Stable Generic Structures

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1 Overview

We describe the construction of a model M which depends on a collection of finite structures $\mathbf{K}_0$ and a notion of strong submodel $\leq$. The definition of strong submodel depends in turn on a 'dimension function' δ which assigns a real number to each finite structure.

In this paper we construct for each real α between 0 and 1 and each finite relational language L, a complete stable theory T_α. Based on this result, [6] showed that the almost sure theory of finite graphs with e dge probability $n^{-\alpha}$ (for irrational α between 0 and 1) is stable. In the process we give the first treatment of classes which do not admit finite closures. We describe here a set of axioms for the dimension function δ which all ows a common treatment of the $\aleph_1$-categorical and the strictly stable examples. But this common rubric obscures the fact that quite different methods are used to

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prove amalgamation in the two cases. Our analysis of the noninterpretability of groups clarifies this dist inction. Our axiomatization covers several other variations which are discussed in the conclusion to this paper.

In the early 1970's $2^{\aleph_0}$ distinct $\aleph_0$-categorical theories were constructed by variations on the Fraïssé-Jónsson construction of a countable homogeneous universal model [16], [12], [14]. But these theories were all unstable and for reasons that appeared to be intrinsic to the method.

Hrushovski showed how to vary the construction to obtain stable structures. He introduced the key notion of assigning a dimension to a finite structure in terms of the difference between the number of points in the structure and the total number of tuples in the distinguished relations on the structure. His innovation had two important consequences. First it is possible to control the stability class of the resulting structures and second such structures do not interpret a group.

In Section 2 we discuss the construction of homogeneous universal models in a slightly broader context than that of Fraïssé and Jónsson. We will first examine a class of finite structures $\mathbf{K}_0$ with a notion $\leq$ of *strong submodel*. We review the conditions necessary for the existence of a generic (homogeneous-universal) model and several necessary and sufficient conditions for that model to be ω-saturated. We conclude this section with several examples showing the formulation of this subject solely in terms of the notion of strong submodel is inadequate to understand the stability and saturation properties of the generic model.

We then in Section 3 address the stability of the generic model. To ensure stability we must consider a dimension function d. We give sufficient conditions for generic structures to be stable and for the notion of d-independence to coincide with nonforking. In Section 5 we show the noninterpretability of infinite groups into the structures defined by these methods. This turns out to be much simpler for the unrestricted case (see below) and illustrates the difference in the amalgamation procedures for the stable and $\aleph_1$-categorical cases. Previous work dealt only with classes that had finite closures (Definition 2.10) (or were natural limits of such classes [17]). In order to analyze classes without finite closures we need to investigate the concept of amalgamation over closed sets; we verify this hypothesis for an important case in Section 4.

Up to this point, we have worked with an arbitrary finite relational language. In Section 6, we investigate more specific results about stable pseudoplanes that arise when we restrict to a single binary relation.

The properties of generic structures constructed by this general method depend on two major factors: the choice of dimension function and the particular set of finite structures used for $\mathbf{K}_0$. For a major set of examples, the choice of the dimension function will depend only on a real parameter α. Having fixed such an α, we call $\mathbf{K}_\alpha$ the class of all finite structures (for a given language) whose dimension is hereditarily nonnegative; this is the *unrestricted* case. *Restricted* cases arise by considering a subclass $\mathbf{P_0}$ of $\mathbf{K}_\alpha$. We will examine the effects of varying each of these parameters.

In addition to the original work of Hrushovski ([19], [18]), this account was influenced by the following papers [21], [25], [17], [3], [15]. It contains material from both [24] and [1]. The general approach depends on [7] and earlier work by Shelah, e.g. [22]. Chris Laskowski and David Kueker have been very helpful both in general conversation and with specific suggestions. We would like to thank the referee for an extremely careful reading that not only improved some arguments but forced the clarification of the general aims of the paper.

2 Strong Submodels

We work throughout with a finite relational language. Both 'finite' and 'relational' are important restrictions although much of the development could proceed with classes of algebras that are uniformly locally finite.

We are going to construct from a collection of finite structures a 'generic' model which will have interesting properties. This construction will be a variant on the Fraïssé-Jónsson construction. Initially we will only vary the notion of submodel. Later we will introduce an abstract dimension and a notion of 'stable amalgamation'.

2.1 Isomorphism Convention. Classes of structures will be denoted by $\mathbf{K}, \mathbf{K}_0, \mathbf{P}$ etc. Each such class of structures denoted by a bold Roman capital (possibly with subscripts) is assumed to be closed under isomorphism. Any binary relation $\leq$ on such a class will be closed under isomorphism in the sense that if A_i, $B_i \in \mathbf{K_0}$ and $A_i \subseteq B_i$ for $i = 1$, 2, and $f : B_1 \rightarrow B_2$ is an isomorphism such that $f \mid A_1$ is an isomorphism of A_1 onto A_2, then $A_1 \leq B_1$ if and only if $A_2 \leq B_2$.

2.2 Notation. i) We write AB for $A \cup B$.

ii) $A \subseteq_\omega B$ means A is a finite subset of B.

iii) By $\Re^+$ we mean the nonnegative real numbers.

iv) For any class of structures $\mathbf{K}$, $S(\mathbf{K})$ is the class of substructures of members of $\mathbf{K}$.

Suppose $\mathbf{K}_0$ is a class of finite structures. Let $\leq$ be a binary relation defined on $\mathbf{K}_0$ with the properties described in Axiom Group A. If $A \leq B$ we say A is a *strong* submodel of B; we also say A is *closed* in B.

First we give A1 through A5; axiom A6 will be introduced later. A class satisfying A1-A4 is essentially what [21] calls a smooth class. A5 is a technical convenience.

Axiom Group A
A1. If $M \in \mathbf{K}_0$ then $M \leq M$.
A2. If $M \leq N$ then $M \subseteq N$.
A3. If $A, B, C \in \mathbf{K}_0$, then $A \leq B \leq C \Longrightarrow A \leq C$.
A4. If $A, B, C \in \mathbf{K}_0$, $A \leq C$, $B \subseteq C$ and $A \subseteq B$ then $A \leq B$.
A5. $\emptyset \in \mathbf{K}_0$ and $\emptyset \leq A$ for all $A \in \mathbf{K}_0$.

2.3 Definition. Let $\mathbf{K}_0$ be a collection of finite structures and $\mathbf{P}$ a class of structures. $\mathbf{K}_0$ is *cofinal* in the structure $A \in \mathbf{P}$ if for each finite $A_0 \subseteq A$ there is an $A' \in \mathbf{K}_0$ with $A_0 \subseteq A' \subseteq A$. $\mathbf{K}_0$ is *cofinal* in $\mathbf{P}$ if for each $A \in \mathbf{P}$, $\mathbf{K}_0$ is cofinal in A.

2.4 Context. $\mathbf{K}_0$ is a class of finite structures satisfying A1-A5. $\overline{\mathbf{K}}_0$ is the class of all structures A with $\mathbf{K}_0$ cofinal in A. $\mathbf{P}$ is a subclass of $\overline{\mathbf{K}}_0$ containing $\mathbf{K}_0$.

2.5 Definition. For all A, $C \in \overline{\mathbf{K}}_0$, we say that A can be *embedded* in C if there is $f : A \longrightarrow C$, an isomorphism from A onto some $C' \subseteq C$. We say that A can be *strongly embedded* in C if there is $f : A \longrightarrow C$, an isomorphism from A onto some $C' \leq C$.

Now if $\mathbf{K}_0$ is cofinal in $\mathbf{P}$ we extend the notion of $\leq$ to $\mathbf{K}_0 \times \mathbf{P}$.

2.6 Definition. Let $\mathbf{K_0}$ be cofinal in $\mathbf{P}$. For $A \in \mathbf{K_0}$ and $B \in \mathbf{P}$ such that $A \subseteq B$, we say that A *is strong in* B if $A \leq B'$ for every $B' \in \mathbf{K_0}$ such that $A \subseteq B' \subseteq B$.

In dealing with this material Kueker and Laskowski [21] demanded that the notion of strong substructure be axiomatized by a universal type. Since in a finite relational language the atomic diagram of any structure is expressible by a single sentence, Definition 2.6 and our axioms can be quickly seen to imply the same definability condition (Lemma 2.8). Intuitively, this holds because Axiom A4 asserts that $\leq$ is preserved under substructure. This is a stronger requirement than the similar axioms in e.g. [7] which have the additional hypothesis that $B \leq C$.

2.7 Notation. Let C be a finite structure enumerated by $\overline{c}$. Then $\delta_C(\overline{c})$, the atomic diagram of C, denotes the conjunction of atomic sentences describing C. If $A \subseteq B$ and $\overline{a}$ enumerates A, $\overline{b}$ enumerates $B - A$, $\delta_B(\overline{a}, \overline{b})$ will denote the atomic diagram of B with the elements of A distinguished. When we write $\delta_B(\overline{x}, \overline{y})$ it is understood that $\overline{x}$ is substituted for $\overline{a}$ and $\overline{y}$ for $\overline{b}$.

2.8 Lemma. *Let* $A \in \mathbf{K}_0$ *and* $C \in \mathbf{P}$. *Then,* $A \leq C$ *if and only if* $C \models \Gamma_A(\overline{a})$*, where* $\Gamma_A(\overline{x})$ *is the following* Π_1*-type*

$$\{\forall \overline{y} \neg \delta_B(\overline{x}, \overline{y}) : A \subseteq C, B \in \mathbf{K}_0, A \not\leq B\}.$$

Proof. Note that for finite A, $A \leq C$ means that there is no finite B such that $A \subseteq B \subseteq C$ and $A \not\leq B$.

