leqslant B$ and $B \subsetneq C$ contradicting that C is primitive over the empty set.
This does not forbid the existence of $C \in \boldsymbol{K}_0$ such that $\delta(C/\emptyset) = 0$ but C is not primitive over $\emptyset$; on this see Lemma 5.29.

{fano}

Example 4.3. Some sets are based on the empty set. In particular, if C is the τ-structure representing the unique 7 point projective plane (often called the Fano plane), then $\delta(C) = 0$. And it is easy to see $(\emptyset, C)$ is a good pair.

In earlier variants of the Hrushovski's construction one was able to prove the existence of a *unique* base B' for *any* given 0-primitive extension A/B. Unfortunately, this assertion is *false* in the current situation, cf. Example 4.4. We will make up for this with a careful examination of the structure of good pairs that almost regains uniqueness.

{example_bases}

Example 4.4. For $A \in \boldsymbol{K}_0$ containing $m + 2$ points $p_1, ..., p_{m+2}$ on a line ℓ and for some c such that $c \notin \{p_1, ..., p_{m+2}\}$ but c is on ℓ in $A \cup \{c\}$; we have that c is 0-primitive over A, and any pair of points in $\ell \cap A$ constitutes a base for c/A.

The following preparatory results allow us to characterize primitive extensions and eventually prove amalgamation for $(\mathbf{K}_\mu, \leqslant)$ (cf. Conclusion 5.15).

{oddpointout}

Proposition 4.5. *Let $B \in \boldsymbol{K}_0$ and $b \in B$ such that b does not occur in any R-tuple from B, then $\delta(B) = \delta(B - \{b\}) + 1$.*

Proof. As b is on no line based in $B - \{b\}$ this follows from Definitions 3.3 and 3.4. ■

Using the above proposition, we can see:

{uniqueness_base}

Proposition 4.6. *Let $A, B \in \boldsymbol{K}_0$ with $A \cap B = \emptyset$, $AB \in \boldsymbol{K}_0$ and $B \leqslant AB$. Then:*

(1) if there exists $b \in B$ such that b does not occur in any R-tuple from AB, and B' denotes $B - \{b\}$, then $\delta(A/B) = \delta(A/B')$.
(2) if the 0-primitive pair A/B is good (cf. Definition 4.1(2)), then for every $b \in B$ we have that b occurs in an R-tuple from AB.

Proof. It suffices to prove (1), and (1) is clear by applying Proposition 4.5 to AB as follows:

$$\delta(A/B) = \delta(AB) - \delta(B) = (\delta(AB') + 1) - (\delta(B') + 1) = \delta(AB') - \delta(B').$$

■

We use the following technical lemma to prove Lemma 4.8, which characterizes good pairs. Lemma 4.8 will be of crucial importance in the next section.

{techforZiegler}

Lemma 4.7. *Suppose C is a primitive extension of B such that $|\hat{C}| \geqslant 2$, then every non-trivial line ℓ with $\ell \cap \hat{C} \neq \emptyset$ intersects B in at most one point. Furthermore, if C is 0-primitive, then any point in $\hat{C}$ lies on two lines based in $\hat{C}$.*

Proof. Let ℓ be a line that intersects $\hat{C}$. Then ℓ is not based in B since, if so, for any $c \in \ell \cap \hat{C}$, Bc would contradict the primitivity of C. But then, if C is 0-primitive, any $c \in \hat{C}$ must lie on a line based in $\hat{C}$, as otherwise, letting $C' = \hat{C} - \{c\}$, Proposition 4.5 implies $\delta(C'/B) = \delta(C/B) - 1 = 0 - 1 < 0$, contradicting $B \leqslant C$. But, in fact, $c \in \hat{C}$ must lie on two lines based in $\hat{C}$. If it is based on only one, deleting c decrements both the number of points and the sum of the nullities of lines based in $\hat{C}$ by 1. So $\delta(C'/B) = 0$, contradicting that C is 0-primitive over B. ■

The next lemma is the *fundamental* tool for our analysis of primitive extensions.

{primchar}

Lemma 4.8. *Let $B \leqslant C \in \boldsymbol{K}_0$ be a primitive extension. Then there are two cases:*
(1) $\delta(C/B) = 1$ and $C = B \cup \{c\}$;
(2) $\delta(C/B) = 0$.
- *(2.1) There is $c \in \hat{C}$ incident with a line ℓ based in B if and only if $|\hat{C}| = 1$. In that case, any $B' \subseteq B$ with $B' \subseteq \ell$ and such that $|B'| = 2$ yields a good pair (B', c). Furthermore, c is in the relation R with an element $b \in B$ if and only if b is on the unique line based in B'.*
- *(2.2) If $|\hat{C}| \geqslant 2$ then there is a unique base B_0 in B for C. Moreover, suppose $b \in B$ and $c \in \hat{C}$. If b and c lie on a nontrivial line, then $b \in B_0$. And every $b \in B_0$ lies on such a line, which must be based in $\hat{C}$.*

Proof. We follow the case distinction of the statement of the lemma:
Case 1. Suppose $\delta(C/B) > 0$ and there are distinct elements in $\hat{C}$ that are not on lines based in B, then any one of them gives a proper intermediate strong extension of B that is strong in C. Thus C must add only one element to B yielding Case 1.
Case 2. Suppose $\delta(C/B) = 0$.
Case 2.1. Suppose there is an element $c \in \hat{C}$ which is on a line with two points in B, say b_1, b_2, and $|\hat{C}| \geqslant 2$. Then clearly Bc is a primitive extension of B and $Bc \lneq BC$. Thus, $\hat{C}$ must be $\{c\}$. Furthermore, $(\{b_1, b_2\}, c)$ is a good pair. So C is based on $\{b_1, b_2\}$ and for any $b \in B$, b is R-related to c if and if $R(b_1, b_2, b)$; otherwise c would be on two lines based in B (contradicting $B \leqslant C$). Conversely, if $|\hat{C}| = 1$ then c must be on a line based in B since $\delta(C/B) = 0$.

Case 2.2 $|\hat{C}| \geqslant 2$ and $\delta(C/B) = 0$.
By Lemma 4.7, each line $\ell \in L(\hat{C})$ intersects B in at most one point b_ℓ. If there is no such b_ℓ, then there is no R-relation between $\hat{C}$ and B, so by Proposition 4.6(2), $B = \emptyset$ and C is based on $\emptyset$. As argued in Remark 4.2, that base must be unique. If there is, let B_0 be the collection of all the b_ℓ, $\ell \in L(\hat{C})$. By Lemma 4.6.(1), $\delta(C/B_0) = \delta(C/B)$, and so (B_0, C) is a good pair. Further B_0 is the unique base for C as these are the only elements of B on lines that intersect $\hat{C}$. ■

Omer Mermelstein provided us with an example showing there are infinitely many primitives based on a single three element set. But the study of (a, b) cycles in [Bal19] led to stronger and simpler examples over smaller base sets. Recall that any linear space with 3-point lines is an example of Steiner triple system (i.e. in Definition 2.6 we have $K = \{3\}$). The following definition will be used to prove Lemma 4.11.

