The Stability spectrum for classes of atomic models

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Abstract

We prove two results on the stability spectrum for $L_{\omega_1,\omega}$. Here $S_i^m(M)$ denotes an appropriate notion of Stone space of m-types over M. Theorem A. Suppose that for some positive integer m and for every $\alpha < \delta(T)$, there is an $M \in \boldsymbol{K}$ with $|S_i^m(M)| > |M|^{\beth_\alpha(|T|)}$. Then for every $\lambda \geq |T|$, there is an M with $|S_i^m(M)| > |M|$. Theorem B. Suppose that for every $\alpha < \delta(T)$, there is $M_\alpha \in \boldsymbol{K}$ such that $\lambda_\alpha = |M_\alpha| \geq \beth_\alpha$ and $|S_i^m(M_\alpha)| > \lambda_\alpha$. Then for any μ with $\mu^{\aleph_0} > \mu$, $\boldsymbol{K}$ is not stable in μ. These results provide a new kind of sufficient condition for the unstable case and shed some light on the spectrum of strictly stable theories in this context. The methods avoid the use of compactness in the theory under study.

1 Context

For many purposes, e.g. the study of categoricity in power, the class of models of a sentence ϕ of $L_{\omega_1,\omega}$ can be profitably translated to study the class of models of a first order theory T that omit a collection Γ of first order types over the empty set. In particular, if ϕ is complete (i.e. a Scott sentence) Γ can be taken as the collection of all non-principal types and the study is of the atomic models of T. This translation dates from the 60's; it is described in detail in Chapter 6 of [Bal]. The study of finite diagrams (see below) is equivalent to studying sentences of $L_{\omega_1,\omega}$; the study of atomic models of a first order theory is equivalent to studying *complete* sentences of $L_{\omega_1,\omega}$.

The stability hierarchy provides a crucial tool for first order model theory. Shelah [She78] and Keisler [Kei76] shows the function $f_T(\lambda) = \sup\{|S(M)| : |M| =$

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$\lambda, M \models T\}$ has essentially only six possible behaviors (four under GCH). In [She70], Shelah establishes a similar result for *homogeneous* finite diagrams. The homogeneity assumption is tantamount to assuming amalgamation over all sets. This is a strong hypothesis that is avoided in Shelah's further investigation of categoricity in $L_{\omega_1,\omega}$ ([She83a, She83b]) which is expounded as Part IV of [Bal]. As we explain below, this investigation begins by identifying the appropriate notion of type over a set (and thus of ω-stable). It is shown that ω-stability implies stability in all powers. And assuming $2^{\aleph_0} < 2^{\aleph_1}$, ω-stability is deduced from $\aleph_1$-categoricity. But further questons questions concerning the stability hierarchy for this notion of type for arbitrarysentences of $L_{\omega_1,\omega}$ had not been investigated. We do so now. In fact our results hold for *arbitrary finite diagrams*, the class of models of first order theory that omit a given set of types over the empty set. But our results are by no means as complete as in homogeneous case.

There are (at least) two *a priori* reasonable notions of Stone space for studying atomic models of a first order theory.

Definition 1.1 *1. Let A be an atomic set; $S_{\mathrm{at}}(A)$ is the collection of $p \in S(A)$ such that if $\boldsymbol{a} \in \mathbb{M}$ realizes p, $A\boldsymbol{a}$ is atomic.*

2. Let A be an atomic set; $S_{\mathrm{mod}}(A)$ is the collection of $p \in S(A)$ such that p is realized in some $M \in \boldsymbol{K}$ with $A \subseteq M$.

In [Bal] we write S^* for the notion called S_{mod} here. The latter notation is more evocative. We will simultaneously develop the results for *unstable* theories for both notions of Stone space and indicate the changes required to deal with the two cases. We will write $S_i(M)$ where i can be either at or mod. The section on *superstable* theories does *not* admit such a parallel development. We discuss the distinction in Section sscase.

Remark 1.2 In [She70], Shelah has a stronger requirement on stability; it implies *by definition* the existence of homogeneous models in certain cardinals. We do not make that assumption here so we are considering a larger class of theories.

We sometimes write $|T|$ for $|\tau|$ where τ is the vocabulary of T. $\boldsymbol{K} = \boldsymbol{K}_T$ is the class of atomic models of T. We write $H = H(\mu)$ for the Hanf number for atomic models of all theories with $|T| = \mu$. By [She78] H equals $\beth_{\delta(T)}$, where $\delta(T)$, the well-ordering number of the class of models of a theory omitting a family of types, is defined in VII.5 of [She78]. It is also shown there that if T is countable, H evaluates as $\beth_{\omega_1}$ while for uncountable T it is $\beth_{(2^{|T|})^+}$. Fix $\mu_\alpha = \beth_\alpha(|T|)$.

Definition 1.3 *1. $\boldsymbol{K}$ is stable in λ (for $i = \mathrm{at}$ or mod) if for every m, and $M \in \boldsymbol{K}$ with $|M| = \lambda$, $|S_i^m(M)| = \lambda$.*

2. Stability classes. For either $i = \mathrm{at}$ or mod,

(a) $\boldsymbol{K}$ is stable if it is stable in some λ.

(b) $\boldsymbol{K}$ is superstable if it is stable in all $\lambda \geq H$.

(c) $\boldsymbol{K}$ is strictly stable if it is stable but not superstable.

(d) *For countable languages, $\boldsymbol{K}$ is ω-stable if it is stable in all $\lambda \geq \aleph_0$.*

We prove Theorem A in Section 2 and Theorem B in Section 3. The proof of Theorem B uses an application of omitting types in Ehrenfeucht-Mostowski models generated by trees of the form ${}^{<\omega}\lambda$. This is by no means new technology but we weren't able to locate an explicit proof of the result so we include one in Section 4.

2 Unstable $\boldsymbol{K}$

We first show that if there are cardinals λ_α in which $\boldsymbol{K}$ is 'sufficiently unstable', then $\boldsymbol{K}$ is not stable in any cardinal.