2.9 Definition. Suppose $\mathbf{K}_0$ is cofinal in M.

i) The class $(\mathbf{K}_0, \leq)$ satisfies the *amalgamation property* (AP) if whenever there are embeddings f_0, g_0 of A into B and C such that $\operatorname{rng} f_0 \leq B$ and $\operatorname{rng} g_0 \leq C$, then there are $D \in \mathbf{K}_0$ and embeddings f_1, g_1 of B, C into D with $\operatorname{rng} f_1 \leq D$, $\operatorname{rng} g_1 \leq D$ and $f_1 \circ f_0 = g_1 \circ g_0$.

ii) The class $(\mathbf{K}_0, \leq)$ of finite relational structures satisfies the *joint embedding property* (JEP) if for any $A, B \in \mathbf{K}_0$ there is a $C \in \mathbf{K}_0$ such that both A and B can be strongly embedded in C.

iii) M is $\mathbf{K}_0, \leq$-universal if every member of $\mathbf{K}_0$ can be strongly embedded in M.

iv) M is $\mathbf{K}_0, \leq$-homogeneous if whenever A and B are finite strong submodels of M that are isomorphic (by f) then there is an automorphism of M extending f.

Obviously, if a class satisfies A5 and the amalgamation property then $(\mathbf{K}_0, \leq)$ has JEP.

2.10 Definition. The structure $M \in \mathbf{P}$ *has finite closures* with respect to $(\mathbf{K}_0, \leq)$ if for every finite $A \subseteq M$ there is a finite $B \leq M$ with $A \subseteq B$. The class $\mathbf{P}$ *has finite closures* with respect to $(\mathbf{K}_0, \leq)$ if each member of $\mathbf{P}$ does.

The word 'closure' is being used in a somewhat ambiguous manner here. We have not asserted that there is a unique minimal closed set containing each finite A but only that there is some finite closed set containing A. The assumption that the entire class $\mathbf{K}$ has finite closures is quite strong but in fact will be easy to verify for many of our examples. We will consider later the impact of a further axiom A6 which implies that there is a unique closure.

We are using 'has finite closures' for a concept that is sometimes referred to as 'locally finite' [24]. But we reserve 'locally finite' for the more common meaning that every finitely generated structure is finite (of course this is trivially true in a relational language) and use 'finite closures' to mean the closure system is locally finite. Note that a countable model M has finite closures if and only if $M = \bigcup_{i<\omega} A_i$, where $A_0 \leq A_1 \leq A_2 \leq ...$ and each A_i is in $\mathbf{K}_0$.

2.11 Definition. The countable model $M \in \overline{\mathbf{K}}_0$ is $(\mathbf{K}_0, \leq)$-*generic* if

i) If $A \leq M, A \leq B \in \mathbf{K}_0$, then there exists $B' \leq M$ such that $B \cong_A B'$,

ii) M has finite closures.

It is easy to see (assuming A1-A5) that if the model M is $(\mathbf{K}_0, \leq)$-*generic* then M is $(\mathbf{K}_0, \leq)$-universal and $(\mathbf{K}_0, \leq)$-homogeneous. We have the following slight generalization of the Jónsson-Fraïssé construction.

2.12 Theorem. *If $(\mathbf{K}_0, \leq)$ is a collection of finite relational structures that satisfies A1-A5 and has the amalgamation property then there is a countable $(\mathbf{K}_0, \leq)$-generic model M.*

Since we have restricted ourselves to a finite relational language, $\mathbf{K}_0$ is cofinal in the class of finite substructures of models of Th(M). Some important properties of the class $(\mathbf{K}_0, \leq)$ and some other definitions are required for further discussion. The following condition makes sense only if $\mathbf{K}_0$ is closed under substructure and we make that assumption in Convention 2.15.

2.13 Axiom A6. Let $A, B, C \in \mathbf{K}_0$. If $A \leq B$, $C \subseteq B$ then $A \bigcap C \leq C$.

2.14 Future Considerations. The exact formulation of the general theory appropriate for constructions of the sort considered in this paper is problematic. Both Baldwin and Shelah [6] and Baudisch [8] deal with classes $\mathbf{K}_0$ which are not closed under substructure. But it easy to use a dimension function on $S(\mathbf{K}_0)$ to define a notion of strong submodel that satisfies A6 on $S(\mathbf{K}_0)$ even though it does not hold on $\mathbf{K}_0$. The class $\mathbf{K}_0$ may satisfy the amalgamation property although $S(\mathbf{K}_0)$ does not. For the purposes of this paper we restrict our generality as follows.

2.15 Convention. Except where other assumptions are explicitly stated, $\mathbf{K}_0$ denotes a class of finite relational structures closed under isomorphism and substructure which satisfies A1 through A6 and the amalgamation property. M denotes the countable generic model and $\mathbf{K} \subseteq \overline{\mathbf{K}}_0$ denotes the class of all models of $T = \mathrm{Th(M)}$.

2.16 Definition. Let $A, B \in \mathbf{K}_0$. We say (A, B) is a *minimal pair* if $A \subseteq B$, A is not a strong substructure of B but for every proper substructure B' of B, $A \leq B'$.

We extend the definition of $\leq$ to $\overline{\mathbf{K}}_0 \times \overline{\mathbf{K}}_0$ as follows.

2.17 Definition. For $A, B \in \overline{\mathbf{K}}_0$, $A \leq B$ if and only if for every minimal pair (A', B') with $A' \subseteq B' \subseteq B$, if $A' \subseteq A$ then $B' \subseteq A$.

Using A6 we see that $\leq$ defined in this way is the same as the given relation when restricted to $\mathbf{K_0}$ and also that this definition is consistent with Definition 2.6.

Note that Axiom A6 now immediately extends to infinite A, C and $A \cap C$.

The following result was pointed out in [25] and [17].

2.18 Lemma. *The following are equivalent.*

i) There does not exist an infinite increasing chain of minimal pairs in $\mathbf{K}_0$.

ii) $\mathbf{K}$ *has finite closures.*

iii) Every ω*-saturated member of* $\mathbf{K}$ *has finite closures.*

iv) Some ω*-saturated member of* $\mathbf{K}$ *has finite closures.*

Proof. i) implies ii) implies iii) implies iv) is obvious. But iv) implies i) since if there is an infinite increasing chain of minimal pairs in $\mathbf{K}_0$ it can be embedded in any ω-saturated member of $\mathbf{K}$.

The following corollary is immediate.

2.19 Corollary. *If the generic model* M *is saturated then* $\mathbf{K}$*, the class of models of* Th(M)*, has finite closures.*

We now present two conditions which are equivalent to the generic M being ω-saturated when $\mathbf{K} = \mathrm{mod}(\mathrm{Th}(\mathrm{M}))$ has finite closures.

2.20 Definition. The pair $(\mathbf{K}_0, \leq)$ is said to have *amalgamation over closed sets* if for any infinite models $N_0, N_1 \in \mathbf{K}$, any set A in $S(\mathbf{K})$, if there are strong embeddings $f : A \to N_0$ and $g : A \to N_1$, then there are $S \models T$ and elementary embeddings $f' : N_0 \to S$, $g' : N_1 \to S$ such that $f'f = g'g$. If this condition holds for all $A \in \mathbf{K}_0$ then $(\mathbf{K}_0, \leq)$ is said to have *amalgamation over finite closed sets.*

An easy compactness argument shows that if T has finite closures then amalgamation over finite closed sets implies amalgamation over arbitrary closed sets. However, in the absence of finite closure, amalgamation over closed sets is strictly weaker than the existence of a countable saturated model (See Example 2.34).

The following theorem depends only on Axioms A1-A5.

2.21 Lemma. *Let* M *be* $(\mathbf{K}_0, \leq)$*-generic and* $T = \mathrm{Th}(\mathrm{M})$*. Assume* $\mathbf{K} = \mathrm{mod}(T)$ *has finite closures. The following are equivalent.*

i) M *is* ω*-saturated*

ii) T *has amalgamation over finite closed sets.*

iii) M is ω_1-universal.

Proof. i) implies ii). Fix strong embeddings $f : A \rightarrow N_0$ and $g : A \rightarrow N_1$. By compactness we may assume N_0 and N_1 are countable. Since M is saturated there are elementary embeddings f_1 and g_1 of N_0 and N_1 into M. But then, $f_1 f A$ and $g_1 g A$ are isomorphic strong submodels of M. So there is an automorphism h of M mapping $f_1 f A$ to $g_1 g A$. Now g_1 and hf_1 witness that M is the required amalgam of N_0 and N_1. ii) implies iii). Let N be a countable model of T. We claim that there is an elementary map from N into M. Let $N = \bigcup A_i$ such that $\langle A_i : i < \omega \rangle$ is a $\leq$-increasing chain of finite closed sets. By genericity we can successively strongly embed the A_i into M. Let N' be the union of the images of A_i. We claim $N' \prec M$. It suffices to show that for any $\overline{a} \subseteq N'$, $(N', \overline{a}) \equiv (M, \overline{a})$. Note that since the class $(\mathbf{K}, \leq)$ has finite closures, $\overline{a}$ can be extended to some finite $\overline{b}$ which is an enumeration of the image of an A_i and is closed in both N' and M. Thus since T has amalgamation over finite closed sets, $(N', \overline{b}) \equiv (M, \overline{b})$, and so $(N', \overline{a}) \equiv (M, \overline{a})$. iii) implies i). Let A be a finite subset of M and $p \in S(A)$. Then p is realized by some b in a countable elementary extension N of M. There is an elementary embedding f of N into M. Note, fA and A are isomorphic strong submodels of M so there is an automorphism g of M mapping fA to A. Then, $gf(b)$ is the required realization of p.