{abgraph}

Definition 4.9 ([CW12]). *We define the notion of (a, b)-cycle graphs in Steiner triple systems. Fix any two points a, b of a Steiner triple system $\mathcal{S} = (P, L)$. The cycle graph $G(a, b)$ has vertex set $P - \{a, b, c\}$ where (a, b, c) is the unique block containing the points a and b. There is an edge coloured a (resp. b) joining x to y if and only if axy is a block (resp. bxy is a block) and the colors alternate.*

{defcycle}

Definition 4.10. *Fix any two points a, b of a Steiner m-system $\mathcal{S} = (P, L)$. We can build an (a, b)-cycle, C_k, $c_1, c_2, \ldots c_{4k}$ of length $4k$ by demanding $R(a, c_{2n+1}, c_{2n+2})$ for $0 \leqslant n \leqslant 2k$, $R(b, c_{2n+2}, c_{2n+3})$ for $0 \leqslant n < 2k$, and $R(b, c_1, c_{4k})$.*

In the Steiner triple system case a triple a, b, c_1 with c_1 not on (a, b) determines a unique cycle as described in Definition 4.10. For m-Steiner systems with $m > 3$, we can choose such cycles but not uniquely. Note that the lines determined by the pairs of points c_n, c_{n+1} in Definition 4.10 must be distinct.

{2prim}

Lemma 4.11. *There are infinitely many mutually non-embeddable primitives in $\boldsymbol{K}_0$ over a two-element set. In fact, there are infinitely many mutually non-embeddable primitives in $\boldsymbol{K}_0$ over the empty set and similarly over a 1-element set.*

Proof. Over any a, b for each k build an (a, b)-cycle C_k , as in Definition 4.10. C_k has $4k$ points and $(\{a, b\} \cup C_k) \in \boldsymbol{K}_0$ has $4k$ 3-element lines. So $\delta(\{a, b\} \cup C_k)) = 2 = \delta(\{a, b\})$. Primitivity easily follows since if the cycle is broken, the δ-rank goes up. So $(\{a, b\}, C_k))$ is a good pair whose isomorphism type we denote by $\boldsymbol{\gamma}_k$.
To get primitives over $\emptyset$, let c be on ab and add the relations $R(c, c_1, c_{2k+1})$ and $R(c, c_{k+1}, c_{3k+1})$. Now the entire structure D_k has $4k + 3$ points and $4k + 3$ lines and can easily be seen to be 0-primitive over the empty set. (Note that for $k = 1$, this is another avatar of the Fano plane.)
Now remove one of the last two instances of R and the result is primitive over a or b. ■

5. The Class $\boldsymbol{K}_\mu$

{Kmusec}

We now introduce the new classes of structures needed to obtain strong minimality. Recall that we have two classes: (i) $\boldsymbol{K}_0$ is a class of finite structures; (ii) $\hat{\boldsymbol{K}}_0$ is the universal class generated by $\boldsymbol{K}_0$. The new class $\boldsymbol{K}_\mu \subseteq \boldsymbol{K}_0$ adds additional restrictions so that the generic model for $\boldsymbol{K}_\mu$ is a strongly minimal linear space, and, in fact, a Steiner k-system for some k. Using Definition 5.7, we axiomatize the subclass $\boldsymbol{K}_d^\mu$ of $\hat{\boldsymbol{K}}_\mu$ (the universal class generated by $\boldsymbol{K}_\mu$) of those models that are elementarily equivalent to the generic for $\boldsymbol{K}_\mu$. We extend Table 3 to a Table 4 including the new classes defined in this section.

Notation	References
$\mathbf{K}^*$	Definition 2.1
$\mathbf{K}_0^*$	Definitions 2.1 and 3.1(2)
$\mathbf{K}_0$	Definition 3.7
$\hat{\boldsymbol{K}}_0$	Definitions 3.7 and 3.1(1)
$\boldsymbol{K}_\mu$	Definition 5.2(3)
$\hat{\boldsymbol{K}}_\mu$	Definition 5.2(4)
$\boldsymbol{K}_d^\mu$	Definition 5.4(4)

TABLE 4. The classes of structures relevant to our construction. table4

The following notation singles out the effect of the fact that our rank depends on line length rather than the number of occurrences of a relation.

{linelength}

Notation 5.1 (Line length)**.** *We write $\boldsymbol{\alpha}$ for the isomorphism type of the good pair $(\{b_1, b_2\}, a)$ with $R(b_1, b_2, a)$ (cf. Lemma 4.8(2.1)).*

{Kmu}

Definition 5.2. *Recall the characterization of primitive extensions from Lemma 4.8.* {itemKmu}

(1) Let $\mathcal{U}$ be the collection of functions μ assigning to every isomorphism type β of a good pair (B, C) in $\boldsymbol{K}_0$ (we write $\mu(B, C)$ instead of $\mu((B, C))$):
   (i) an integer $\mu(\beta) = \mu(B, C) \geqslant \delta(B)$, if $|C - B| \geqslant 2$;
   (ii) an integer $\mu(\beta) \geqslant 1$, if $\beta = \boldsymbol{\alpha}$ (cf. Notation 5.1).

(2) For any good pair (B, C) with $B \subseteq M$ and $M \in \hat{\boldsymbol{K}}_0$, $\chi_M(B, C)$ denotes the number of disjoint copies of C over B in M. Of course, $\chi_M(B, C)$ may be 0. {Kmuitem}

(3) Let $\boldsymbol{K}_\mu$ be the class of structures M in $\boldsymbol{K}_0$ such that if (B, C) is a good pair, then $\chi_M(B, C) \leqslant \mu(B, C)$. {Kmuhatitem}

(4) $\hat{\boldsymbol{K}}_\mu$ is the universal class generated by $\boldsymbol{K}_\mu$ (cf. Notation 3.1(1)).

In [Bal19], we change the set $\mathcal{U}$ in various ways (and explore the combinatorial consequences of this change in the resulting generic model). In this paper, we assume $\mu \in \mathcal{U}$ unless specified otherwise.

The value of $\mu(\boldsymbol{\alpha})$ is a fundamental invariant in determining the possible complete theories of generic structures; in particular we will see that it determines the length of every line in the generic and thus in any model elementary equivalent to it.

{primline}

Remark 5.3. We analyze the structure of extensions governed by good pairs with isomorphism type $\boldsymbol{\alpha}$ from Notation 5.1. Suppose $\{b_1, b_2, a\} \subseteq F \in \boldsymbol{K}_\mu$ with $R(b_1, b_2, a)$. The 0-primitive extensions C of $B = \{b_1, b_2\}$ with $|\hat{C}| = 1$ are exactly the points on the line ℓ through b_1, b_2. Any pair of points e_1, e_2 from F that are on ℓ form a base witnessed by $(\{e_1, e_2\}, a)$ with $R(e_1, e_2, a) \wedge R(b_1, b_2, a)$.

Most arguments for amalgamation in Hrushovski constructions (e.g. [Bal88, Hol99, Hru93, Zie13]) depend on a careful analysis of the location of the *unique* base of a good pair. Here, when $|\hat{C}| = 1$, the uniqueness disappears and one must focus on the line rather than a particular base for it.

There are two general approaches to showing existence of complete strongly minimal theories by the Hrushovki construction. One divides the construction into two pieces, free and collapsed [Goo89, Zie13]. The final theory is taken as the sentences true in the generic model. The second, as the original [Hru93], provides a direct construction of the strongly minimal set. We choose here to follow the Holland's version of this approach. She insightfully emphasised axiomatizing the theory of the class $\boldsymbol{K}_d^\mu$ of d-closed structures [Hol99],which we now define, by clearly identifiable π_2-sentences. This established the model completeness which was left open in [Hru93]. In fact, we axiomatize the theory T_μ of the class $\boldsymbol{K}_d^\mu$, prove it is strongly minimal, and then observe that the generic satisfies T_μ.