Theorem 2.1 *Suppose that for some positive integer m and for every $\alpha < \delta(T)$, there is an $M \in \boldsymbol{K}$ with $|S_i^m(M)| > |M|^{\beth_\alpha(|T|)}$. Then for every $\lambda \geq |T|$, there is an M with $|S_i^m(M)| > |M|$.*

Remark 2.2 (Proof Sketch) Before the formal proof we outline the argument. We start with a sequence of models M_α and many distinct types over each of them. By an argument which is completely uniform in α, we construct triples $\langle \boldsymbol{a}_{\alpha,i}, \mathbf{b}_{\alpha,i}, \mathbf{d}_{\alpha,i}\rangle$ for $i < \mu_\alpha^+$ with the $\boldsymbol{a}_{\alpha,i}, \mathbf{b}_{\alpha,i} \in M_\alpha$ and $\mathbf{d}_{\alpha,i}$ in an elementary extension M'_α of M_α of the same cardinality and so that $M_\alpha \mathbf{d}_{\alpha,i}$ atomic and the distinctness of the types is explicitly realized. Then we apply Morley's omitting types theorem to the M'_α and extract from this sequence a countable set of indiscernibles with desirable properties. Finally, this set of indiscernibles easily yields models of all cardinalities with the required properties.

Remark 2.3 The idea of the proof can be seen by ignoring the α and proving the result with one model of size $\beth_{\delta(T)}$.

Notation 2.4 $\lambda_\alpha = |M_\alpha|^{\beth_{\alpha+2}(|T|)}$. $\mu_\alpha = \beth_\alpha(|T|)$. $\kappa_\alpha = \beth_{\alpha+2}(|T|)$.

Lemma 2.5 *For some τ_Φ extending τ with $|\tau_\Phi| = |\tau|$, that there is Φ, proper for linear orders, with fixed additional unary predicate P, P_1 and binary R such that:*

1. *For every linear ordering I, $N_I = EM_\tau(I, \Phi) \models T$ and $M_I = EM_\tau(I, \Phi) \restriction P \in \boldsymbol{K}$. Naturally $J \subset I$ implies $N_J \prec N_I$ and $M_J \prec M_I$.*
2. *The skeleton of N_I is $\langle \boldsymbol{a}_i\frown\mathbf{b}_i\frown\mathbf{c}_i : i \in I\rangle$ and $\lg(\mathbf{c}_i) = m$.*
3. *For some first order ϕ:*

$$N_I \models (\phi(\mathbf{c}_t, \boldsymbol{a}_s) \equiv \phi(\mathbf{c}_t, \mathbf{b}_s)) \text{ iff } s <_I t.$$

4. *$M_I \cup \mathbf{c}_i$ is atomic (in N_I)*
5. *For $S_{\mathrm{mod}}(M)$, we add the requirement that for each $s \in I$,*

$$M_{I,s} = N_I \restriction \{d : N_I \models R(d, \mathbf{c}_s)\}$$

is an atomic elementary submodel of N_I containing $M_I\mathbf{c}_s$.

Proof. The proof of Lemma 2.5 requires a number of steps. Fix for each $\alpha < \delta(T)$, $M_\alpha \in \boldsymbol{K}$ such that $|S_i^m(M_\alpha)| > \lambda_\alpha = |M_\alpha|^{\beth_{\alpha+2}(|T|)}$. Fix $p_{\alpha,i}$ for $i < \lambda_\alpha^+$, a list of distinct types in $S_i^m(M_\alpha)$. We work throughout in a monster model $\mathbb{M}$ of T.

Notation 2.6 *In the following construction, we choose by induction triples $\langle \boldsymbol{a}_{\alpha,i}, \mathbf{b}_{\alpha,i}, \mathbf{d}_{\alpha,i}\rangle$ for $i < \mu_\alpha^+$. We use the following notation for initial segments of the sequences.*

1. $D_{\alpha,i} = \{\mathbf{d}_{\alpha,j} : j < i\}$.
2. $X_{\alpha,i} = \{\boldsymbol{a}_{\alpha,j}, \mathbf{b}_{\alpha,j} : j < i\}$.
3. *$q_{\alpha,i}$ is the type of $d_{\alpha,i}$ over $X_{\alpha,i}$.*

The following variant on splitting is crucial to carry out the construction. We call it ex-splitting (for external) because the elements which exemplify splitting are required to satisfy the same type over a set which is not in the model.

Definition 2.7 *Let M be a model, $X \subset M$ and $D \subset \mathbb{M}$. We say that $p \in S_i^m(M)$* ex-splits *over (D, X) if there exist $\boldsymbol{a}, \mathbf{b} \in M$, $\mathbf{d} \in \mathbb{M}$ so that $\mathbf{d}$ realizes $p \restriction X$, $\boldsymbol{a} \equiv_D \mathbf{b}$ but $(\boldsymbol{a}, \mathbf{d})$ and $(\mathbf{b}, \mathbf{d})$ realize different types.*

Claim 2.8 *The number of types in $S_i^m(M_\alpha)$ that do not ex-split over a pair (D, X) with $|X| = |D| = \beth_\alpha$ is $\mu_{\alpha+2}$.*

Proof. Since it only strengthens the result while simplifying the exposition we let $X = \emptyset$. Let P_α denote the collection of $\mathrm{tp}(\mathbf{e}/M_\alpha)$ with $\lg(\mathbf{e}) = m$ such that if $\boldsymbol{a}, \mathbf{b} \in M_\alpha$ realize the same type over D then $(\boldsymbol{a}, \mathbf{e})$ and $(\mathbf{b}, \mathbf{e})$ realize the same type. Each type r in P_α is determined by knowing for each $i < |T|$ its restriction to one k-tuple from each equivalence class of the equivalence relation E_k on M_α defined by $\boldsymbol{a} E_k \mathbf{b}$ if $\boldsymbol{a}$ and $\mathbf{b}$ realize the same type over D. So, since $|D| = \mu_\alpha$, there are at most $(2^{2^{\mu_\alpha}})^{|T|} = \mu_{\alpha+2}$ possible such r. $\square_{2.8}$

As noted, for each M_α we will be constructing by induction on $i < \mu_\alpha^+$, sets $X_{\alpha,i}, D_{\alpha,i}$ of cardinality μ_α. We need to choose in advance a type p_α which does not ex-split over any $(X_{\alpha,i}, D_{\alpha,i})$ that arises. In order to do that we restrict the source of $D_{\alpha,i}$; clearly $X_{\alpha,i} \subset M_\alpha$. That is, we will fix M'_α with $M_\alpha \prec M'_\alpha$, $|M'_\alpha| = \lambda_\alpha$, and M'_α is μ_α^+ saturated. (Note then that M'_α is *not* in general atomic.)