Kueker and Laskowski [21] prove the stronger result that if the generic structure M is weakly saturated then M is saturated.

Using Axiom A6, we can show the generic is homogeneous. Returning to the convention established by Paragraph 2.15 we make the following definition which is justified by the immediately following lemma.

2.22 Notation. Let $\mathbf{K}$ have finite closures. For $N \in \mathbf{K}$ and $X \subseteq N$ the least set containing X that is strong in N is denoted $\mathrm{icl_N(A)}$ or $\overline{A}$, if the ambient model is clear, and is called the intrinsic closure in N of A.

2.23 Theorem. *Let $\mathbf{K}$ have finite closures. For all $N \in \mathbf{K}$ and each finite $A \subseteq N$, amongst the finite sets X such that $A \subseteq X \leq N$ there is one $\overline{A}$, which is least with respect to inclusion. Thus, $\overline{A}$ is contained in the algebraic closure of A in N.*

Proof. If A_1 and A_2 are extensions of A that are strong in N, their intersection is also strong in N by A6. So there is a unique substructure of N extending A with minimal cardinality that is strong in N. Let N_1 be an ω-saturated elementary extension of N. If $p = \mathrm{tp}(\overline{A}/A)$ is not algebraic, there is a realization A' of p in N_1 other than $\overline{A}$. B ut $\mathrm{icl}_N(A) = \mathrm{icl}_{N_1}(A) = \overline{A}$ so this contradicts the uniqueness of $\mathrm{icl}_{N_1}(A)$.

Although this argument uses the fact that **K** has finite closures, we establish the conclusion of Lemma 2.23 in a more general situation by other methods in Section 3.

2.24 Corollary. *With the same assumptions, the generic model M is homogeneous.*

Proof. Note that by definition the generic model has finite closures. If two finite sequences in M realize the same type they must have isomorphic algebraic closures. In particular, their closures in the sense of $\mathbf{K}_0$ are isomorphic. But by genericity, this isomorphism extends to an automorphism.

2.25 Definition. i) $A \prec_1 B$ if $A \subseteq B$ and for every $\overline{a} \in A$ and every universal formula $\phi(\overline{x})$, $A \models \phi(\overline{a})$ if and only if $B \models \phi(\overline{a})$.

ii) T is 1-model complete if $A, B \models T$ then $A \prec_1 B$ implies $A \prec B$.

Standard methods show that if the generic model is saturated then $T = \mathrm{Th}(M)$ is 1-model complete. Kueker and Laskowski [21] show a converse: if T is 1-model complete, every countable model can be embedded as a 1-substructure of a model with finite closures, and the generic realizes every consistent universal type, then M is saturated. We have shown that if **K** has finite closures then amalgamation over finite closed sets implies 1-model completeness. Must a class with finite closures that is 1-model complete have amalgamation over finite closed sets?

In this paper we establish the stability of the theories T_α (for each real α) of certain generic structures M_α without determining the quantifier complexity of T. In fact, Baldwin and Shelah [6] have modified the argument of [23] to show that each theory T_α is 1-model complete.

2.26 Definition. We say A is n-strong in B, written $A \leq_n B$, if for any B' with $A \subseteq B' \subseteq B$ and $|B' - A| \leq n$, $A \leq B'$.

For each finite A there is a universal formula $\theta_{n,A}(\overline{x})$ which holds of $\overline{a}$ in a structure N just if $\overline{a}$ enumerates a structure isomorphic to A which is n-strong in N. Let $\theta_n(\overline{x})$ denote the disjunction of these over the structures A with cardinality n. So $N \models \theta_n(\overline{a})$ just if $\overline{a}$ enumerates an n-strong substructure of N.

2.27 Definition. $(\mathbf{K}_0, \leq)$ has the *uniform amalgamation property* (u.a.p.) if the following condition holds for every $A \leq B \in \mathbf{K}_0$. For every $m \in \omega$ there is an $n = f_B(m)$ such that if $A \leq_n C$ then there is a D, a strong embedding of C into D and an m-strong embedding of B into D that complete a commutative diagram with the given embeddings of A into B and C.

If $(\mathbf{K}_0, \leq)$ has the uniform amalgamation property then for each B with $A \leq B$ and each m there is an $n = f_B(m)$ such that any n-strong embedding g of A into the generic model M extends to an m-strong embedding of B into M. (Since M has finite closures there is a finite E with $gA \leq_n E \leq M$. By uniform amalgamation find D with $B \leq_m D$ and $E \leq D$. By the definition of generic D can be embedded into M over E.) We can write the uniform amalgamation property syntactically: For all A, $B \in \mathbf{K_0}$ such that $A \leq B$ and all m,

$$M \models \forall \overline{x} \left(\theta_{f_B(m)}(\overline{x}) \wedge \delta_A(\overline{x}) \rightarrow (\exists \overline{y}) \left(\theta_m(\overline{x}, \overline{y}) \wedge \delta_B(\overline{x}, \overline{y}) \right) \right). \qquad (1)$$

As pointed out by Herwig, Poizat, Wagner if $\mathbf{K}$ has finite closures then $(\mathbf{K}_0, \leq)$ has uap iff M is ω-saturated. We prove one direction of this equivalence.

2.28 Theorem. *If $(\mathbf{K}_0, \leq)$ has the uniform amalgamation property and $\mathbf{K}$ has finite closures then M is ω-saturated.*

Proof. We know there is an ω-saturated model N (possibly uncountable) that has finite closures. We will show that N is $L_{\infty,\omega}$-equivalent to M and hence M is ω saturated. It suffices to construct a back and forth system (see e.g. [20, pages 22 and 53]. Let $\mathbf{I} = \{p | p{:}A \approx A', A \leq M, A' \leq N, |A| < \omega\}$. If $A' \leq B' \leq N$ for a finite B', the map p^{-1} extends to B' by the definition of generic. Now suppose $A \leq B \leq M$. We show the type

$$\{\delta_B(\overline{a}', \overline{y})\} \cup \{\theta_m(\overline{a}', \overline{y}) : m < \omega\}$$

is consistent. Then by ω-saturation of N, the appropriate B' can be chosen. But the consistency is clear since $N \models \theta_{f_B(m)}(\overline{a}') \wedge \delta_A(\overline{a}')$ for each n and $M \equiv N$ so

$$N \models (\forall \overline{x}) \left(\theta_{f_B(m)}(\overline{x}) \wedge \delta_A(\overline{x}) \rightarrow (\exists \overline{y}) \delta_B(\overline{x}, \overline{y}) \wedge \theta_m(\overline{y}) \right).$$

2.29 Notation. Let $B \cap C = A$. The *free amalgam* of B and C over A, denoted $B \bigotimes_A C$, is the structure with universe BC but no relations not in B or C.

2.30 Definition. B is a *primitive extension* of A if $A \leq B$ but there is no B_0 properly between A and B with $A \leq B_0 \leq B$.

Noticing that if $A \leq B$ and $A \leq_{|B|-|A|+m} C$, implies $B \leq_m B \otimes_A C \in \mathbf{K}_0$ we formulate the following variant of the amalgamation property. This condition is slightly weaker than the property established as the main step in the proof of the amalgamation in the strongly minimal case (as expounded e.g in [3]).

2.31 Definition. $(\mathbf{K}_0, \leq)$ has the *sharp amalgamation property* if for every A, B, C in $\mathbf{K}_0$ with $A \leq B$ and $A \leq_{|B|-|A|} C$, if B is a primitive extension of A, then either $B \otimes_A C \in \mathbf{K}_0$ or there is a strong embedding of B into C over A.

Note that any one-point extension must be primitive. It is now straightforward to prove by induction that

2.32 Proposition. *If* $(\mathbf{K}_0, \leq)$ *has the sharp amalgamation property then* $(\mathbf{K}_0, \leq)$ *has the uniform amalgamation property with* $f_B(m) = m + |B - A|$.

In this section we have discussed the construction of a generic model in term of a notion of strong submodel. We have given several conditions for the generic model to be ω-saturated. We have not considered the stability class of the structure. In order to do so, we introduce in Section 3 a dimension function defined on finite structures.

A necessary and sufficient condition for ω-saturation of the generic is 'uniform amalgamation'. Can one find a class with finite closures where the generic is not ω-saturated? For example can the class $\mathbf{K}_0$ be derived from a rational dimension function as in Section 3 and have the amalgamation property but not the uniform amalgation property?

In the following four examples we show that with slight variations on the notion of strong substructure we can create generic models with quite different properties. Convention 2.15 is suspended for the presentation of these examples which show both the necessity of assuming $\mathbf{K}$ has finite closures in Lemma 2.28 and the difficulty in establishing stability if we only invoke the notion of strong submodel. In these examples we say two points u, v in a graph are *connected* if there exists a path from u to v. In all three of these examples each class is the closure under isomorphism of the structures specifically listed. The various examples are derived from (expansions of) (Z, S), the structure of the integers with the successor relation.

2.33 Example. Let S be the adjacency relation on Z defined by $S(x, y)$ holds if and only if x and y are two consecutive integers.