{defd-cl}

Definition 5.4. *Fix the class $(\boldsymbol{K}_0, \leqslant)$ of τ-structures as defined in Definition 3.7.*

(1) For $A \in \hat{\boldsymbol{K}}_0$, $X \subseteq_\omega A$ and $a \in A$, we let:

$$d_A(X) = \min\{\delta(Y) : X \subseteq Y \subseteq_\omega A\},$$

and

$$d_A(a/X) = d_A(aX) - d_A(X).$$

{item_defcl}

(2) For $M \in \hat{\boldsymbol{K}}_\mu$, and $X \subseteq_\omega M$:

$$\mathrm{cl}^{\mathrm{d}}_{\mathrm{M}}(X) = \{a \in M : d_M(aX) = d_M(X)\}.$$

For infinite X, $a \in \mathrm{cl}^{\mathrm{d}}_{\mathrm{M}}(X)$ if $a \in \mathrm{cl}^{\mathrm{d}}_{\mathrm{M}}(X_0)$ for some $X_0 \subseteq_\omega X$.

(3) For $M \in \hat{\boldsymbol{K}}_\mu$ and $X \subseteq M$, X is d-closed in M if $d(a/X) = 0$ implies $a \in X$ (equivalently, for all $Y \subseteq_\omega M - X$, $d(Y/X) > 0$).

{mu-d}

(4) Let $\boldsymbol{K}_d^\mu$ consist of those $M \in \hat{\boldsymbol{K}}_\mu$ such that $M \leqslant N$ and $N \in \hat{\boldsymbol{K}}_\mu$ imply M is d-closed in N.

The switch from δ to d is designed to ensure that $X \subseteq Y$ implies $d(X) \leqslant d(Y)$; the submodularity of d is verified as in e.g. [BS96, Hol99, Zie13], and so the function d is truly a dimension function, thus inducing a matroid structure.

{geomexists}

Fact 5.5. *The d-closure operator $\mathrm{cl}^{\mathrm{d}}_{\mathrm{M}}$ (cf. Definition 5.4(2)) induces a combinatorial pregeometry on any $M \in \hat{\boldsymbol{K}}_\mu$.*

We use good pairs to build our axiomatization, Σ_μ, of the theory of the class $\boldsymbol{K}_d^\mu$. We write Σ_μ as the union of four sets of first-order τ-sentences: Σ^0_μ, Σ^1_μ, Σ^2_μ and Σ^3_μ. Before listing them, we explain the origin of the third group: Σ^2_μ. We would like to just assert the collection of *universal-existential* sentences: for all good pairs (B, C) with $B \subseteq M$, $\chi_M(B, C) = \mu(B, C)$. Unfortunately, some good pairs may conflict with each others, and so, as far as we know, the equality may fail for some good pairs when the base B is not strong in the model. Basically, this could happen because if (P, G) and (Q, F) are good pairs with QF contained in PG then realizing (P, G) implies that (Q, F) is automatically realized. In particular, note that the C of the good pair (B, C) of Example 5.8 contains a new good pair (B', C').

The distinguishing property of models $M \in \boldsymbol{K}_d^\mu$ is that since every 0-primitive extension over a finite *strong* subset of M can be embedded in M, by Lemma 5.16, no proper 0-primitive extension of M is in $\hat{\boldsymbol{K}}_\mu$. In fact, this property characterizes the models that are elementarily equivalent to the generic.

Crucially, Holland[10] expresses this failure by a clearly motivated π_2-sentence, which we expound in Remark 5.9. A salient point about the generic for $\boldsymbol{K}_\mu$, denoted $\mathcal{G}_\mu$ (Notation 5.18), is that $\mathcal{G}_\mu \in \boldsymbol{K}_d^\mu$. This fact is not used directly in the proof of strong minimality of T_μ; we will observe it in Proposition 5.20.

{notation}

Notation 5.6. *Note that the diagram of a finite structure A is expressed by a quantifier-free first-order formula $\alpha_A(\overline{x})$ which is satisfied by a sequence $\overline{a}$ enumerating A.*

One reason for the difficulty in the axiomatization is that the function μ is defined on arbitrary substructures, not strong substructures. *Restricting to strong substructure would inhibit if not prevent the π_2-axiomatization as the strong substructure relation ($A \leqslant M$) is only type-definable.* Thus, in Lemma 5.10, we cannot assume D is strong in both E and F. In the following definition we rely on the terminology introduced in Definitions 4.1 and 5.2.

{ax}

Definition 5.7. *Σ_μ is the union of the following four sets of sentences:*

(1) Σ_μ^0 is the collection of universal sentences axiomatizing $\boldsymbol{K}_0$ as in Definition 3.7.

(2) Σ_μ^1 is the collection of universal *sentences that assert:*

$$B \subseteq M \quad \Rightarrow \quad \chi_M(B,C) \leqslant \mu(B,C).$$

{item_sigma2}

(3) Σ_μ^2 is the collection of universal-existential *sentences $\psi_{B,C}$, depending on the good pair (B,C), such that for every occurrence of B if $M \models \psi_{B,C}$ then for some good pair (A,D) with $AD \subseteq BC$, any structure N containing MC satisfies $\chi_N(A,D) > \mu(A,D)$ and so violates Σ_μ^1. See Lemma 5.21 for the explicit formulation of these sentences.*

(4) Σ_μ^3 is the collection of existential sentences asserting that every model is infinite.

The argument in Lemma 5.10 that provides both the axiomatization of $\boldsymbol{K}_d^\mu$ and the amalgamation for $(\boldsymbol{K}_\mu, \leqslant)$ differs from a mere amalgamation argument in one significant way: $D \subseteq F$ but $D \leqslant F$ is not assumed (on the other hand, $D \leqslant E$ *is* assumed). We require several technical lemmas to address the difficulties arising from this fact. On the other hand, we will see that models that satisfy Σ_μ are in $\boldsymbol{K}_d^\mu$. Our argument shows that if there is a model F that satisfies Σ_μ^i for $i < 2$, then we can find sentences to prevent extensions in which F is not d-closed. The following example shows the necessity for the complications in proving Lemma 5.10: new primitives can occur in many ways.

{iclosedmatters}

Example 5.8. Construct the isomorphism type $\boldsymbol{\beta}$ of a good pair (B,C) defined as follows. Let B be two points d_1, d_2 and C consists of six points c_i for $i = 1, \ldots 6$. Let the non-trivial lines be $\{d_1, c_1, c_2, c_3\}$,$\{d_2, c_4, c_5, c_3\}$, $\{c_4, c_1, c_6\}$ and $\{c_5, c_2, c_6\}$. So C has 6 points and 4 lines each of nullity 1 so rank 2. And BC has 8 points and 4 lines, 2 of nullity 1 and 2 of nullity 2 so BC also has rank 2. Check primitivity by inspection.

Now turn this example on its head. Consider the following example of the setting of Lemma 5.10. Set $\mu(\boldsymbol{\alpha}) = 4$ and $\mu(\boldsymbol{\beta}) = 2$ Let $D = \{c_1, c_2\}$, $F = D \cup \{c_3, c_4, c_5, c_6, d_2\}$ and $E = D \cup \{d_1\}$. So $F = \mathrm{icl(D)}$. (D,E) is a good pair. By

[10]Holland provides a common framework for both *ab initio* constructions and fusions. The generality introduces considerations that are not relevant here, and our new predimension and the restriction to linear spaces introduce complications to her argument. Thus, for the convenience of the reader, we rephrased the argument for our situation.

adding the single point d_1 we have gotten a new realization (B', C') of the *isomorphism type* $\mu(\boldsymbol{\beta}$ of the good pair (B, C), which is not contained in either D or E, but in $F \cup E$. This example does not violate Lemma 5.10 as $\mu(\boldsymbol{\alpha}) = 2$ (and has to be since there are 4-element lines in F).