Note that the number of types in $S_i^m(M_\alpha)$ that do not ex-split over *any* pair (D, X) with $|X| = |D| = \beth_\alpha$ is bounded by the number of such sets, $|M_\alpha|^{\mu_\alpha}$, times the number of types in $S_i^m(M_\alpha)$ that do not ex-split over a particular choice of (D, X), which is $\mu_{\alpha+2}$ by Claim 2.8. That is, the bound is $|M'_\alpha|^{\mu_\alpha} \times \mu_{\alpha+2}$. Since this number is less than λ_α^+, we can fix a type $p_\alpha \in S_i^M(M_\alpha)$ which does not ex-split over any of the relevant (D, X).

Definition 2.9 *For each $\alpha < \delta(T)$, fix M'_α with $M_\alpha \prec M'_\alpha$, $|M'_\alpha| = \lambda_\alpha$, and M'_α is μ_α^+ saturated. Choose, by induction on $i < \mu_\alpha^+$, triples $\mathbf{e}_{\alpha,i} = \langle \boldsymbol{a}_{\alpha,i}, \mathbf{b}_{\alpha,i}, \mathbf{d}_{\alpha,i}\rangle$ where*

a) $\mathbf{d}_{\alpha,j} \in M'_\alpha$.

b) $\boldsymbol{a}_{\alpha,i}, \mathbf{b}_{\alpha,i}$ *are sequences of the same length from* M_α *that realize the same type over* $D_{\alpha,i} = \{\mathbf{d}_{\alpha,j} : j < i\}$.

c) The types over the empty set of $(\boldsymbol{a}_{\alpha,i}, \mathbf{d}_{\alpha,i})$ *and* $(\mathbf{b}_{\alpha,i}, \mathbf{d}_{\alpha,i})$ *differ.*

d) $q_{\alpha,i} = p_\alpha \restriction X_{\alpha,i} = \mathrm{tp}(\mathbf{d}_{\alpha,i}/X_{\alpha,i})$ *so if* $j < i$, $q_{\alpha,j} \subseteq q_{\alpha,i}$.

e) $M_\alpha \mathbf{d}_{\alpha_i}$ *is an atomic set for each* i. *(In the* $*$*-version* $N_{\alpha,i}$ *is an atomic model containing* $M_\alpha \mathbf{d}_{\alpha_i}$.*)*

Construction 2.10 Choose $\mathbf{d}_{\alpha,i}$ to realize $p_\alpha \restriction X_{\alpha,i}$. By Claim 2.8, we can choose $\boldsymbol{a}_{\alpha,i}$ and $\mathbf{b}_{\alpha,i}$ to satisfy conditions a) and b). So we have

$$\mathrm{tp}(\mathbf{d}_{\alpha,i}, \boldsymbol{a}_{\alpha,j}) = \mathrm{tp}(\mathbf{d}_{\alpha,i}, \mathbf{b}_{\alpha,j}) \text{ if and only if } i < j. \tag{1}$$

But we want this order condition for a single formula. We have for each $i < \mu_\alpha^+$, the types of $(\boldsymbol{a}_{\alpha,i}, \mathbf{d}_{\alpha,i})$ and $(\mathbf{b}_{\alpha,i}, \mathbf{d}_{\alpha,i})$ differ. That is, $\phi_{\alpha,i}(\boldsymbol{a}_{\alpha,i}, \mathbf{d}_{\alpha,i})$ and $\neg\phi_{\alpha,i}(\mathbf{b}_{\alpha,i}, \mathbf{d}_{\alpha,i})$ for some $\phi_{\alpha,i}$. By the pigeon-hole principal we may assume the $\phi_{\alpha,i}$ is always the same ϕ_α. (Further, since $|T|$ is not cofinal in $\delta(T)$, we can assume the ϕ_α is the same ϕ for all α.)

Suppose the construction is completed. We expand τ to a language $\tau_\Phi \supset \tau$ by adding predicates $P, <, R$ and Skolem functions. We add Skolem axioms to T to get a theory T_1 that admits quantifier elimination, requiring that these Skolem functions applied to elements of P give an element of P so that P will pick out an elementary submodel. (We make a similar requirement for $R(x, \mathbf{y})$ in the mod-case.) Let M_α^+ be a model of T_1 (submodel of M'_α) with cardinality μ_α^+ containing M_α and all the $\mathbf{d}_{\alpha,i}$ ($N_{\alpha,i}$ in the $*$-case). Interpret P as the model M_α and the relation $<$ as the ordering on the triples $\langle \mathbf{e}_{\alpha,i} : i < \mu_\alpha \rangle$ imposed by μ_α.

Assign the Skolem functions so that the $\mathbf{e}_{\alpha,i}$ generate M'_α and interpret R by

$$R = \{e\widehat{\ }\mathbf{d}_{\alpha,i}; e \in M_\alpha, i < \mu_\alpha^+\}.$$

(In the $S_*(M)$ case, interpret R as $\{e\widehat{\ }\mathbf{d}_{\alpha,i} : i < \mu_\alpha^+, e \in N_{\alpha,i}\}$.)

Notation 2.11 *Let* Γ *be the collection of types:*

1. $\mathcal{P}_n = \{\bigwedge_{i<n} P(x_i)\} \cup \{q(\mathbf{x}) : q$ is a non-principal n-type $\}$
2. $\mathcal{Q}_n = \{\bigwedge_{i<n} R(x_i, y)\} \cup \{q(\mathbf{x}, y) : q$ is a non-principal n-type $\}$

Now apply Morley's omitting types theorem[1] to the τ_Φ-theory T_1 and the collection of M_α^+ to get a countable sequence I of order indiscernibles and an extension Φ of T_1, (the EM-template) such that Φ is realized in each M_α^+ and such that for every linear order J, $EM_\tau(J, \Phi) \models T_1$ and omits Γ.

[1]See Appendix A.3.1 of [Bal] for a precisely tailored version. See [She78] or [Hod93], page 587 for a version with the role of the ordering more explicit. The latter two sources make the connection with the well-ordering number clear.

Remark 2.12 (Morley's Method) The next observation requires a little care in proving Morley's theorem rather than just quoting it. The M'_α are generated by the $\mathbf{e}_{\alpha,i}$ and we have interpreted $<$ so that these are exactly the domain of $<$. So in proving the omitting types theorem, all witnesses for the consistency of the template $\Phi(\mathbf{c})$ can be chosen from the domain of $<$. We use this fact now.