Let $\mathbf{K}^1 = \mathrm{mod}(\mathrm{Th}(Z, S))$ and $\mathbf{K}_0^1$ be the collection of finite substructures of members of $\mathbf{K}^1$. Define $A \leq^1 B$ to mean $A \subseteq B$ and points not connected in A are still not connected in B.

It is easy to see that Axioms A1 through A6 are true in the class $(\mathbf{K}_0^1, \leq^1)$.

Note that any element A in $\mathbf{K}_0^1$ is isomorphic to a finite substructure of (Z, S). But for any finite substructure of (Z, S), say A', there is a consecutive segment B of Z containing A', so $A' \subseteq B \leq Z$. Hence $(\mathbf{K}_0^1, \leq^1)$ has finite closures.

It is routine to check that $(\mathbf{K}_0^1, \leq^1)$ has the amalgamation property. Since our notion of strong submodel guarantees that distinct components remain distinct in the amalgam, it is not difficult to see that the $(\mathbf{K}_0^1, \leq^1)$-generic model is the union of infinitely many copies of Z, and is ω-saturated.

2.34 Example. Let $\mathbf{K}^2 = \mathbf{K}^1$ and $\mathbf{K}_0^2 = \mathbf{K}_0^1$. But, define $A \leq^2 B$ to mean A is a substructure of B. It is easy to check that Axioms A1 through A6 hold. And again it is easy to check the amalgamation property.

Note that $(\mathbf{K}_0^2, \leq^2)$ does not have finite closures, since if $N \in \mathbf{K}^2$ is $Z \times \{0\} \bigcup Z \times \{1\}$, $A = \{\langle 3, 0\rangle, \langle 3, 1\rangle\}$, then $A \subseteq N$, but we can not find a finite $A' \in \mathbf{K}^2$ such that $A \subseteq A' \leq^2 N$.

It is easy to check that (Z, S) satisfies conditions i) and ii) in the definition of generic model, hence it is the $(\mathbf{K}_0^2, \leq^2)$-generic model M_2. But (Z, S) is not ω-saturated.

2.35 Example. Let A_n be the structure whose universe is the set of integers from $-n$ to n in Z with the restriction of the ternary relation $+$ and the binary relation S (successor) on Z.

Let $\mathbf{K}^3 = \mathrm{mod}(\mathrm{Th}(Z, 0, +, S))$ and $\mathbf{K}_0^3 = \{A_n : n \in \omega\} \bigcup \{\emptyset\}$. $\leq^3$ just means substructure.

Every finite $A \in \mathbf{K}^3$ is a symmetric (with respect to zero) segment. Let $A = (-m, +m)$, $B = (-n, +n)$, then $A \leq^3 B$ if and only if $m \leq n$. Axioms

A1 through A5 hold in $(\mathbf{K}_0^3, \leq^3)$. Since A_n can be uniquely imbedded in A_m if $n \leq m$ and no other embeddings are possible the amalgamation property is immediate. Note that $\mathbf{K}_0^3$ is not closed under substructure and A6 is not satisfied. The class $(\mathbf{K}_0^3, \leq^3)$ does not have finite closures for the same reason as in Example 2.34. The $(\mathbf{K}_0^3, \leq^3)$-generic model $M_3 = (Z, 0, +, S)$ is also not ω-saturated. T=Th(M_3) is superstable but not ω-stable.

2.36 Example. Let A_n be the structure whose universe is the set of integers from $-n$ to n in Z with the restriction of the ternary relation $+$, the restriction of the ternary relation $\cdot$, and the binary relation S (successor) on Z. Let $\mathbf{K}^4 = \mathrm{mod}(\mathrm{Th}(\mathrm{Z}, 0, +, \cdot, \mathrm{S}))$ and $\mathbf{K}_0^4 = \{A_n : n \in \omega\} \bigcup \{\emptyset\}$. $\leq^4$ just means substructure.

The existence of a generic model is established exactly as in the previous example but now the generic is not even stable.

3 Dimension Functions

In this section we introduce a 'dimension function' on finite structures. This dimension will allow us to define a notion of strong submodel that we treat as in the previous case. But it will also allow us to define a notion of 'free amalgamation'. With it we can investigate the stability class of the generic models constructed in Section 2. The basics of this theory are due to Hrushovski, [18], [19]. However, all other published accounts (except Baldwin and Shelah [6]) are based on classes which have finite closures. Thus, the proof that amalgamation over closed sets implies stability and the proof that T_α has amalgamation over closed sets are new here.

We now describe the fundamental examples of the formalism elaborated in this paper.

3.1 Example. Fix a finite relational structure B for a finite relational language L. Without loss of generality we assume that each relation of L holds of a tuple $\overline{a}$ only if the elements $\overline{a}$ are distinct and if $R(\overline{a})$ holds, $R(\overline{a}')$ holds for any permutation $\overline{a}'$ of $\overline{a}$. $e(B)$ denotes the number of subsets $B_0 = \{b_1, \ldots b_n\}$ of B such that for some ordering $\overline{b}$ of B_0, $B \models R(\overline{b})$ for some relation symbol R of L.

We will write $R(A)$ for the collection of tuples in A that satisfy some relation of the language. (This is quite natural notation if the language has only one relation symbol. It does not lead to trouble for an arbitrary finite relational language because we are only interested in the number of tuples that satisfy some relation.) Let A, B, C be disjoint. We write $R(A, B)$ for the collection of tuples from AB that contain at least one member of A and one of B. Write $e(A, B)$ for $|R(A, B)|$. Let A, B, C be disjoint sets. We write $R(A, B, C)$ for the collection of tuples from ABC that contain at least one member of A and one of C. Write $e(A, B, C)$ for $|R(A, B, C)|$. Fix a function $\delta_{\beta,\alpha}(u, v)$ with the form $\beta u - \alpha v$ where $\alpha, \beta \in \Re^+$. We will write $\mathbf{v}(B)$ for $\langle |B|, \delta_{\beta,\alpha}(|B|, e(B))\rangle$ and $\delta_{\beta,\alpha}(B)$ for $\delta_{\beta,\alpha}(|B|, e(B))$.

We will usually work with functions of the form $\delta_{1,\alpha}(u, v) = u - \alpha v$. Note that if $\alpha = m/n$ is rational, we can use $nu - mv$ with the same effect. When α is rational the methods of Section 2 will apply and the generic model will be ω-saturated and in fact ω-stable. For irrational α, further complications arise but we will find that the generic model is stable. (Wagner pointed out another variant: let the coefficient α depend on which relation R_i is satisfied. This notion was conceived independently by Shelah and exploited heavily in [6].) The crucial properties of this function are listed in Axiom 3.9.

3.2 Definition. Fix a finite relational language L. For any α, $0 < \alpha \leq 1$,

$$\mathbf{K}_\alpha = \{A : |A| < \omega \text{ and for any } A' \subseteq A, 0 \leq \delta_{1,\alpha}(A')\}.$$

We will show that $\mathbf{K}_\alpha$ has the amalgamation property; T_α is the theory of the associated generic. In Section 6 we specialize this convention and restrict from an arbitrary finite language to graphs.

These examples and others that we encounter later all fit into the following framework.

3.3 Context. Let $\mathbf{K}_0$ be a class of finite structures closed under substructure and (as always) under isomorphism. Let $\overline{\mathbf{K}}_0$ be the universal class determined by $\mathbf{K}_0$.

Let $\delta : \mathbf{K}_0 \to \Re^+$ be an arbitrary function with $\delta(\emptyset) = 0$. Extend δ to $d : \overline{\mathbf{K}}_0 \times \mathbf{K}_0 \to \Re^+$ by for each $N \in \overline{\mathbf{K}}_0$, $d(N, A) = \inf\{\delta(B) : A \subseteq B \subseteq_\omega N\}$. We usually write $d(N, A)$ as $d_N(A)$.

We will attach other requirements to δ shortly but some basic properties of d rely only on the range of δ being the positive real numbers.

3.4 Monotonicity. Let $A, B \in \mathbf{K}_0$ and $N \in \overline{\mathbf{K}}_0$. If $A \subseteq B$ then $d_N(B) \geq d_N(A)$.

3.5 Definition. Now for finite A, B contained in N, define the relative dimension of A over B, $d_N(A/B)$ as $d_N(A/B) = d_N(AB) - d_N(B)$. If $B \subseteq A \subseteq_\omega N$ this simplifies to $d_N(A/B) = d_N(A) - d_N(B)$.

The following facts are immediate from the definition.

3.6 Lemma. *Let $A, B, C \in \mathbf{K}_0$ all be subsets of $N \in \overline{\mathbf{K}}_0$.*

i) $d_N(A/C) \geq 0$.

ii) $d_N(AB/C) = d_N(A/BC) + d_N(B/C)$.

iii) If $A \subseteq A'$ then $d_N(A'/C) \geq d_N(A/C)$.

iv) For every A and every $\epsilon > 0$ there is a finite $A_0 \subseteq N$ such that $\delta(A_0) < d_N(A) + \epsilon$.

We call clause iv) the *approximation principle.*

3.7 Definition. For $A \in \mathbf{K}_0$ and $B \subseteq N \in \overline{\mathbf{K}}_0$, extend the notion of relative dimension to allow an infinite base set by setting for arbitrary B,

$$d_N(A/B) = \inf\{d_N(A/B_0) : B_0 \subseteq_\omega B\}.$$

Here are the additional properties we promised to require of δ. We first fix a bit more notation.