{Hollsent} **Remark 5.9.** Example iclosedmatters shows that good pairs can conflict so we don't know in general that a model M of T_μ will satisfy $\chi_M(B, C) = \mu(B, C)$ for all good pairs (B, C) that appear in M. We first prove in Lemma 5.10 that each good pair (B, C) can only conflict with finitely many pairs (B', C') and that that can happen only if one pair is included in the other. Following [Hol99], to guarantee that $M \in \boldsymbol{K}_d^\mu$, we assert by the formula $\psi_{B,C}$ (cf. Definition 5.7(3)) that each conflicting pair (A, D) is 'almost realized' in M so that adding points from C contradicts Σ_μ^1.

In Lemma 5.10 the fact that (D, E) is a good pair implies that $D \leqslant E$, and so we can use Lemma 3.14 and thus consider $G = E \oplus_D F$.

{isoqe} **Lemma 5.10.** *Let $F, E \models \Sigma_\mu^i$, for $i < 2$, and $D \subseteq F$, and suppose that (D, E) is a good pair (and so in particular $D \leqslant E$). Now, if $G = E \oplus_D F$ and for some good pair $(B, C) \subseteq G$ we have $\chi_G(B, C) > \mu(B, C)$, then:*

(A) if $|C| = 1$, $C = \{c\}$ and c is on a line based on some $B' \subseteq D$;
(B) if $|C| \geqslant 2$ then $B \subseteq E$ and there exists C' with $BC' \backsimeq BC$, with $C' \subseteq (E - D)$. Further, if $D \leqslant F$, there is a copy C'' of C over B with $C'' = (E - D)$, and $B \subseteq D$.

Proof. Since $G = E \oplus_D F$ we can use the notation and results of 3.12-3.14. Note that F, D, E are in $\hat{\boldsymbol{K}}_\mu$ by the definition of the axioms Σ_μ. Furthermore, $D \leqslant E$ and $E \in \boldsymbol{K}_\mu$, by the definition of good pair. Let $\mathcal{C}$ be a set of $\mu(B, C) + 1$ disjoint copies of C over B in G, and list $\mathcal{C}$ as $(C_1, ..., C_m)$, for $m = \mu(B, C) + 1$.

Case A. $|C| = 1$.
Then (B, C) witnesses the isomorphism type $\boldsymbol{\alpha}$ from Definition 5.1. So, there must be a line ℓ of size $\mu(B, C) + 3$ in G. Since E and F satisfy Σ_μ^1, there must be $d \in F - D$ and $c \in E - D$ that lie on ℓ. By Definition 3.12(2) ℓ must contain two points (say, comprising B') in D that are connected to $c \in E - D$. Since $\{c\}$ is then primitive over D, $E - D$ must be $\{c\}$. We finish the first claim. Note $\chi_F(B', C) = \mu(B, C)$ as ℓ has $\mu(\boldsymbol{\alpha}) + 2$ points in F.

Case B. $|C| \geqslant 2$.

Case B.1. Suppose $B \subseteq F$.
Since $\chi_F(B, C) \leqslant \mu(B, C)$, there must be a $C_i \in \mathcal{C}$ that intersects $\hat{G} = G - F = E - D$. So, since $F \leqslant G$ and C/B is primitive, $C_i \subseteq \hat{G} = G - F = E - D$. But, since E is primitive over D, FE is primitive over F, so $C_i = E - D$.

By Proposition 4.6(2), each element of B is R-related to some element of C_i. So, if $B \cap (F - D) \neq \emptyset$, there is a line ℓ from $F - D$ to $C_i = G - F$. Let $c \in C_i$ be on ℓ. By Definition 3.12, ℓ must be based in D and since $C_i = \hat{G}$ is primitive over D $C_i = \{c\}$ and so $|C_i| = 1$, contradicting the choice of case.

Under the assumption that $B \subseteq F$, we are then reduced to the case that $B \subseteq D$. By Case 2.2 of Lemma 4.8, B is the only subset of F on which C_i is based. Hence, as $BC_i \subseteq E$, we finish Case B.1 without using the supplemental hypothesis for the 'further' in Case (B).

Case B.2. $B \not\subseteq F$

We proceed along the lines of the argument for Lemma 35 of [Hol99][11], starting with her Claim 3.

{kh3}

Claim 5.11. *If $C_i \in \mathcal{C}$ and $\delta(C_i \cap E/(C_i \cap D)B) \geqslant 0$, then C_i does not split over F, i.e. $C_i \subseteq F$ or $C_i \subseteq (E - D)$.*

Proof. If not, we have:

$$0 \leqslant \delta(C_i \cap E/(C_i \cap D)B) = \delta(C_i \cap E/(C_i \cap F)B) = \delta(C_i/(C_i \cap F)B),$$

where the inequality restates the hyypothesis, the first equality holds since $F \leqslant G$, and the second holds since for any $X \subsetneq Y$, $\delta(Y/X) = \delta((Y - X)/X)$ and $C_i \cap E = C_i - (C_i \cap F)B$. Basic properties of δ yield:

$$\delta(C_i/(C_i \cap F)B) = \delta(C_i/B) - \delta(C_i \cap F/B).$$

Since $\delta(C_i/B) = 0$, we conclude that $\delta(C_i \cap F/B) \leqslant 0$. Since (B, C_i) is a good pair entails every non-empty proper subset of C_i has positive rank over B, this inequality implies that C_i does not split over F yielding the claim. ∎

{noneinF}

Claim 5.12. *No $C_j \in \mathcal{C}$ satisfies $C_j \subseteq F$.*

Proof. Suppose there is such a C_j. Since we are in Case B.2, there is a $b_0 \in \hat{E} \cap B$. By Lemma 4.7, as C_j is primitive over B, b_0 lies on a line ℓ_j based in $C_j \subseteq F$. But then as in Case B.1, ℓ_j is based in D and $E - D$ must be a singleton, $\{b_0\}$. Thus, all of the $m = \mu(B, C) + 1$ copies C_i of C over B in G are contained in F.

Further, b_0 is on a line ℓ_i based on C_i for each i. If any two of these lines are distinct then $b_0 \in \mathrm{icl(F)}$; this contradicts $F \leqslant G$. So, we now know there is a single line ℓ with b_0 on ℓ which intersects each C_i in at least two points. Since the line ℓ intersects both C_i and C_{i+1}, $\delta(C_{i+1}/BC_1 \ldots C_i) < 0$. I.e, the sequence $\delta(BC_1 \ldots C_i)$ is strictly decreasing as i increases. Since $\delta(B) = \delta(C_1B)$ and $m \geqslant \delta(B)$, this implies $\delta(BC_1 \ldots C_m) < 0$. This contradiction yields the result. ∎

We now translate Claim 2 (in support of Lemma 35) of [Hol99] to our notation; the proof is the same. We include the short computation.