Note that any τ_Φ formula $\phi(\mathbf{x})$ is in Φ if it is true of every tuple $\langle \mathbf{e}_{\alpha,i_1}, \ldots \mathbf{e}_{\alpha,i_1}\rangle$ with $i_1 < i_2 < \ldots i_n$. We describe a crucial such sentence.

Let $\mathbf{x}^1\mathbf{x}^2\mathbf{x}^3$ be a triple of sequences with the first two having the same length as $\lg(\boldsymbol{a}) = \lg(\mathbf{b})$ and the third has length m. Let $\psi(\mathbf{x}, \mathbf{y})$ denote:

$$\phi(\mathbf{y}^3, \mathbf{x}^1) \equiv \phi(\mathbf{y}^3, \mathbf{x}^2).$$

Let ψ_1 be the assertion that ϕ defines a linear order on its domain; this directly translates precisely Lemma 2.5.3 and is true by the displayed statement 1. These structures clearly satisfy all the conditions of the requirements in Lemma 2.5 and we complete the proof.

$\square_{2.5}$

Proof of Theorem 2.1: To show instability in λ, let I be a dense linear ordering with cardinality λ and choose $J \supset I$, that realizes more than $|I|$ cuts over I. Then $EM_\tau(J, \Phi)$ realizes more than λ types in $S^m_i(P(EM_\tau(I, \Phi))$. To see this, consider for any cut in I realized by an element $j \in J$ the type:

$$\{\psi(\langle \boldsymbol{a}_i, \mathbf{b}_i, \mathbf{c}_i\rangle, \mathbf{x},) : i < j\} \cup \{\psi(\langle \boldsymbol{a}_i, \mathbf{b}_i, \mathbf{c}_i\rangle, \mathbf{x},) : i \geq j\}.$$

Then $\langle \boldsymbol{a}_j, \mathbf{b}_j, \mathbf{c}_j\rangle$ realizes the type in $EM_\tau(J, \Phi)$ and $P(EM_\tau(J, \Phi))\mathbf{c}_j$ is an atomic set since $\mathcal{Q}$ was omitted. For the *-case, use the interpretation of R to define $N_{\alpha,i}$. $\square_{2.1}$

Question 2.13 *Must an atomic class that is unstable in all λ have the order property?*

We say a class of atomic models has the *order property* if there is a sequence as in Lemma 2.5.3 but with the set of all the sequences being contained in atomic set. Condition 3) only requires each triple to be atomic. In particular we don't know the various c's can appear together in any atomic model.

3 Strictly stable case

As the following examples show, it is easy to have superstable sentences of $L_{\omega_1,\omega}$ that are not superstable for arbitrary values below H. The following theorem has two easily stated corollaries. If $\boldsymbol{K}$ is not superstable then it is not stable in every λ with $\lambda^\omega > \lambda$. If $\boldsymbol{K}$ is superstable then it is stable in some $\lambda < H$.

In this section we restrict to $S_{at}(M)$. At a crucial point we need that the class of types is 'local', closed under unions of increasing chains. This hold for S_{at} but not for S_{mod}.

The results here are related to those in [GS86] but the combinatorics here is considerably simpler than in [GS86] for two related reasons. First, we construct tree indiscernibles indexed by ${}^{<\omega}\lambda$ while they are concerned with ${}^{\leq\omega}\lambda$; the limit node is much more difficult to handle. Second, they are constructing many non-isomorphic models, we only construct many different types. Thus, there results depend on the existence of large cardinals while this paper is in ZFC.

Example 3.1 For $\alpha < \omega_1$, let ϕ_α be Morley's sentence that has a model in $\beth_\alpha$ but no larger model. It is easy to see that the sentences are not stable in the cardinalities where they have models. Let ψ be the Scott sentence of an infinite set with only equality. Now let ψ_α assert that either a structure has a nontrivial relation and obeys ϕ_α or just ψ. Then ϕ_α is $\beth_\alpha$-unstable but stable (indeed categorical in all cardinals beyond $\beth_{\omega_1}$.

If one adds even joint embedding such trivial examples are no longer apparent.

Theorem 3.2 *Suppose that for every $\alpha < \delta(T)$, there is $M^\alpha \in \boldsymbol{K}$ such that $\lambda^\alpha = |M^\alpha| \geq \beth^\alpha$ and $S^m_{at}(M^\alpha) > \lambda_\alpha$. Then for any μ with $\mu^{\aleph_0} > \mu$, $\boldsymbol{K}$ is not stable in μ.*

Proof. Fix for each $\alpha < \delta(T)$, $M^\alpha \in \boldsymbol{K}$ such that $|S^m_i(M^\alpha)| > \lambda_\alpha$. Fix $p_{\alpha,i}$ for $i < \lambda^+_\alpha$, a list of distinct types in $S^m_i(M^\alpha)$. We work throughout in a monster model $\mathbb{M}$ of T.

To prepare for the application of Morley's omitting types theorem we must construct a sequence of models and certain types. For this, we construct trees of types that arise from failure of stability. The combinatorics slightly extends the classical arguments and avoids compactness. Note that this stage of the construction takes place in the original language. We will apply the following general result uniformly to each M^α.

Fact 3.3 *Suppose $|M| \geq \mu_{\alpha+1}$ and $\mathcal{P}$ is a collection of $> \lambda_\alpha = |M|$ members of $S^m_i(M)$. Then there exists a sequence $\langle \mathbf{b}_j : j < \mu_\alpha \rangle$ with each $\mathbf{b}_j \in M$ and a formula $\phi(\mathbf{x}, \mathbf{y}) = \phi_{\mathcal{P}}$ such that for each $j < \mu_\alpha$,*

$$|\{p \in \mathcal{P} : i < j \rightarrow \phi(\mathbf{x}, \mathbf{b}_i) \in p \text{ but } \neg\phi(\mathbf{x}, \mathbf{b}_j) \in p\}| > \lambda_\alpha. \qquad (2)$$

Proof. We consider many possibilities for ϕ and prove one works. We choose $\{\phi_\eta : \eta \in T_i\}$ by induction on $i < \mu_\alpha$ where each T_i is a subset of ${}^i2$ and each $\mathbf{b}_\eta \in M^\alpha$ so that

1. $j < i$ and $\eta \in T_i$ implies $\eta \restriction j \in T_j$.
2. if $\eta \in T_i$ then $p_\eta = \{\phi_{\eta\restriction j}(\mathbf{x}, \mathbf{b}_{\eta\restriction j})^{\eta(j)} : j < i\}$ is included in $> \lambda_\alpha$ members of $\mathcal{P}$.
3. For limit i,

   $$T_i = \{\eta \in {}^i2 : (\forall j < i)\eta \restriction j \in T_j \text{ and } p_\eta \text{ is included in } > \lambda_\alpha \text{ members of } \mathcal{P}\}$$

4. if $i = j + 1$ then $T_i = \{\eta\widehat{\ }0, \eta\widehat{\ }1 : \eta \in T_j\}$.