3.8 Notation. Let $\delta(B/A)$ denote $\delta(AB) - \delta(A)$. Note that $\delta(B/A)$ and $\delta(BA/A)$ are equal.

Just rewriting the definition, we have $\delta(AB/C) = \delta(A/BC) + \delta(B/C)$.

3.9 Axioms. Let $N \in \overline{\mathbf{K}}_0$ and A, B, $C \in \mathbf{K_0}$ be substructures of N.

i) If A, B, and C are disjoint then $\delta(C/A) \geq \delta(C/AB)$.

ii) For every n there is an $\epsilon_n > 0$ such that if $|A| < n$ and $\delta(A/B) < 0$ then $\delta(A/B) \leq -\epsilon_n$.

iii) There is a real number ϵ independent of N, A, B, C such that if A, B, C are disjoint subsets of a model N and $\delta(A/B) - \delta(A/BC) < \epsilon$ then $R(A, B, C) = \emptyset$ and $\delta(A/B) = \delta(A/BC)$.

Clause i) can be rephrased as: $B \subseteq C$ and $A \cap C = \emptyset$ implies $\delta(A/B) \geq \delta(A/C)$. We want to show that these properties hold of our main examples.

3.10 Proposition. *Let $N \in \overline{\mathbf{K}}_0$, and A, $B \in \mathbf{K_0}$ be disjoint substructures of N. If $\delta = \delta_{\beta,\alpha}$ and $A \cap B = \emptyset$, then $\delta(A/B) = \delta(A) - \alpha e(A, B)$.*

Proof.

$$\begin{aligned} \delta(A/B) &= \delta(AB) - \delta(B) \\ &= \beta|AB| - \alpha(e(A) + e(B) + e(A,B)) - (\beta|B| - \alpha e(B)) \\ &= \beta|A| - \alpha(e(A) + e(A,B)) \\ &= \delta(A) - \alpha e(A,B). \end{aligned}$$

Using Proposition 3.10 it is easy to check that Axiom 3.9 holds when $\delta = \delta_{\beta,\alpha}$. Firstly, i) reduces to the observation that $e(A, BC) \geq e(A, B)$. In ii) we may assume that $B \cap A = \emptyset$. Then Proposition 3.10 gives $\delta(A/B) = \delta(A) - \alpha\, e(A, B)$. But there are only a finite number of possibilities for A with $|A| \leq n$. Thus

$$\min \{|\delta(A) - \alpha k| : k \in \mathbf{Z},\ |A| \leq n,\ \text{and } \delta(A) < \alpha k\}$$

will do for ϵ_n. For iii), from Proposition 3.10 we have

$$\begin{aligned} & \delta(A/B) - \delta(A/BC) \\ =\ & (\delta(A) - \alpha\, e(A,B)) - (\delta(A) - \alpha\, e(A,BC)) \\ =\ & \alpha\, (e(A,BC) - e(A,B)) \\ =\ & \alpha e(A,B,C). \end{aligned}$$

Thus, $\delta(A/B) - \delta(A/BC) < \alpha$ only if $e(A, B, C) = 0$ as required. But then $\delta(A/B) = \delta(A/BC)$. So α is the required ϵ.

3.11 Definition. For $A \subseteq B$, $A, B \in \mathbf{K}_0$, define $A \leq B$ if for each $X \subseteq A$, $d_A(X) = d_B(X)$.

Equivalently we have for $A, B \in \mathbf{K}_0$, $A \leq B$ if and only if

$$A \subseteq B \wedge \forall X\, (A \subset X \subseteq B \Rightarrow \delta(X) \geq \delta(A))\,.$$

With this definition we return to the situation of Section 2.

3.12 Theorem. $(\mathbf{K}_0, \leq)$ *satisfies Axioms A1 through A6.*

Proof. A1-A5 hold just by examination of the definition. For A6, just note that if $A, B \in \mathbf{K}_0$ and $A \subseteq B$ then $A \leq B$ if and only for each $C \subseteq B$, $\delta(A \cap C) \leq \delta(C)$.

3.13 Context. Henceforth, $(\mathbf{K}_0, \leq)$ is class of finite structures closed under isomorphism and substructure with $\leq$ induced by a function δ obeying Axioms 3.9. Moreover, we assume $(\mathbf{K}_0, \leq)$ satisfies the amalgamation property and $\mathbf{K}$ is the class of models of the theory of the generic model M of $(\mathbf{K}_0, \leq)$. $\mathcal{M}$ is a large saturated model of $T = \mathrm{Th(M)}$. In the absence of other specification, the dimension function d is the function induced on $\mathcal{M}$ by δ.

Now we can describe minimal pairs in terms of the dimension function in the following straightforward way.

3.14 Lemma. *Let* A, $B \in \mathbf{K_0}$ *and* $A \subseteq B$. *Then* (A, B) *is a minimal pair if and only if* $\delta(B) < \delta(A)$ *and* $\delta(A) \leq \delta(C)$ *for all* C *with* $A \subseteq C \subset B$.

Just as in Definition 2.17, we extend $\leq$ to a relation on $\overline{\mathbf{K}}_0 \times \overline{\mathbf{K}}_0$. The next lemma shows that extending in this way provides the same relation as if we had used the idea of Definition 3.11 directly on $\overline{\mathbf{K}}_0$ instead of passing through the minimal pairs.

3.15 Lemma. *Let* A, $B \in \overline{\mathbf{K_0}}$ *and* $A \subseteq B$. *Then* $A \leq B$ *if and only if for all* $X \subseteq_\omega A$, $d_A(X) = d_B(X)$.

Proof. If for some finite $X \subseteq A$, $d_A(X) > d_B(X)$ it is straightforward to find Y with $X \subseteq Y \subseteq B$, $\delta(Y) < d_A(X) \leq \delta(Z)$ for every Z with $X \subseteq Z \subseteq A$ and a sequence of $Y_i \subseteq Y$ with $Y_0 = X$, (Y_i, Y_{i+1}) a minimal pair and $\delta(Y_n) \leq \delta(Y)$. Then some Y_i is not contained in A. For the least such i, (Y_i, Y_{i+1}) witness that A is not strong in B (by the minimal pairs definition).

Conversely, suppose there exists a minimal pair (X, B') with $X \subseteq B' \subseteq B$, $X \subseteq A$ but B' is not contained in A. Let

$$\epsilon = \min\{|\delta(B'/B'')| : X \subseteq B'' \subseteq B'\}$$

and choose X^* containing X with $\delta(X^*) < d_A(X) + \epsilon$. Now

$$\delta(B'/X^*) \leq \delta(B'/(X^* \cap B')) \leq -\epsilon.$$

So

$$d_B(X) \leq \delta(B'X^*) = \delta(B'/X^*) + \delta(X^*) < -\epsilon + d_A(X) + \epsilon = d_A(X).$$

In the case where **K** admitted finite closures the notion of the closure of a set was easy to understand. The closures can be more complicated in the current situation.

3.16 Definition. For finite $A \subseteq N$, let

$$\Im_0(A) = \cup\{B : A \subseteq B \subseteq N, \text{ and } (A, B) \text{ is a minimal pair}\}.$$

For each $n < \omega$, let

$$\Im_{n+1}(A) = \cup\{B \subseteq N : \text{ there is an } A' \subseteq \Im_n(A) \text{ and } (A', B) \text{ is a minimal pair}\}.$$

Let $\Im_N(A) = \bigcup_{n<\omega} \Im_n(A)$.

.

Clearly $\Im_N(A)$ is intrinsically closed and the definition of $\leq$ yields immediately the following representation of the closure.

3.17 Lemma. *For finite $A \subseteq N$, $\Im_N(A) = \mathrm{icl_N(A)}$.*

Note that this formulation shows that for arbitrary X,

$$\mathrm{icl}_{\mathrm{N}}(\mathrm{X}) = \cup\{\mathrm{icl}_{\mathrm{N}}(\mathrm{X}') : \mathrm{X}' \subseteq_\omega \mathrm{X}\} = \cup\{\Im_{\mathrm{N}}(\mathrm{X}') : \mathrm{X}' \subseteq_\omega \mathrm{X}\}.$$

3.18 Lemma. *Let $N \in \overline{\mathbf{K_0}}$, A, B, $C \subseteq N$, (A, B) be a minimal pair, $A \subseteq C$, and $B \not\subseteq C$. Then $\delta(B/C) \leq \delta(B/A) < 0$. In particular, $\delta(BC) < \delta(C)$.*

Proof. $\delta(B/C) \leq \delta(B/B \cap C) \leq \delta(B/A) < 0$ since (A, B) is a minimal pair and B is not a subset of C. But $\delta(BC) = \delta(B/C) + \delta(C)$.

3.19 Lemma. *If (A, B) is a minimal pair then there is an n such that for any $N \in \mathbf{K}$ and any embedding f of A into N, there are at most n extensions f_i of f mapping B into N.*

Proof. Suppose f embeds A into N. Let $B_1, \ldots B_k$ be images of extensions of f to embeddings of B into N such that B_i is not contained in the union of the B_j for $j < i$. By Lemma 3.18 and induction,

$$\delta(B_k B_{k-1} \ldots B_1/A) = \sum_{i=1}^{k-1} \delta(B_i/AB_{i-1} \ldots B_1) + \delta(B_1/A).$$

By Lemma 3.18 and induction each $\delta(B_i/AB_{i-1} \ldots B_1) < \delta(B_1/A)$. Axiom 3.9 iii) guarantees that $\delta(B_1/A) \leq \epsilon_{|B_1-A|}$. So $\delta(B_k B_{k-1} \ldots B_1/A) < -k \cdot \epsilon_{|B_1-A|}$. If $k > \delta(A)/\epsilon_{|B_1-A|}$, this implies $\delta(B_1, \ldots B_k A) < 0$ which is impossible.