{kh2}

Claim 5.13. *At most $\delta(B \cap E/D)$ copies of the C_i satisfy:*

$$\delta(C_i \cap E/(C_i \cap D)B) < 0.$$

Proof. Write C_i^* for $C_i \cap E$. Suppose $\delta(C_i^*/(C_i \cap D)B) < 0$ for $1 \leqslant i \leqslant m$. Since $D \leqslant E$, we have:

$$\begin{aligned} 0 &\leqslant \delta(C_1^*, \ldots C_m^*(B \cap E)/B \cap D) \leqslant \Sigma_i \delta(C_i^*/(B \cap E)D) + \delta(B \cap E/D) \\ &= \Sigma_i \delta(C_i^*/BD) + \delta(B \cap E/D) \leqslant -m + \delta(B \cap E/D). \end{aligned}$$

∎

Recall that $\mu(B, C) \geqslant \delta(B)$. By the freeness of the amalgamation $\delta(B/B \cap F) = \delta(B \cap E/B \cap D)$. So, by Definition 5.2, and since $|C| \geqslant 2$, we have:

$$(*) \qquad \delta(B \cap E/B \cap D) \leqslant \delta(B) \leqslant \mu(B, C).$$

Claims 5.11 and 5.12 imply for every $1 \leqslant i \leqslant m$, one of the following happens:

[11] We follow the spirit of Holland's proof. We are able to give a more conceptual proof of Claim 1 of her argument because of the special role of the good pair $\boldsymbol{\alpha}$. For convenience, we wrote out the computations for the other two claims from [Hol99].

(a) $\delta(C_i \cap E/(C_i \cap D)B) < 0$;
(b) $C_i \subseteq (E - D)$.

By Claim 5.13 and $(*)$, there are at most $\mu(B, C)$ copies of the C_i satisfying (a). So there is a $C_j \subseteq (E - D)$. We show $B \subseteq E$. If not, there is a $b_1 \in B \cap (F - E)$ and since $C_j \subseteq (E - D)$ a line from b_1 to some $c \in C_j$. Thus c is on a line based on D and so $C_j = E - D = \{c\}$. This contradicts $|C| \geqslant 2$ so $B \subseteq E$, giving the first assertion of Item (B).

Finally, we establish the 'further' in assertion (B). There must be a $C' \in \mathcal{C}$ that intersects $F - D$, since $E \in \boldsymbol{K}_\mu$. But C' cannot split over E since, with, the additional assumption, $E \leqslant G$; so $C' \subseteq (F - D)$. But such a C' is based on a unique $B' \subseteq D$ since $D \leqslant F$. So $B = B' \subseteq D$. But then C_j (from the previous paragraph) is primitive over D and based on $B \subseteq D$, and so $C_j = E - D$. Hence, C_j is the required C''.

This concludes the proof of Lemma 5.10. ∎

After one more lemma, we obtain amalgamation (with identifications, i.e. *not* disjoint) for $(\boldsymbol{K}_\mu, \leqslant)$.

{getmap}

Lemma 5.14. *Suppose A and A' are primitive over Y with $\delta(A/Y) = \delta(A'/Y) = 0$ and both are based on $B \subseteq Y$ with isomorphic good pairs $(B, \hat{A})$ and $(B, \hat{A}')$, where $\hat{A} = A - Y$ and $\hat{A}' = A' - Y$. Then the map fixing Y and taking A to A' is an isomorphism.*

Proof. There are two cases depending on the cardinality of $\hat{A}$.

Case 1. $|\hat{A}| = 1$.

As in case 1.1 of Lemma 4.8 let ℓ be a line which is based in Y and suppose $\hat{A} = \{a\}, \hat{A}' = \{b\}$ are each on ℓ but neither is in Y. Then, since both a and b are R-related only to the points on ℓ the map fixing Y and taking a to b is an isomorphism.

Case 2. $|\hat{A}| \geqslant 2$.

Applying Lemma 4.8(2.2), there is a unique base B (the B_0 of the lemma) and there is a bijection f between $|\hat{A}|$ and $|\hat{A}'|$ such that for each $b \in B$, $R(c_1, c_2, b)$ if and only $R(f(c_1), f(c_2), b)$. The union of that map with the identity on Y is as required. ∎

We now show that any element of $\hat{\boldsymbol{K}}_\mu$ (not just $\boldsymbol{K}_\mu$) can be amalgamated (possibly with identifications) over a (necessarily finite) *strong* substructure D with a strong extension of D to a member E of $\boldsymbol{K}_\mu$. The extension is in

{conclude}

Conclusion 5.15. *If $D \leqslant F \in \hat{\boldsymbol{K}}_\mu$ and $D \leqslant E \in \boldsymbol{K}_\mu$ then there is $G \in \hat{\boldsymbol{K}}_\mu$ that embeds (possibly with identifications) both F and E over D. Moreover, if $F \in \boldsymbol{K}^\mu_d$, then $F = G$. In particular, $(\boldsymbol{K}_\mu, \leqslant)$ has the amalgamation property, and there is a generic structure $\mathcal{G}_\mu \in \hat{\mathbf{K}}_\mu$ for $(\boldsymbol{K}_\mu, \leqslant)$.*

Proof. Let D, E, F satisfy the hypotheses. Clearly, we can assume that $D \leqslant E$ is a primitive extension. If $\delta(E/D) = k > 0$, Lemma 4.8 implies $k = 1$ and $E - D = \{a\}$. Now the disjoint amalgamation $E \oplus_D F$ is in $\hat{\boldsymbol{K}}_\mu$ since a is not R-related to any other element. So, we are reduced to 0-extensions and can refine the induction to assume (D, E) is a good pair. We have an amalgam $G \in \hat{\boldsymbol{K}}_0$ such that $G = E \oplus_D F$, $F \leqslant G$, and $E \leqslant G$. If $G \in \boldsymbol{K}_\mu$, we finish. If not, there is an isomorphism type β of a good pair (B, C) and $(C_i : i < m)$ with $(B, C_i) \subseteq G$ realizing β and such that

$m > \mu(B, C)$. We now make a case distinction and show that in both cases we can embed E into F over D.

Case 1. $|C| = 1$.

By Lemma 5.10(A) and by primitivity of $D \leqslant E$, we have that $|E - D| = 1$. But $E \in \boldsymbol{K}_\mu$, and so $\chi_D(\boldsymbol{\alpha}) < \mu(\boldsymbol{\alpha})$, from which it follows that the element of $E - D$ can be embedded in $F - D$ over B.

Case 2. $|C| > 1$.

The 'further' clause of Lemma 5.10(B) shows that there must be a copy of C equal to $E - D$. Thus, using Lemma 5.14 and the argument in the last paragraph of the proof of Lemma 5.10 (there is a copy of C in $F - D$), we can conclude that we can embed E into F over D.

For the 'moreover', note that $M \in \boldsymbol{K}_d^\mu$ implies that every proper extension N of M with $N \in \hat{\boldsymbol{K}}_d^\mu$ satisfies $d(N/M) > 0$. ■

{getmax}

Corollary 5.16. *If $M \in \boldsymbol{K}_d^\mu$ and $B \leqslant M$, then for any good pair (B, C) with $C \cap M = B$, we have:*

$$\chi_M(B, C) = \mu(B, C).$$

Proof. By Conclusion 5.15, since $B \leqslant M$, there is an amalgamation in $\hat{\boldsymbol{K}}_\mu$ of C and M over D. But, M and MC cannot be freely amalgamated over B. As, in a putative amalgam N, $d_N(C/M) = 0$. Whence since M is d-closed, $C \subseteq M$, contradicting free amalgamation. By the 'further' of Lemma 5.10(B).2, the violation of Σ_1 is given by the new copy of the pair (B, C), and so $\chi_M(B, C) = \mu(B, C)$. ■

Question 5.17. *Is $D \leqslant M$ essential for the conclusion of Lemma 5.16?* The necessity of this hypothesis example in the fusion case appears in [BH00, §4]. The proof relies on that assumption both in using the 'further' of Lemma 5.10 and Conclusion 5.15. Looking carefully at the proof of Lemma 5.10 reveals that if there is a counterexample (D, E), the failure is witnessed by a (B, C) with $m = \mu(B, C) + 1$ such that $B \subseteq E$, $B \not\subseteq F$, no $C_i \subseteq F$ and some $C_i \subseteq E - D$. Thus, $|E - D| \geqslant |C| + m$. It is unclear whether B might be contained in $D - E$. Thus, we need something far different from Example 5.8 where we showed new isomorphism types of good pairs could appear in an amalgam but $|E - D| = 1$.