For the successor step in the induction recall the following crucial observation of Morley. Suppose there are more than $|M|$ types over M extending a partial type p. Then there exists a formula $\phi(\mathbf{x}, \boldsymbol{a})$ with $\boldsymbol{a} \in M$ such that both $p \cup \{\phi(\mathbf{x}, \boldsymbol{a})$ and

$p \cup \{\neg\phi(\mathbf{x}, \boldsymbol{a})\}$ have more than $|M|$ extensions to complete types over M. (We are extending Morley's analysis to types in $S_i^m(M)$ but the argument is just counting; otherwise there is a unique type obtained by always choosing the big side.)

The interesting point in the induction is the limit stage. We cannot guarantee that individual paths survive. But at each stage in the induction, we have defined types over a set of cardinality μ_α. So there are at most $\mu_{\alpha+1}$ types over $\{\mathbf{b}_\eta : \lg(\eta) < \delta\}$. So one of the paths must have more than λ_α extensions to $S_i^m(M)$.

So $T_{\mu_\alpha} \neq \emptyset$. Choose $\eta \in T_{\mu_\alpha}$. Let $\phi'_j(\mathbf{x}, \mathbf{b}_j) = \phi_{\eta\restriction j}(\mathbf{x}, \mathbf{b}_{\eta\restriction j})^{\eta(j)}$ for $j < \mu_\alpha$. But since the path has length $\mu_\alpha = \beth_\alpha(T)$, by the pigeonhole principle we may assume there is a single formula ϕ. This completes the construction of the ϕ and the $\mathbf{b}_j$. $\square_{3.3}$

Now we apply this fact by constructing from the original M^α given in Theorem 3.2 a sequence of models $\hat{M}^\alpha$ and associated sequences $\mathbf{c}_{\alpha,\rho}$ for $\rho \in {}^{<\omega}\mu_\alpha$.

Definition 3.4 *Let $\hat{M}^\alpha$ be a μ_α^+ saturated elementary extension of M^α. We construct for each α by induction on $n < \omega$, submodels M_n^α of M^α and types $\{q_\nu^\alpha : \nu \in {}^{<\omega}\mu_\alpha\}$ with $q_\nu^\alpha \in S_i^m(M_{\lg(\nu)}^\alpha)$ and realizations $\mathbf{c}_{\alpha,\nu}$ of $q_\nu^\alpha \in M^\alpha$ satisfying the following conditions.*

1. *M_k^α is an increasing chain of submodels of M^α, each with cardinality μ_α.*
2. *Suppose $k < m \leq n$, $\nu \in {}^k\mu_\alpha$, then $q_\nu^\alpha \in S_i^m(M_k^\alpha)$.*
3. *Each $q_\nu^\alpha \in S_i^m(M_k^\alpha)$ has $> \lambda_\alpha$ extensions to $S_i^m(M^\alpha)$*
4. *Suppose $k < m \leq n$, $\nu \in {}^k\mu_\alpha$, $\rho \in {}^m\mu_\alpha$ and ρ extends ν:*
$$q_\nu^\alpha \subseteq q_\rho^\alpha.$$
5. *If $\nu \in {}^k\lambda_\alpha$, $k < n$, $i \neq j$, then*
$$q_{\nu\frown i}^\alpha \neq q_{\nu\frown j}^\alpha.$$
6. *$\mathbf{c}_{\alpha,\nu} \in \hat{M}_\alpha$ realizes q_ν^α.*

Construction 3.5 We use Fact 3.3 to construct objects meeting this definition. By induction, for each $\rho \in {}^n\mu_\alpha$ the type $q_\rho^\alpha \in S_i^m(M_n^\alpha)$ has $> \lambda_\alpha$ extensions to $S_i^m(M^\alpha)$. Let $\mathcal{P}_\rho = \{r \in S_i(M_\alpha) : q_\rho^\alpha \subseteq r\}$ so $|\mathcal{P}_\rho| > \lambda_\alpha$. By Fact 3.3, we find $\langle \mathbf{b}_{\alpha,\rho\frown j} : j < \mu_\alpha\rangle$ and ϕ_ρ satisfying displayed statement 2.

Let M_{n+1}^α be a submodel of M^α with $M_n^\alpha \cup \{\mathbf{b}_{\alpha,\rho} : \rho \in {}^{n+1}(\mu_\alpha)\} \subseteq M_{n+1}^\alpha$ and has cardinality μ_α. $M_{n+1}^\alpha \subset M_\alpha$ so is an atomic model and each q_ρ^α extends to an atomic type over M^α so $M_n^\alpha \mathbf{c}_{\alpha,\rho}$ is an atomic set.

For $\rho \in {}^n(\mu_\alpha)$ and $i < \mu_\alpha$ first define
$$p'_{\rho\frown i} = q_\rho^\alpha \cup \{\phi_\rho(x, \mathbf{b}_{\alpha,\rho\frown j}) : j < i\} \cup \{\neg\phi_\rho(x, \mathbf{b}_{\alpha,\rho\frown i})\}.$$

Since $\lambda_\alpha < |\{r \in S_i(M^\alpha) : p'_\rho \subseteq r\}|$, we can find $p_{\rho\frown i}^\alpha \in S_i^m(M_n^\alpha)$ extending $p'_{\rho\frown i}$ such that $\mathcal{P}_{\rho\frown i} = \{r \in S_i(M^\alpha) : p_{\rho\frown i} \subseteq r\}$ has cardinality $> \lambda_\alpha$. Note that
$$p_{\rho\frown i}^\alpha \supseteq q_\rho^\alpha \cup \{\phi_\rho(x, \mathbf{b}_{\alpha,\rho\frown j}) : j < i\} \cup \{\neg\phi_\rho(x, \mathbf{b}_{\alpha,\rho\frown i})\}.$$