We can immediately conclude the following generalization of Theorem 2.23.

3.20 Corollary. *If $A \subseteq N$ then the intrinsic closure of A in N, $\mathrm{icl_N(A)}$ is contained in the algebraic closure of A in N, $\mathrm{acl}_N(A)$.*

3.21 Definition. For $A, B \in S(\mathbf{K}_0)$, $A \leq_i B$ if $A \subseteq B$ but there is no A' with $A \subseteq A' <_s B$. If $A \leq_i B$, we say B is an *intrinsic* extension of A.

By letting $t(|A|, |B|) = \delta(A)/\epsilon_{|B-A|}$, we find the following more uniform version of Lemma 3.19.

3.22 Lemma. *If $A \leq_i B$, there is a function $t(|A|, |B|)$ from $\omega \times \omega$ to ω such that for any $N \in \mathbf{K}$, if f maps A into $N \in \mathbf{K}_0$ there are at most $t(|A|, |B|)$ extensions of f mapping B into N.*

.

In general the converse to Corollary 3.20 fails. We discuss this situation in Lemma 4.5.

3.23 Notation. Through Lemma 3.27, we deal with an arbitrary $N \in \overline{\mathbf{K}}_0$. A, B, C, A^*, B^*, C^* denote finite subsets of N. For simplicity, we write d for d_N and $\overline{A}, \overline{B}$ etc. to denote $\mathrm{icl_N(A)}, \mathrm{icl_N(B)}$ etc.

We can improve the approximation principle to the following *strong* approximation principle.

3.24 Lemma. *Let $A \subseteq A' \subseteq_\omega \overline{A}$. Then $d(A) = d(A')$.*

Proof. Clearly, $d(A') \geq d(A)$. We show that for every $\epsilon > 0$ for any N there is a finite $A^* \supseteq A'$ with $\delta(A^*) < d(A) + \epsilon$. By the approximation principle, we can choose $A_0 \supseteq A$ with $\delta(A_0) < d(A) + \epsilon$. If $A' \subseteq A_0$, we are finished. If not, since $A' \subseteq \overline{A}$, there is a sequence $A = B_0 \subseteq B_1 \ldots \subseteq B_n$ such that $A' \subseteq B_n \subseteq \overline{A}$, eventually B_i is not contained in A_0,and each (B_i, B_{i+1}) is a minimal pair. By induction and Lemma 3.18 $\delta(B_iA_0) \leq \delta(A_0)$. Thus B_nA_0 contains A' and satisfies $\delta(B_nA_0) \leq \delta(A_0) < d(A) + \epsilon$ as required.

The difficulty of the next lemma is to guarantee that $A^* = B^* \cap \overline{A}$.

3.25 Lemma. *If $A \subseteq B$ are finite subsets of N, then for every $\epsilon > 0$ there exist $A^* \subseteq B^*$ with $A^* \subseteq \overline{A}$, $B^* \subseteq \overline{B}$ and $A^* = B^* \cap \overline{A}$ such that $\delta(B^*) < d(B) + \epsilon$ and $\delta(A^*) < d(A) + \epsilon$. Consequently, $|\delta(B^*/A^*) - d(B/A)| < \epsilon$.*

Proof. First choose any B^* with $\delta(B^*) < d(B) + \epsilon$ by the approximation principle. If $\delta(B^* \cap \overline{A}) < d(A) + \epsilon$ we are done. If not, choose by the strong approximation principle $A^* \supseteq (B^* \cap \overline{A})$ with $A^* \subseteq \overline{A}$ and $\delta(A^*) < d(A) + \epsilon$. Now, $A^* \cap B^* = B^* \cap \overline{A}$ and $\delta(A^*/A^* \cap B^*) \leq 0$. But then

$$\delta(A^*B^*) = \delta(A^*/B^*) + \delta(B^*) \leq \delta(A^*/(A^* \cap B^*)) + \delta(B^*) \leq \delta(B^*)$$

so we can use A^*B^* for B^* and A^* as chosen.

We now have $d(B) \leq \delta(B^*) < d(B) + \epsilon$, $d(A) \leq \delta(A^*) < d(A) + \epsilon$. Subtracting the second inequality from the first yields the final conclusion.

3.26 Lemma. *Let $B = A \cap C$ with A and C finite. Then for any $\epsilon > 0$, there exist A^*, B^*, C^* with $B^* = A^* \cap C^*$ and*

$$\begin{array}{ccccc} 0 & < & \delta(A^*) - d(A) & < & \epsilon \\ 0 & < & \delta(B^*) - d(B) & < & \epsilon \\ 0 & < & \delta(C^*) - d(C) & < & \epsilon. \end{array}$$

Consequently,

$$0 < \delta(A^*/C^*) - c(A/C) < \epsilon.$$

Proof. For this, first choose $B^* \subseteq \overline{B}$ and $C^* \subseteq \overline{C}$ by Lemma 3.25 applied to $B \subseteq C$ such that $d(B) \leq \delta(B^*) < d(B) + \epsilon$, $d(C) \leq \delta(C^*) < d(C) + \epsilon$, $C^* \cap \overline{B} = B^*$. (Note that B^* and C^* were chosen together, not independently.) Now, $\mathrm{icl_N(B^*)} = \overline{\mathrm{B}}$ and $\mathrm{icl_N(AB^*)} = \overline{\mathrm{A}}$ and so $d(AB^*) = d(A)$. Applying Lemma 3.25 to $B^* \subseteq AB^*$, choose A^* and B^*_0 so that $A^* \cap \overline{B} = B^*_0$, $B^*_0 \supseteq B^*$ and $\delta(A^*) < d(AB^*) + \epsilon = d(A) + \epsilon$. Since $\overline{A} \cap \overline{C} = \overline{B}$, $C^* \cap A^* = C^* \cap A^* \cap \overline{B} = B^*$. Again, the last statement follows by subtracting inequalities.

3.27 Lemma. *For finite A and C, $d(A/(A \cap C) \geq d(A/C)$.*

Proof. Let $\epsilon > 0$ be given. Choose A^*, B^*, C^* satisfying the conclusion of Lemma 3.26. From i) of Axioms 3.9 (applied to $A^* - C^*$ and $B^* = A^* \cap C^*$)

$$\delta(A^*C^*) - \delta(C^*) = \delta(A^*/C^*) \leq \delta(A^*/A^* \cap C^*) = \delta(A^*) - \delta(B^*).$$

Therefore $d(AC) - \delta(C^*) \leq \delta(A^*) - \delta(B^*)$. Letting $\epsilon \to 0$, we have $d(AC) - d(C) \leq d(A) - d(B)$ which is the required result.

With the previous sequence of lemmas we can conclude:

3.28 Theorem. *Let $N \in \overline{\mathbf{K}}_0, A \subseteq_\omega N$, and $B \subseteq C \subseteq N$. Then $d_N(A/B) \geq d_N(A/C)$.*

Proof. First assume B and C are finite. Then,

$$\begin{array}{rcl} d(A/B) & = & d(AB) - d(B) \\ & \geq & d(AB) - d((AB) \cap C) \\ & = & d(AB/(AB) \cap C). \end{array}$$

By Lemma 3.27 and since $B \subseteq C$ we have $d(AB/(AB) \cap C) \geq d(AB/C) = d(A/C)$. We extend to the case in which B is finite and C infinite as follows.

$$\begin{array}{rcl} d(A/C) & = & \inf\{d(A/C_0) : C_0 \subseteq_\omega C\} \\ & = & \inf\{d(A/C_0) : B \subseteq C_0 \subseteq_\omega C\} \quad \leq \quad d(A/B). \end{array}$$

Finally, when both B and C are infinite the result follows from the previous case and the definition of $d(A/C)$.

Now we can extend the observation that $d(A/C)$ depends only on $A - C$ to infinite C.

3.29 Corollary. *Let $N \in \overline{\mathbf{K}}_0$. For finite $A_1, A_2 \subseteq N$ and arbitrary $C \subseteq N$, if $A_1 - C = A_2 - C$ then $d(A_1/C) = d(A_2/C)$.*

Proof. Suppose for contradiction that for some positive δ, $d(A_1/C) = d(A_2/C) + \delta$. Choose finite $C_1, C_2 \subseteq C$ such that for $i = 1, 2$, $d(A_i/C_i) < d(A_i/C) + \delta$. Let X denote $(A_1 \cap C)(A_2 \cap C)C_2$. Then $d(A_1/C) \leq d(A_1/X) = d(A_2/X) \leq d(A_2/C_2) < d(A_2/C) + \delta$ contradicting the choice of δ.

We now turn to considering the stability of T. For the following, recall the stipulations of Context 3.13. Lemma 3.29 shows that in the following definition $A \downarrow^d{}_C B$ if and only if $(A - C) \downarrow^d{}_C (B - C)$. Lemma 3.31 shows that the union of independent closed sets is closed.

3.30 Definition. i) We say the finite sets A and B are *independent over* C and write $A \downarrow^d_C B$ if

(a) $d(A/C) = d(A/CB)$.