{gennot}

Notation 5.18. *Let $\mathcal{G}_\mu$ denote the generic for $(\boldsymbol{K}_\mu \leqslant)$ (cf. Conclusion 5.15).*

Notice that it follows from Corollary 5.15 that every member of $\boldsymbol{K}_\mu$ is strongly embeddable in $\mathcal{G}_\mu$.

{def_rich}

Definition 5.19. *Let $(\mathbf{K}_0, \leqslant)$ be as in the context of Fact 3.6. The structure M is rich for the class $(\hat{\mathbf{K}}_0, \leqslant)$ (or $(\hat{\mathbf{K}}_0, \leqslant)$-rich) if for any finite $A, B \in \mathbf{K}_0$ with $A \leqslant M$ and $A \leqslant B$ there is a strong embedding of B into M over A.*

Clearly, a generic is rich. Even more, since the definition of $\boldsymbol{K}_d^\mu$ requires the embedding only of finite extensions with dimension 0, we have:

{dcl}

Proposition 5.20. *Every rich model, and so in particular $\mathcal{G}_\mu$, is in $\boldsymbol{K}_d^\mu$.*

Proof. We show that every $(\boldsymbol{K}_\mu, \leqslant)$-rich model M is in $\boldsymbol{K}_d^\mu$. Suppose for contradiction that there is an $N \in \hat{\boldsymbol{K}}_\mu$ with $M \leqslant N$ and there is a $C \subseteq (N - M)$ such that C is 0-primitive over M. By Lemma 4.8, C is based on some finite $B \subseteq M$. Since $M \leqslant N$, C is also primitive over $B_0 = \mathrm{icl}_M(B)$. Since M is rich there is a copy $C_1 \subseteq M$ of C over B_0. Now let $B_1 = \mathrm{icl}_M(C_1)$. Applying richness again we can choose another embedding C_2 of C into M over B_1. Continuing in this fashion, after less than $\mu(B,C)+1$ steps we have contradicted $M \in \hat{\boldsymbol{K}}_\mu$. ∎

Note that by Lemma 2.12 we can impose a rank 3 matroid structure on the generic $\mathcal{G}_\mu$. Alternatively, the arguments from Section 3 up until here would go through just as well if we worked with $\check{\boldsymbol{K}}_0$ (Definition 2.11) rather than $\boldsymbol{K}_0$, and so we could have insisted on working with rank 3 matroids throughout. By either approach, we are justified in calling the final object a plane in the sense of Definition 2.11.

Now we explain the interaction between the axioms Σ^1_μ and Σ^2_μ. No extension of a model of Σ^2_μ by a good pair is in $\hat{\boldsymbol{K}}_\mu$. This will yield the axiomatization of the theory of the d-closed structures and thus of the generic (by Proposition 5.20).

{proveax} **Lemma 5.21.** *The family of first-order sentences Σ_μ (Definition 5.7) satisfies the following: if $M \in \hat{\boldsymbol{K}}_\mu$ and $M \models \Sigma_\mu$ (and so in particular Σ^2_μ), then no extension N of M by a good pair satisfies Σ^1_μ.*

Proof. We use the notation of Lemma 5.10. For each duo of good pairs (D,E) and (B,C) with $BC \subseteq DE$ and with $\chi_N(B,C) > \mu(B,C)$, define the formula $\varphi_{(D,E),(B,C)}$ as follows. Suppose $C_1, \dots C_r$ (where $r = \mu(B,C)+1$) are disjoint copies of C over B and let $\overline{s}$ enumerate $H = \bigcup_i C_i - D$. Let $\chi(\overline{v}, \overline{x})$ be a possible atomic diagram of $H \cup D$, where $\lg(\overline{v}) = \lg(\overline{s})$. Further, let $\varphi_{(D,E),(B,C)}$ be the formula $\bigvee_i (\exists \overline{v})\chi_i(\overline{v}, \overline{x})$ where the χ_i are the finitely many possible such diagrams. Let $\rho(\overline{x})$ be the atomic diagram of D. Now Σ^2_μ is the collection of all formulas (for all good pairs (D,E)):

$$\psi_{(D,E)}: \quad \rho(\overline{x}) \to \bigvee_{BC \subseteq DE} \varphi_{(D,E),(B,C)}(\overline{x}). \tag{3}$$

Clearly any extension N of M by (B,C) will violate Σ^1_μ, since $M \models \Sigma^2_\mu$. That is, if $M \models \varphi_{(D,E),(B,C)}$, then any extension of M by (D,E) forces the realization of too many copies of C over B. ∎

Note that we can calculate the size of lines in the generic $\mathcal{G}_\mu$. Our structures $\mathcal{G}_\mu$ uniformize the result that there are only finitely many finite line lengths in any strongly minimal linear space (cf. Fact 2.3). Since the following result puts only one constraint on μ, for each $k \geqslant 3$, we can show in Corollary 5.26 that there are continuum-many strongly minimal theories T_μ such that in each of them all lines have fixed length k.

{maxlen} **Lemma 5.22.** *Let $\boldsymbol{\alpha}$ be the isomorphism type of the good pair $(\{b_1, b_2\}, c)$ (Definition 5.1) where $R(b_1, b_2, c)$ holds. If $\mu(\boldsymbol{\alpha}) = m$, then the length of every line in $\mathcal{G}_\mu$ is $m+2$ (recall that in Definition 5.2(1)(ii) we ask that $m \geqslant 1$).*

Proof. Clearly there can be no line in $\mathcal{G}_\mu \in \hat{\boldsymbol{K}}_\mu$ with cardinality greater than $\mu(\boldsymbol{\alpha})+2$. We show that for that for any $F \in \boldsymbol{K}_\mu$, any $b_1, b_2 \in F$, if the maximal clique X in F containing b_1, b_2 satisfies $|X| < \mu(\boldsymbol{\alpha})+2$, then there is a $G \in \boldsymbol{K}_\mu$ with $F \leqslant G$ and an element $a \in G - F$ with $X \cup \{a\}$ a clique. By the amalgamation, this suffices. Take D as $\{b_1, b_2\}$ and E as $\{b_1, b_2, a\}$ where $E \models R(b_1, b_2, a)$ and a satisfies only the relations completing $X \cup \{a\}$ to a clique in $G = F \oplus_D E$. Clearly, $\chi_G(\{b_1, b_2\}, a) \leqslant \mu(\boldsymbol{\alpha})+2$. But there can be no (B,C) contained in G with $\chi_G(B,C) \geqslant \mu(B,C)$ with $|C| > 1$. As, by Lemma 5.10(B), such a pair would have to be contained in E and the only isomorphism type of a good pair embedded in E is $\boldsymbol{\alpha}$. ∎

Recall (Definition 5.4) that a finite set X is d-independent when each $x \notin \mathrm{cl}^{\mathrm{d}}(X - \{x\})$, i.e. $d(X) > d(X - \{x\})$ for each $x \in X$. It is then easy to establish the first of the following assertions by induction and the others follow.

{basicd}

Lemma 5.23. *Let $M \in \hat{\boldsymbol{K}}_\mu$ and let Y be d-independent in M. For every finite $X \subseteq Y$ we have:*
(i) $d(X) = |X|$;
(ii) $X \leqslant M$, and so $\mathrm{icl}_M(X) = X$;
(iii) there are no R-relations among elements of X.