This completes the $n+1$st stage of the construction. So we can construct the M_n^α and $\{q_{\nu,i} : \nu \in {}^{<\omega}\mu_\alpha\}\rangle$, $\hat{M}^\alpha$ and by μ_α^+-saturation choose $\mathbf{c}_{\alpha,\rho} \in \hat{M}^\alpha$. Note

$$\{\phi_\rho(\mathbf{c}_{\alpha,\rho\hat{\ }i}, \mathbf{b}_{\alpha,\rho\hat{\ }j}) : j < i\} \cup \{\neg\phi_\rho(\mathbf{c}_{\alpha,\rho\hat{\ }i}, \mathbf{b}_{\alpha,\rho\hat{\ }i})\}. \tag{3}$$

We expand τ to a language $\tau_\Phi \supset \tau$ by adding predicates $P, P_n, <, <^*, R$ and Skolem functions. We add Skolem axioms to T to get a theory T_1 that admits quantifier elimination, requiring that these Skolem functions applied to elements of P give an element of P so that P will pick out an elementary submodel.

Let M_α^+ be a model of T (submodel of $\hat{M}^\alpha$) with cardinality μ_α containing M_n^α for $n < \omega$ and all the $\mathbf{c}_{\alpha,\rho}$ Assign the Skolem functions so that M_α^+ is generated by the $\mathbf{b}_{\alpha,\rho}$ for $\rho \in {}^{<\omega}\mu_\alpha$. Interpret $<$ as the partial order on the $\langle \mathbf{b}_{\alpha,\rho} : \rho \in {}^{<\omega}(\mu_\alpha)\rangle$ given by inclusion on the indices. Let $<^*$ be a linear order of the $\langle \mathbf{b}_{\alpha,\rho} : \rho \in {}^{<\omega}(\mu_\alpha)\rangle$ given by lexiocographic order on the indices.

Interpret P as the model $\bigcup_{n<\omega} M_n^\alpha$ and R as

$$\{e\hat{\ }\mathbf{c}_{\alpha,\rho} : \rho \in {}^{<n}(\mu_\alpha), e \in \bigcup_{n<\omega} M_n^\alpha\}.$$

Further add predicates P_n and function symbols F_n. Let P_n denote M_n^α. Define $F_n(\mathbf{b}_{\alpha,\rho}) = \mathbf{c}_{\alpha,\rho}$.

Now let T_1 be the collection of all $L(\tau_\Phi)$-sentences that are true in each $\hat{M}_\alpha$.

Observe:

Claim 3.6 *For each n, the following informal assertion can be formalized as a sentence in $L(\tau_\Phi)$:*

For any finite linearly ordered initial $<$-segment of the tree with length n, $\mathbf{x}_0, \ldots \mathbf{x}_n$, (so $P_i(\mathbf{x}_i)$):

1. $\bigwedge_{i \le n} P_i(\mathbf{z}) \wedge \mathbf{z} <^* \mathbf{x}_i \wedge \phi_i(F_n(x_n), \mathbf{z})$
2. $\bigwedge_{i \le n} \neg\phi_i(F_n(x_n), \mathbf{x}_i)$.

The universal quantification of each such sentence is true in each $\hat{M}_\alpha$ and so is in T_1.

Let Γ be the collection of types:

1. $\mathcal{P}_n = \{\bigwedge_{i<n} P(x_i)\} \cup \{q(\mathbf{x}) : q$ is a non-principal n-type $\}$
2. $\mathcal{Q}_n = \{\bigwedge_{i<n} R(x_i, y)\} \cup \{q(\mathbf{x}, y) : q$ is a non-principal $n+m$-type $\}$

Now apply the omitting types theorem (as stated Section 4) to the τ_Φ-theory T_1 and the collection of M_α^+ to get a countable set of tree-indiscernibles in order type ${}^{<\omega}\omega$ and an extension Φ of T_1, (the EM-template) such that Φ is realized in each M_α^+ and such that for every tree of J of order ${}^{<\omega}\lambda$, $EM_\tau(J, \Phi) \models T_1$ and omits Γ.

Finally we must show:

Claim 3.7 *If $\lambda^\omega > \lambda$ then there is an I with $|I| = \lambda$ such that $S^m_{\rm at}(M_I) > \lambda$, where $M_I = EM(I,\Phi) \restriction P$.*

Note that by 3 and Claim 3.6 we have:

1. If $\rho, \hat{\rho\, i} \in I$ $\mathrm{tp}(F_n(\rho)/P_n(M)) \subseteq \mathrm{tp}(F_{n+1}(\hat{\rho\, i})/P_{n+1}(M))$.
2. If $\rho \in I$ and $i \neq j$,

$$\mathrm{tp}(F_{n+1}(\hat{\rho\, j})/P_{n+1}(M)) \neq \mathrm{tp}(F_{n+1}(\hat{\rho\, i})/P_{n+1}(M)).$$

Now in any $M_I = EM(I,\Phi)$ for any $\rho \in J$ define $p_\rho \in S^m_{\rm at}(P_N(M_I) = \mathrm{tp}(F_n(\rho), P_n(M))$. Now letting $p_\eta \in S^m_{\rm at}(P(M_I))$ be $\bigcup_{i<\omega} p_{\eta\restriction n}$, we find λ^ω members of $S^m_{\rm at}(P(M_I))$. The definition of $S^m_{\rm at}$ guarantees the union is in $S^m_{\rm at}$.

$\Box_{3.7}$

4 Tree Indiscernibility

The theorem reported here is implicit in the literature (e.g. [She78, GS86]) but we could not find an explicit statement. Theorem VII.3.6 of [She78] finds an indiscernible tree in the first order case but we want to omit types as well. The basic plan of the proof dates to Morley [Mor65] and, as in [Bal], we follow the clear exposition of this plan in [Mar02], indicating the modifications needed for the more complicated combinatorics to build models over indiscernible trees to omit types instead of over linear orders.

Many variants of tree indiscernibles are used in various parts of model theory; we sketch the contexts to point out where the current version lies.