(b) $\overline{AC} \cap \overline{BC} \subseteq \overline{C}$.

ii) We say the (arbitrary) sets A and B are *independent over* C and write $A \downarrow^d_C B$ if for every finite $A' \subseteq A$ and $B' \subseteq B$, $A' \downarrow^d_C B'$

iii) The class $\mathbf{K}$ is *countably (finitely) based* if for every finite A and arbitrary B there is a countable (finite) $C \subseteq B$ such that $A \downarrow^d_C B$.

iv) If $A \downarrow^d_C B$, we may also write tp(A; B) is *d-free over* C.

3.31 Lemma. *Suppose A, B and $C = A \cap B$ are closed and $A \downarrow^d_C B$. Then AB is closed.*

Proof. If not there exist finite $A_0 \subseteq A$ and $B_0 \subseteq B$ and a $\mathbf{K}$ such that (A_0B_0, K) is a minimal pair and K is not contained in AB. Let $\epsilon = \delta(A_0B_0) - \delta(K)$. Note that for any A^*, B^* with $A_0 \subseteq A^* \subseteq A$ and $B_0 \subseteq B^* \subseteq B$, since (A_0B_0, K) is a minimal pair and K is not contained in AB, $\delta(K/A^*B^*) \leq -\epsilon$. Thus $d(KA^*B^*) < \delta(A^*B^*) - \epsilon$. From the independence, we are going to choose a pair (A_2, B_2) with $A_2 \subseteq A, B_2 \subseteq B$ such that $d(KA_2B_2) \geq \delta(A_2B_2) - \epsilon$. This contradiction will prove the lemma.

By the definition of independence $d(A_0/B_0C) = d(A_0/C)$. Choose finite $C_0 \subseteq C$ with $d(A_0/C_0) < d(A_0/C) + \epsilon/4$.

Now let $A_1 = A_0C_0$ and $B_1 = B_0C_0$ and $C_1 = A_1 \cap B_1$. Observe that $d(A_0/C) = d(A_1/C)$ by Lemma 3.29. Moreover, $A_0C_0 = A_1$ implies $d(A_0/C_0) = d(A_1/C_0) \geq d(A_1/C_1)$, so $d(A_1/C_1) < d(A_1/C) + \epsilon/4$.

Then choose finite C_2 with $C_1 \subseteq C_2 \subseteq \overline{C}_1 \subseteq C$ and finite A_2 with $A_1C_2 \subseteq A_2 \subseteq \overline{A}_1 \subseteq A$ satisfying $\delta(A_2) < d(A_1) + \epsilon/4$ and $\delta(C_2) < d(C_1) + \epsilon/4$. Thus, $|\delta(A_2/C_2) - d(A_1/C_1)| < \epsilon/4$. Finally, choose finite B_2 with $B_1C_2 \subseteq B_2 \subseteq \overline{B}_1$ with $\delta(B_2) < d(B_1) + \epsilon/4$. (We were able to choose $C_2 \subseteq A_2 \cap B_2$ using the strong approximation principle because $\overline{C}_1 \subseteq (\overline{A}_1 \cap \overline{B}_1)$.) It remains to verify that A_2, B_2 are as required. $d(KA_2B_2) \geq d(A_1B_1)$ so it suffices to show $d(A_1B_1) \geq \delta(A_2B_2) - \epsilon$.

Now

$$\begin{aligned}
d(A_1B_1) &= d(A_1/B_1) + d(B_1)\\
&\geq d(A_1/CB_1) + d(B_1)\\
&= d(A_1/C) + d(B_1)\\
&> d(A_1/C_1) - \epsilon/4 + d(B_1)\\
&\geq (\delta(A_2/C_2) - \epsilon/4) - \epsilon/4 + d(B_1)\\
&\geq (\delta(A_2/C_2) - \epsilon/4) - \epsilon/4 + (\delta(B_2) - \epsilon/4)\\
&\geq \delta(A_2/C_2) + \delta(B_2) - 3\epsilon/4.
\end{aligned}$$

The choice of C_2 guarantees that if $C_2 \subseteq X$, $\delta(X/C_2) \geq -\epsilon/4$. In particular, $\delta(A_2 \cap B_2/C_2) \geq -\epsilon/4$.

Thus,

$$\begin{aligned}
\delta(A_2/C_2) + \delta(B_2) - 3\epsilon/4 &= \delta(A_2/(A_2 \cap B_2)) + \delta(A_2 \cap B_2/C_2) + \delta(B_2) - 3\epsilon/4\\
&\geq \delta(A_2/(A_2 \cap B_2)) - \epsilon/4 + \delta(B_2) - 3\epsilon/4.
\end{aligned}$$

Combining with the first computation, we have

$$d(A_1B_1) \geq \delta(A_2/(A_2 \cap B_2)) + \delta(B_2) - \epsilon = \delta(A_2B_2) - \epsilon.$$

We have fulfilled our commitments and completed the proof.

Now we show that any $\mathbf{K}$ satisfying Context 3.13 is countably based.

3.32 Lemma. *Let X be closed and $a \notin X$. Then there is a countable closed $X' \subseteq X$ such that $\overline{aX'} \cap X = X'$ and $d(a/X) = d(a/X')$. That is, $a \downarrow^d_{X'} X$. Moreover, if the range of δ is well ordered (in the natural order on the reals) then $\mathbf{K}$ is finitely based.*

Proof. Suppose that for $i < \omega$, an increasing sequence of finite sets X_i has been chosen with $d(a/X_i) - d(a/X) < 1/i$. Let $U = \bigcup_{i<\omega} X_i$ and $X' = X \cap \overline{aU}$. Lemma 3.17 shows that the intersection of closed sets is closed and Corollary 3.19 yields the closure of countable set is countable. Finally applying Lemma 3.28, we see that X' is as required. If the range of δ and thus the numbers $d(A/B)$ as A, B range through $\mathbf{K}_0$ is a well ordered subset of the reals, it is straightforward to find a finite base.

3.33 Lemma. *Let B be closed and suppose A and C are finite sets disjoint from B such that $A \downarrow^d_B C$. Then*

$$R(\overline{AB} - B, B, C) = \emptyset.$$

Proof. The assumption of independence implies $\overline{AB} \cap \overline{CB} = B$. Fix an arbitrary finite subset A_0 of $\overline{AB} - B$ that contains A. We will show $R(A_0, B, C) = \emptyset$. Let $\epsilon \in \Re^+$ be the parameter from Axiom 3.9 iii).

Applying Lemma 3.17, choose $B_0 \subseteq_\omega B$ such that

$$d(A_0/B_0) < d(A_0/B) + \epsilon/4 \tag{2}$$

and

$$AB_0 \subseteq A_0B_0 \subseteq \overline{AB_0}. \tag{3}$$

Now, apply Lemma 3.26 to A_0B_0 and CB_0, we obtain A^* and C^* such that, letting B^* denote $A^* \cap C^*$,

$$A_0B_0 \subseteq A^* \subseteq \overline{A_0B_0},\ CB_0 \subseteq C^* \subseteq \overline{CB_0},$$

$$\delta(A^*/B^*) - d(A_0/B_0) < \epsilon/4. \tag{4}$$

and

$$0 < \delta(A^*/C^*) - d(A_0/B_0C) < \epsilon/4. \tag{5}$$

Since $B^* \subseteq \overline{AB} \cap \overline{CB} = B$ the result follows if we show

$$R(A^* - B^*, B^*, C^* - B^*) = \emptyset.$$

This is clear from Axiom 3.9 iii) and the following claim.

Claim. $\delta(A^*/B^*) < \delta(A^*/C^*) + \epsilon$.

The claim follows from the following four inequalities.

$$|\delta(A^*/B^*) - d(A_0/B_0)| < \epsilon/4$$
$$|d(A_0/B_0) - d(A_0/B)| < \epsilon/4$$
$$|d(A/B) - d(A/B_0C)| < \epsilon/4$$
$$|d(A/B_0C) - \delta(A^*/C^*)| < \epsilon/4.$$

The first inequality follows from (4) and the second is (2). Note that $d(A_0/B_0) = d(A/B_0)$ since $\overline{A_0B_0} = \overline{AB_0}$ by (3). From the second inequality we have $|d(A/B) - d(A/B_0)| < \epsilon/4$. But by Lemma 3.28, $d(A/B_0C) \leq d(A/B_0)$. But also $d(A/B_0C) \geq d(A/BC) = d(A/B)$. This establishes the third inequality. Since, $A_0B_0 \subseteq A^* \subseteq \overline{A_0B_0}$, $\overline{A_0B_0C} = \overline{AB_0C}$; thus $d(A/B_0C) = d(A_0/B_0C)$. But the last inequality for A_0 is (5).

Now combining the four inequalities we have the Claim.

3.34 Theorem. *Let M be $(\mathbf{K}_0, \leq)$-generic and $T = \mathrm{Th(M)}$ where $\leq$ is derived from δ satisfying Axiom 3.9. Suppose $(\mathbf{K}_0, \leq)$ has amalgamation over closed sets.*

i) T is stable.

ii) If the range of δ is well-ordered then T is ω-stable.

Proof. Fix a model N of T; we will count the 1-types over N.