Using Lemmas 5.23 and 5.15, we follow Holland's proof showing that Σ_μ axiomatizes the complete theory of $\boldsymbol{K}_d^\mu$ (See Lemma 23 of [Hol99].).

{axcomp}

Lemma 5.24. *If $M \models \Sigma_\mu$ then $M \in \boldsymbol{K}_d^\mu$. Moreover, Σ_μ is an axiomatization of the complete theory T_μ of the class $\boldsymbol{K}_d^\mu$.*

Proof. Suppose $M \models \Sigma_\mu$. Then $M \models \Sigma_\mu^2$. By Lemma 5.21, there is no extension $N \in \hat{\boldsymbol{K}}_\mu$ of M by a good pair. That is, $M \in \boldsymbol{K}_d^\mu$.
[Why? If $M \leqslant N$ and there exists $C \subseteq N - M$ with $d_N(C/M) = 0$, there exists $B \subseteq M$ such that (B, C) is a good pair. Σ_μ^2 asserts there is a good pair (A, D) with $AD \subseteq BC$ and $\chi_N(A, D) > \mu(A, D)$. So $N \notin \hat{\boldsymbol{K}}_\mu$.]
Suppose now that $M, M' \models \Sigma_\mu$. By Σ_μ^3 and by taking elementary extensions we may assume that both have cardinality $\kappa > \aleph_0$. We show that Σ_μ is κ-categorical and so complete. As geometries, M and M' have bases X, X' of the same cardinality. By Lemma 5.23, $X \leqslant M$ and $X' \leqslant M'$ and they are isomorphic by any bijection f. The isomorphism f extends to one between M and M' since M and M' are built from X and X' by a sequence of 0-primitive extensions and each step can be extended by Lemma 5.15. ∎

Having followed the outline of her proof, we have the analog to Holland's result [Hol99] that the strongly minimal Hrushovski constructions are model complete.

Remark 5.25. *Since the axioms Σ_μ are universal-existential and T_μ is $\aleph_1$-categorical, it is model complete by Lindstroms's 'little theorem': that π_2-axiomatizable theories that are categorical in some infinite power are model complete [Lin64].*

We prove we have $2^{\aleph_0}$-many distinct theories using Lemma 4.11.

{contmu}

Corollary 5.26. *There are continuum-many $\mu \in \mathcal{U}$ (cf. Definition 5.2(1)) which give distinct first-order theories of Steiner systems. That is, there is $\mathcal{V} \subseteq \mathcal{U}$ such that $|\mathcal{V}| = 2^{\aleph_0}$ and $\mu \neq \nu \in \mathcal{V}$ implies that $Th(G_\mu) \neq Th(G_\nu)$ (recall Notation 5.18).*

Proof. For any $X \subseteq \omega$, let μ_X assert that $\mu(\boldsymbol{\gamma}_k)$ (from the proof of Lemma 4.11) is 3 if $k \in X$ and 2 if not (recall that it must be at least 2). Then, if $k \in X \setminus Y$, then $T_{\mu_X} \not\equiv T_{\mu_Y}$ (cf. Notation 5.18), since there are three extensions in the isomorphism type $\mu(\boldsymbol{\gamma}_k)$ of some pairs $\{a, b\}$ in models of T_{μ_X} but not in models of T_{μ_Y}. ∎

{dclinacl}

Lemma 5.27. *If $M \in \boldsymbol{K}_d^\mu$, then for every $X \subseteq M$, $\mathrm{cl}^{\mathrm{d}}(X) = \mathrm{acl}_M(X)$. Thus, T_μ is strongly minimal.*

Proof. We first show that for $M \in \hat{\boldsymbol{K}}_\mu$ the left hand side is contained in the right. If Y is a finite subset of M,$\delta(Y/X) = 0$, Y is a union of a finite chain with length $k < \omega$ of extensions by good pairs (B_i, C_i); each is realized by at most $\mu(B_i, C_i)$ copies, and so:

$$|Y| \leqslant \sum_{i<k} \mu(B_i, C_i) \times |C_i|.$$

Thus, $Y \subseteq \mathrm{acl}_M(X)$.

Concerning the other containment, let $M \in \boldsymbol{K}^\mu_d$, $a \in M$ and $X \subseteq_\omega M$. If $d(a/X) > 0$ and X_0 is a maximal d-independent subset of X, then $X_0 \cup \{a\}$ extends to a d-basis for M. Furthermore, in the proof of Lemma 5.24, we observed that any permutation of a d-basis extends to an automorphism of M. Thus, if $a \notin \mathrm{cl}^{\mathrm{d}}(X)$, then $a \notin \mathrm{acl}_M(X)$. Hence, $\mathrm{cl}^{\mathrm{d}}(X) = \mathrm{acl}_M(X)$, as desired.

Strong minimality follows, since for any finite A there is a unique non-algebraic 1-type over A, namely the type p of a point a such that: (i) a is not on any line based in A (and so $\delta(a/A) = 1$); (ii) Aa is strong in any model. Clause (ii) is given by the collection of universal sentences forbidding any $B \supseteq Aa$ with $\delta(B) < \delta(Aa)$. Thus, in $\mathcal{G}_\mu$ we have that $d(a/A) = 1$ for any a realizing p. Hence, any two realizations a and b of p are such that $Aa \leqslant \mathcal{G}$ and $Ab \leqslant \mathcal{G}$, and thus they are automorphic by the genericity of $\mathcal{G}_\mu$ (cf. Conclusion 5.15). Hence, p is a complete type. ∎

{fanoset}

Notation 5.28. *Let F be the Fano plane and $\mathcal{F}$ be the set of $\mu \in \mathcal{U}$ such that:*

$$\mu(\emptyset, F) > 0.$$

Lemma 5.29 shows that for any $\mu \in \mathcal{F}$ and $M \models T_\mu$, we have that $\mathrm{acl}_M(\emptyset)$ is infinite; by Ryll-Nardjewski, T_μ is not $\aleph_0$-categorical. In view of Lemma 5.27, the countable models correspond exactly to the models of dimension α for $\alpha \leqslant \aleph_0$.

{nlf}

Lemma 5.29. *Let $\mu \in \mathcal{F}$. Neither the generic, $\mathcal{G}_\mu$, nor any model of T_μ is locally finite with respect to* $\mathrm{cl}^{\mathrm{d}} = \mathrm{acl}$ *(cf. Lemma 5.27). Thus, T_μ is not $\aleph_0$-categorical and has $\aleph_0$ countable models. Since the generic has infinite dimension, it is ω-saturated.*

Proof. We show that the algebraic closure of the empty set is infinite. Construct a sequence $(A_i : i < \omega)$ in $\mathcal{G}_\mu$ by letting A_0 to be the Fano plane, which (Example 4.3) is easily seen to be 0- primitive over the empty set. Notice that there can only be finitely many realizations of the Fano plane in any model of T_μ, and so A_0 is in the algebraic closure of the empty set. Now let a_0, b_0, c_0 be the vertices of the triangle in the standard picture of the Fano plane. Choose a_1, b_1, c_1 disjoint from A_0 so that (a_0, a_1, c_1), (b_0, b_1, c_1), and (a_1, b_1, c_0) are triples of collinear points. Then, letting $A_1 = \{a_0, b_0, c_0, a_1, b_1, c_1\}$, it is to see that A_1 is a primitive extension of A_0. Now build A_2 by taking a_1, b_1, c_1 as the base and adding a_2, b_2, c_2 as in the construction of A_1 from A_0; and then iterate. Each stage (and hence the union) can be strongly embedded as A'_i in the generic. But then $\delta(A'_{i+1}/A'_i) = d(A_{i+1}/A_i) = 0$. By transitivity, with A_ω denoting $\bigcup_{i<\omega} A_i$, we have that for any finite $X \subseteq A_\omega$, $d(X/A_0) = 0$. Since $\mathrm{cl}^{\mathrm{d}} = \mathrm{acl}$ (Lemma 5.27), we finish. We constructed this sequence in the algebraic closure of the empty set, and so it occurs in the prime model of T. Thus, $\mathrm{acl}_M(\emptyset)$ is infinite for any model M of T_μ. By Ryll-Nardjewski, T_μ is not $\aleph_0$-categorical. In view of Lemma 5.27, as in any strongly minimal theory, these models correspond exactly to models of dimension α for $\alpha \leqslant \aleph_0$. ∎

{fcon}

6. Further Context

In this section we place our work in the context of further work on the model theory of Steiner systems/linear spaces, studies on the consequences of flat geometries, and Hrushovski constructions.