In τ	linear order	$2^{<\omega}$	$\lambda^{<\omega}$
In τ_Φ	linear order	$2^{<\omega}$	$\lambda^{<\omega}$

Indiscernibles may be ordered by linear orders, or trees of the form $2^{<\omega}$, $\lambda^{<\omega}$ or even $2^{\leq\omega}$, $\lambda^{\leq\omega}$. We may want to find find the ordering in the basic language (to witness unstability at some level) or not (to avoid introducing instability). Ehrenfeucht and Mostowski did not introduce the order to the base language (so second row) and built the tree over a linear order (first column). In his constructions of many models of unstable theories, Shelah is in the first column row, second row. To investigate stable but not superstable theories, we want the first row, third column. The exposition here differs from [She78] because in working with $L_{\omega_1,\omega}$, we must omit types. The use of trees indexed by $2^{\leq\omega}$ to construct many models in $\aleph_1$ if a countable theory is not ω-stable appears in [She78]. Shelah has many extensions and some for uncountable languages occur in [Bal89]. There are further applications to two-cardinal models and to Peano arithmetic. Tree indiscernibles on $2^{<\omega}$ rely on Halpern-Lauchli etc; tree indiscernibles on $2^{<\omega}$ rely on Erdos-Rado.

We first establish some background notation.

Notation 4.1 *1. A* tree *is $\boldsymbol{T} \subset {}^{\leq\omega}\lambda$ be a subtree of ${}^{\leq\omega}\lambda$ that is closed under initial segment.*

2. atp *means atomic (quantifier-free) type.*

3. *The vocabulary τ^* will denote the language of trees we use. It contains the partial order on the tree, $<$, the lexiocographic order on the tree $<^*$, and the levels P_n.*

4. *When elements $\boldsymbol{a}_\eta$ and $\boldsymbol{a}_\tau$ in a structure M are indexed by a tree and $\eta, \tau \in \boldsymbol{T}$ realize the same quantifier free τ^*-type in the tree $\boldsymbol{a}_\eta$ then $\boldsymbol{a}_\tau$ have the same length.*

5. *If ν is an n-element sequence from $\boldsymbol{T}$, $\boldsymbol{a}_\nu$ denotes $\langle \boldsymbol{a}_{\nu(0)}, \ldots \boldsymbol{a}_{\nu(n-1)} \rangle$.*

Definition 4.2 *For any vocabulary τ, let M be a τ-structure and Σ a set of τ-formulas. $\langle \boldsymbol{a}_\eta : \eta \in \boldsymbol{T} \rangle \subset M$ is a set of* Σ-tree indiscernibles*:*
if $\mathrm{atp}_{\tau^*}(\eta/\emptyset) = \mathrm{atp}_{\tau^*}(\nu/\emptyset)$ *then* $\mathrm{tp}_\Sigma(\boldsymbol{a}_\tau/\emptyset) = \mathrm{tp}_\Sigma(\boldsymbol{a}_\tau/\emptyset)$ *in* M.
We just say tree indiscernibles *if Σ is all formulas in $L(\tau)$.*

We rely on a combinatorial lemma. The result is proved as Theorem 2.6 in the appendix to [She78]. A stronger result (the bound on $k(m,n)$ is smaller) with a shorter proof appears in the appendix of [GS86].

Lemma 4.3 ([She78]) *For every $n, m < \omega$, there is a $k(n,m) < \omega$ such that if $\lambda = \beth_k(\chi)^+$ the following is true. For any function $f : [{}^{\leq n}\lambda]^m \to \chi$, there exists a $\boldsymbol{T} \subseteq {}^{\leq n}\lambda$ such that*

1. *Each $\eta \in \boldsymbol{T}$ has χ^+ immediate successors in $\boldsymbol{T}$.*

2. *If ν and τ are m-tuples from $\boldsymbol{T}$ with $\mathrm{atp}_{\tau^*}(\eta/\emptyset) = \mathrm{atp}_{\tau^*}(\nu/\emptyset)$, then $f(\tau) = f(\eta)$.*

We now prove the theorem on the existence of tree-indiscernibles. In order to be clear about the definability of the tree in the original language we are quite pedantic about the vocabularies involved. Note that in the basic Ehrenfeuct-Mostowski argument the language ($<$) analogous to τ^* here is only a tool and the ordering is not definable in the final structure; in this context the tree is 'definable' in the final structure.

Notation 4.4

1. *τ is the vocabulary of the theory T we actually want to study; a set Γ of τ-types is omitted.*

2. *τ^* is a language $(<, <^*, P_n)$ of trees.*

3. *τ_Φ includes both τ and τ^* and includes Skolem functions. The Skolem axioms and relations with crucical τ-formulas are axiomatized in T_1.*

4. *The set of constants C which guarantee the consistency of the order are added to τ_Φ.*

5. *$\Phi \subset L(\tau_\Phi \cup C)$ is the template or diagram of the tree-indiscernibles.*

Recall that $\mu_\alpha = \beth_\alpha(|T|)$. Writing μ_α rather than $\beth_\alpha$ and considering M_α for $\alpha < \delta(T) = (2^{|T|})^+$ is the price for dealing with uncountable T.

Theorem 4.5 *Let T_1 be a theory with Skolem functions in a vocabulary τ_Φ. Suppose for $\alpha < \delta(T)$, there exists a model M_α of M_1 with $|M_\alpha| \geq \mu_\alpha$ such that M_α omits a family Γ of τ-types. τ_Φ contains the vocabulary τ^* and $<, <^*, P_n$ define a tree of type ${}^{<\omega}\beth_\alpha$ in M_α.*

Then, there is a countable set of tree indiscernibles $C = \angle \mathbf{c}_\tau : \tau \in I$ with I of order type ${}^{<\omega}\omega$ such that the diagram of C is realized in each M_α and an extension Φ of T such that for every tree J of the form ${}^{<\omega}\lambda$, $EM_\tau(J, \Phi) \models T$ and omits Γ.

Proof. After expanding the language τ_Φ with new constants $\langle c_\rho : \rho \in {}^{<\omega}\omega\rangle$, we need to demonstrate the consistency of the following families of sentences.

1. $c_\rho \neq c_\eta$ if $\rho \neq \eta$.
2. For each τ_Φ-formula $\phi(\mathbf{v})$, for each quantifier-free τ^*-type r. If η, ν both realize r,
$$\phi(\mathbf{c}_\nu) \equiv \phi(\mathbf{c}_\eta).$$
3. For each $p \in \Gamma$, for each τ_Φ-term t and each quantifier-free τ^*-type r, there is a ϕ_p, such that if ν realizes r
$$\neg\phi(t(\mathbf{c}_\nu)).$$
(This should be vectorized as well if p is a type in several variables.)
4. If a ψ is the universal quantification of a τ_Φ-formula $\chi(\mathbf{x}_1, \ldots \mathbf{x}_n)$ that is true in all M_α then $\chi(\mathbf{c}_1, \ldots \mathbf{c}_n) \in \Phi$.