{onbc}

Remark 6.1. We compare our examples with the construction in [BC1x] of structures existentially closed for the class of all Steiner quasigroups. Note that Steiner quasigroups are the quasigroups associated with Steiner *triple* systems in [BC1x].

(i) Their generic, denoted $\mathbb{M}_{\mathrm{sq}}$, has continuum many types over the empty set, satisfies TP_2 and $NSOP_1$, and it is locally finite (but not uniformly locally finite) as a quasigroup. If $\mu \in \mathcal{U}$, then it is obvious that T_μ fails the first three of these properties since it is strongly minimal. Furthermore, we showed in Lemma 5.29 that our examples with $\mu \in \mathcal{F}$ are not locally finite for acl $=$ cl^{d}. Strikingly, in $\mathbb{M}_{\mathrm{sq}}$, the definable closure is equal to the algebraic closure (dcl $=$ acl). In [Bal19] we show that this equality fails drastically in any T_μ with $\mu \in \mathcal{U}$.

(ii) The structure $\mathbb{M}_{\mathrm{sq}}$ is the prime model of its theory; our $\mathcal{G}_\mu$ is saturated. While the example in [BC1x] is quantifier eliminable, ours is only model complete. The first is the model completion of the universal theory of Steiner quasigroups. Since each $M \in \boldsymbol{K}_\mu$ can be extended to $N \in \boldsymbol{K}_d^\mu$, the second is the model completion of the universal theory of $\hat{\boldsymbol{K}}_\mu$ for the relevant μ. Quantifier elimination does not follow since, despite the limited amalgamation in Conclusion 5.15, $\hat{\boldsymbol{K}}_\mu$ does not have amalgamation.

(iii) In the introduction we mentioned further results on the combinatorics of strongly minimal Steiner systems and strongly minimal quasigroups from [Bal19], and compared our approach with that of [BC1x, CK16, HP].

We now show that our examples have all the properties of the *ab initio* Hrushovski construction [Hru93].

Definition 6.2. *Suppose* (A, cl) *is a pregeometry with dimension function* d *and* $F_1, ..., F_s$ *are finite-dimensional closed subsets of* A *(i.e. they are strong in* A*). For* $\emptyset \subsetneq S \subseteq \{1, ..., s\}$*, we let* $F_S = \bigcap_{i \in s} F_i$*. We say that* (A, cl) *is flat if for all such* $F_1, ..., F_s$ *we have:*

$$d\Big(\bigcup_{1\leqslant i\leqslant s} F_i\Big) \leqslant \sum_{\emptyset\neq S} (-1)^{|S|} d(F_S).$$

The crux of this definition is that if the function d is computed on a finite union of finite dimensional closed sets by the inclusion-exclusion principle[12] then the value of d is at worst underestimated. To see this, observe that d is the difference of two functions. In the usual case, the first is the cardinality of the sets, the second the number of occurrences of R in each set. Since inclusion-exclusion is exact for the first and underestimates for the second, the difference is flat. Since the sets are strong in A, we can compute with δ.

To copy the argument for our predimension we only have to note that our δ is the difference of cardinality and a sum of values of the nullity function and inclusion-exclusion underestimates nullity. For this, observe that if A, B are closed sets and ℓ is a line in AB, then:

[12] If $\mathcal{A}$ is a finite collection of finite sets then $|\bigcup \mathcal{A}| = \Sigma\{(-1)^{\mathcal{E}} |\bigcap \mathcal{E}| \colon \emptyset \neq \mathcal{E} \subsetneq \mathcal{A}\}$.

$$n_{AB}(\ell) \leqslant n_A(\ell \cap A) + n_B(\ell \cap B) - n_{A\cap B}(\ell \cap (A \cap B)).$$

(The only way the inequality is sharp is if $|\ell \cap (A \cap B)| < 2$.) Thus, as usual, we have:

Lemma 6.3. *For any $\mu \in \mathcal{U}$, the* acl- *pregeometry associated with T_μ is flat.*

We isolate for strongly minimal sets the following facts scattered in the literature, often in more generality, since they will be relevant for us. We also state one further definition since it is used directly in a proof.

Definition 6.4 ([Pil96, Chapter 4])**.** *A pseudoplane is a system of points and lines such that any two distinct lines intersect in only finitely many points, two distinct points share only finitely many lines, and every line is infinite.*

{thehammer}
{flatnogroup} **Fact 6.5.** *Let T be a strongly minimal theory.*

(1) [Hru93, Lemma 14+remark just after] *If the* acl-*geometry of T is flat, then T does not interpret an infinite group and T is CM-trivial.*
(2) [Pil96, Theorem 5.1.1] *If the* acl-*geometry of T is locally modular and non-trivial, then T interprets an infinite group.*
(3) [Mar98, Proposition 1] *If T is not locally modular, then there is a complete-type definable pseudoplane interpretable in T^{eq}.* {lmpp}
(4) [Pil99, Lemma 1.6] *If* acl($\emptyset$) *is infinite in T, then T admits weak elimination of imaginaries.*

The following lemma is presumably well-known to experts but we didn't see it written down anywhere, and so we decided to state it explicitly.

Lemma 6.6. *If T is strongly minimal and the* acl-*geometry of T is non-trivial but flat, then T violates Zilber's conjecture, T is CM-trivial, and it does not eliminate imaginaries. Moreover, if* acl($\emptyset$) *is infinite, then T admits weak elimination of imaginaries.*

Proof. Assertions (1) and (2) from Fact 6.5 imply the violation of Zilber's conjecture and CM-triviality. By (3) there is a complete-type definable pseudoplane interpretable in T^{eq}. Elimination of imaginaries would imply the interpretation is into the vocabulary τ. In particular, there would be two definable infinite sets (lines) with finite intersection. This contradicts strong minimality. The last claim is immediate from Fact 6.5(4). ■

Thus, using Lemma 5.29 and Fact 6.5, we have immediately in our context:

{nozil}
Corollary 6.7. *If $\mu \in \mathcal{U}$, T_μ violates Zilber's conjecture and does not admit elimination of imaginaries. If, further, $\mu \in \mathcal{F}$, T has weak elimination of imaginaries.*

For any of the variants in [Bal19] of the theory T_μ, any model of T_μ has infinite algebraic closure over finitely many constants, so the conclusion is much broader. But in that paper we do find a variant with trivial acl($\emptyset$).

Question 6.8. *What is the pseudoplane in the T^{eq} that Fact 6.5(2) tells us exists in the models of our $T = T_\mu$?*

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