We begin with pairs (M_α, X_α), a model M_α and a subset $X_\alpha = \{\boldsymbol{a}_\tau : \tau \in \boldsymbol{T}_\alpha\}$ which contains a sufficiently large tree. We show how to choose (M'_α, X'_α) with subset $X'_\alpha = \{\boldsymbol{a}_\tau : \tau \in \boldsymbol{T}'_\alpha\}$ to satisfy the properties specified in the following claims. In particular, they satisfy the properties of the original (M_α, X_α).

Claim 4.6 *Let S_n be the collection of τ_Φ-types over the empty set which are realized in $\bigcup_{i \leq n} P_n(M_\alpha)$. The sequence (M'_α, X'_α) has the property that for each α:*

If $\eta, \nu \in X'_\alpha$ both realize the same quantifier-free τ^*-type r, then for each $\phi \in \tau \cup \{<, <^*\} \cup \{P_i : i \leq n\}$,

$$\phi(\mathbf{c}_\eta) \equiv \phi(\mathbf{c}_\nu). \tag{4}$$

Moreover, $(Y_\alpha, <, <^*) \approx {}^{\leq n}\beth_\alpha$.

Proof. Let $N_\alpha = M_{\alpha+k}$ where $k = k(m,n)$. Let $f : [X_\alpha]^m \to S_n$, where $f(\nu) = s$ if $\mathrm{tp}_{\tau_\Phi}(\boldsymbol{a}_\nu) = s$. Now by Lemma 4.3, there is a X'_α (contained in $\bigcup_{i<n} P_i(M_\alpha)$) and with $(X'_\alpha, <, <^*) \approx {}^{\leq n}\beth_\alpha$ and (4) is true on X'_α.

$\square_{4.6}$

We omit the proof of the next claim which is obtained from Lemma 4.3 in a way similar to Claim 4.6.

Claim 4.7 *Let $\mathcal{T}$ be the collection of terms $t(\mathbf{x}) \in \tau_\Phi$ with at most n-free variables. Suppose $\langle \phi_i : i < |\tau_\Phi| \rangle$ be an enumeration of the formulas in $L\tau_\Phi$). Finally, S is the collection of functions: $\langle t, p \rangle \mapsto \phi_{s(t,p)}$ from $\mathcal{T} \times \Gamma$ to $L(\tau_\Phi)$ such that $\phi_{s(t,p)} \in p$. For each α the sequence (M'_α, X'_α) has the property:*

For each $p \in \Gamma$ and each quantifier-free τ^*-type r, if $\eta \in Y_\alpha$ realizes r

$$M'_\alpha \models \neg\phi_{s(t,p)}(t(\boldsymbol{a}_\eta)).$$

Moreover, $(X'_\alpha, <, <^*) \approx {}^{<\omega}\beth_\alpha$.

Now we describe the actual construction of Φ checking its finite consistency. Let Φ_0 include all τ_Φ sentences true in all M_α and the assertion that the c_ρ are distinct. We construct Φ and a sequence of pairs (M^n_α, X^n_α) by an induction of length ω.

Let Σ_n contain all formulas of τ_Φ with at most n free variables.

At odd stages, we want to guarantee indiscernibility: Each quantifier free τ^*-type is homogenous for each formula ϕ. At stage $2n+1$, we apply Claim 4.6 to $(M^{2n}_\alpha, X^{2n}_\alpha)$ to obtain M^n_α $(M^{2n+1}_\alpha, X^{2n+1}_\alpha)$ and we define Φ_{n+1}. We also refine (and rename for convenience) the index set of ordinals to guarantee that for all α, each τ^*-type in S_n is given the same truth value is for all tuples from Y_α realizing r. This assignment gives us Φ_{n+1}. We can do this because at any stage, the number of Σ_n-theories is at most $2^{|T|}$ which is not cofinal in $(2^{|T|})^+$. (For countable T we could work one formula at a time and $\beth_{\omega_1}$ is sufficient rather than $(2^{|T|})^+$.) Note that as n increases in this induction, the indiscernibility is being insured for larger P_n. Since the Σ_n are increasing this results in a consistent theory Φ giving tree-indiscernibility in $L(\tau_\Phi)$.

On even stages we similarly guarantee the omission of Γ by at stage $2k$ attending to the terms of τ_Φ with at most n arguments and getting indiscernibility with respect to an initial segment of the tree (i.e on $\langle P_j : j < m \rangle$ for some m).

$\square_{4.5}$

References

[Bal] John T. Baldwin. *Categoricity*. Number 51 in University Lecture Notes. American Mathematical Society. 2009, www.math.uic.edu/˜ jbaldwin.

[Bal89] J. Baldwin. Diverse classes. *The Journal of Symbolic Logic*, 54:875–893, 1989.

[GS86] R. Grossberg and Saharon Shelah. A non structure theorem for an infinitary theory which has the unsuperstablity property. *Illinois J Math*, 30:364–390, 1986.

[Hod93] W. Hodges. *Model Theory*. Cambridge University Press, 1993.

[Kei76] H.J. Keisler. Six classes of theories. *J. Australian Math. Soc. Ser. A*, 25:257–266, 1976.

[Mar02] D. Marker. *Model Theory: An introduction*. Springer-Verlag, 2002.

[Mor65] M. Morley. Omitting classes of elements. In Addison, Henkin, and Tarski, editors, *The Theory of Models*, pages 265–273. North-Holland, Amsterdam, 1965.

[She70] S. Shelah. Finite diagrams stable in power. *Annals of Mathematical Logic*, 2:69–118, 1970.

[She78] S. Shelah. *Classification Theory and the Number of Nonisomorphic Models*. North-Holland, 1978.

[She83a] S. Shelah. Classification theory for nonelementary classes. I. the number of uncountable models of $\psi \in L_{\omega_1\omega}$ part A. *Israel Journal of Mathematics*, 46:3:212–240, 1983. paper 87a.

[She83b] S. Shelah. Classification theory for nonelementary classes. I. the number of uncountable models of $\psi \in L_{\omega_1\omega}$ part B. *Israel Journal of Mathematics*, 46;3:241–271, 1983. paper 87b.