By Lemma 3.32, for any a, there is a countable subset $X \subseteq N$ such that X is closed and $\overline{aX} \cap N = X$ and $a \downarrow^d{}_X N$. We see first that tp(a/X) determines tp(a/N). Suppose that $a \equiv_X a'$ and $a' \downarrow^d_X N$. Then $\overline{a'X} \cap N = X$ and $\overline{aX} \equiv_X \overline{a'X}$ since $\overline{a'X} \subseteq \mathrm{acl}(a'X)$ by Corollary 3.20. Now $\overline{aX} \downarrow^d{}_X N$ and $\overline{a'X} \downarrow^d{}_X N$ imply $N\overline{aX} \equiv_X N\overline{a'X}$ by Lemma 3.33; both $N\overline{aX}$ and $N\overline{a'X}$ are closed by Lemma 3.31. Thus by amalgamation over closed sets, $(\mathcal{M}, N\overline{aX}) \equiv (\mathcal{M}, N\overline{a'X})$. In particular, a and a' realize the same type over N. Thus in order to count the number of types over N, it suffices to count the number of types over X and the number of choices of X from N. Since X is countable,

$$| S(N) | \leq 2^{\aleph_0} \cdot | N |^{\aleph_0} .$$

Hence T is stable.

If the range of δ is well-ordered the second part of Lemma 3.32 allows us to choose a finite X and it is easy to see that $(\mathbf{K}_0, \leq)$ has finite closures. By Lemma 2.21, T has an ω-saturated model and so is small. Thus,

$$| S(N) | \leq \aleph_0 \cdot | N |^{<\aleph_0} .$$

and so T is ω-stable.

Is there a class $\mathbf{K}$ which is finitely based but does not have finite closures?

We show that if the base set C is $\aleph_1$-saturated then $A \downarrow^d_C B$ corresponds to the usual notion of nonforking.

3.35 Lemma. *Suppose T satisfies the hypotheses of Theorem 3.34. Suppose further that M is an $\aleph_1$-saturated model of T, A is a finite set and $M \subseteq B$. If $A \downarrow_M B$ (in the nonforking sense) then $A \downarrow^d_M B$.*

Proof. Since $\overline{AM}$ is contained in $\mathrm{acl}(AM)$ and $\overline{B} \subseteq \mathrm{acl}B$, $A \downarrow_M B$ implies $\overline{AM} \cap \overline{B} \subseteq M$.

Suppose for contradiction that $d(A/MB) < d(A/M)$. Fix $\epsilon = d(A/M) - d(A/MB)$ and choose $B_1 \subseteq_\omega MB$ with $d(A/B_1) < d(A/MB) + \epsilon/2$ and then $B_1^* \subseteq \overline{B}_1$, $A^* \subseteq \overline{A}$ with $d(A^*/B_1^*) < d(A/B_1) + \epsilon/2$. Now, since $\overline{A} \downarrow_M \overline{MB}$, $t(B^*/A^*)$ is finitely satisfied in M. Since M is strongly $\aleph_1$-saturated, there is $B' \subseteq M$ realizing the same type. Note that since B' realizes the type of B^* over the empty set $d_M(B') < d(B^*)$. Thus, $d(A^*/B') < d(A/MB) + \epsilon = d(A/M)$. This contradiction yields the result.

3.36 Corollary. *Suppose T satisfies the hypotheses of Theorem 3.34. If M is $\aleph_1$-saturated then for any A and B, $A \downarrow_M B$ if and only if $A \downarrow_M^d B$.*

Proof. We know $t(A/M)$ has a unique nonforking extension over any B and by the proof of Theorem 3.34 there is a unique d-free extension. By Lemma 3.35 these two are the same.

If we strengthen the hypotheses we can guarantee that $A \downarrow_B C$ and $A \downarrow_B^d C$ agree if B is an intrinsically closed set. For this we need a stronger form of amalgamation.

3.37 Definition. The class $(\mathbf{K}_0, \leq)$ is said to have *free amalgamation over closed sets* if for any infinite models $N_0, N_1 \in \mathbf{K}$, any set A in $S(\mathbf{K})$, if there are strong embeddings $f : A \to N_0$ and $g : A \to N_1$, then there are $S \models T$ and elementary embeddings $f' : N_0 \to S$, $g' : N_1 \to S$ such that $f'f = g'g$ and such that if N_0' , A', N_1' are the images of N_0, A, N_1) in S, $R(N_0', A', N_1') = \emptyset$.

The crucial distinction in the proofs of Corollary 3.35 and Lemma 3.38 is that without *free* amalgamation over closed sets we are unable to establish the existence of d-free extensions.

3.38 Lemma. *Suppose T satisfies the hypotheses of Theorem 3.34 but amalgamation over closed sets is strengthened to free amalgamation over closed sets. Suppose C is intrinsically closed, A is a finite set and $C \subseteq B$.*

i) $A \downarrow_C B$ *(in the nonforking sense) if and only if* $A \downarrow_C^d B$.

ii) tp(A/C) *is stationary.*

Proof. Note that if $C \leq B$ and $C \leq CA \subseteq \mathcal{M}$, free amalgamation over closed sets shows there is $A' \subseteq \mathcal{M}$ such that $A' \downarrow_C^d B$. The argument in Theorem 3.34

shows that this free extension is unique. Thus any type $p \in S(C)$ has a unique d-free over C extension to a type $\hat{p}$ over $\mathcal{M}$. By, e.g. 4.14 of [2], this implies $\hat{p}$ is the nonforking extension of p and so A' realizing $\hat{p}|B$ is a nonforking extension of tp(A/C) as required. The second assertion is now clear.

The following result summarises these arguments.

3.39 Corollary. *Suppose $(\mathbf{K}, \leq)$ is derived from a dimension function $\delta = \delta_{1,\alpha}$ and T_α is the theory of the generic model.*

i) T_α is stable; $A \downarrow_B C$ and $A \downarrow^d_B C$ agree if B is an intrinsically closed set.

ii) If α is rational, then T_α is ω-stable.

Proof. By Lemma 4.8, $(\mathbf{K}_\alpha, \leq)$ has free amalgamation over closed sets. Note that when α is rational, say $\alpha = p/q$, the range of α is the set of numbers of the form $(mq - np)/q$ which is well-ordered in the natural order on the reals. The rest follows.

Note that this argument does not apply to the $\aleph_1$-categorical examples which do have the full amalgamation property. We still get ω-stability by Theorem 3.34 but the proof that forking and d-independence are the same is a differen t technical observation.

While the method of construction gives the intuition that the structures constructed in this section are not superstable; it is not so easy to verify this fact. However, Herwig [17] has demonstrated in detail the non-superstability of both the ω-categorical example and his own example of a generic which is strictly stable and ω-saturated. This argument easily transfers to our situation. It also easy to see, by considering the possibilities for the intrinsic closure of a finite set, that in general our examples are not small.

4 Amalgamation over closed sets

We continue with the framework established in Context 3.13. In this section we will show a large class of examples has amalgamation over closed sets. More strongly, we prove that these examples have (free) amalgamation over closed sets. For these arguments we need to describe a further strengthening of the amalgamation condition on $(\mathbf{K}_0, \leq)$. This condition, the full amalgam ation property, holds both for the $\mathbf{K}_\alpha$ and for Hrushovski's $\aleph_0$-categorical stable structures but in general it fails for the ω-stable examples.

The following definition is made for an arbitrary pair $(\mathbf{K}_0, \leq)$ where $\mathbf{K}_0$ is class of finite structures and $\leq$ is a refinement of the substructure relation. We write $B \cap C = A$ to mean that the universes of B and C intersect in a structure whose universe is denoted A and that B and C induce a common structure on that universe which is also denoted by A. (This awkward sentence is our payment for failing to distinguish a model from its universe.)

4.1 Definition. $(\mathbf{K}_0, \leq)$ has the *full amalgamation property* if $B \cap C = A$ and $A \leq B$ imply $B \bigotimes_A C \in \mathbf{K}_0$ and $C \leq B \bigotimes_A C$.

4.2 Lemma. $(\mathbf{K}_\alpha, \leq)$ *has the full amalgamation property and so* $(\mathbf{K}_\alpha, \leq)$ *has the uniform amalgamation property.*

Proof. With the notation of Definition 4.1, let $X \subseteq B \bigotimes_A C$ and to simplify notation, let $X \bigcap B = B_0$, $X \bigcap A = A_0$, $X \bigcap C = C_0$. Since $A, B, C \in \mathbf{K}_\alpha$, $A_0, B_0, C_0 \in \mathbf{K}_\alpha$ by the definition of $\mathbf{K}_\alpha$.

We need to show $X \in \mathbf{K}_\alpha$. Clearly, $\mid X \mid = \mid B_0 \mid + \mid C_0 \mid - \mid A_0 \mid$. Note $R(C_0 - A_0, B_0 - A_0) = \emptyset$, so $e(A_0, C_0 - A_0) = e(B_0, C_0 - A_0)$. Thus,

$$\begin{aligned} e(X) &= e(B_0) + e(C_0 - A_0) + e(B_0, C_0 - A_0) \\ &= e(B_0) + e(C_0) - e(A_0) - e(A_0, C_0 - A_0) + e(B_0, C_0 - A_0) \\ &= e(B_0) + e(C_0) - e(A_0). \end{aligned}$$

Since δ is linear, we have $\delta(X) = \delta(B_0) + \delta(C_0) - \delta(A_0)$. By Axiom A6, $A \leq B$ implies $A_0 \leq B_0$. Hence $\delta(A_0) \leq \delta(B_0)$, so

$$\begin{aligned} \delta(X) &= \delta(B_0) + \delta(C_0) - \delta(A_0) \\ &\geq \delta(B_0) + \delta(C_0) - \delta(B_0) = \delta(C_0) \geq 0. \qquad (*) \end{aligned}$$

Hence $X \in \mathbf{K}_\alpha$ as required. If $X \supseteq C$, then $C_0 = C$, we have $\delta(X) \geq \delta(C)$ by (*). Thus, $C \leq B \bigotimes_A C$.

It is immediate that the full amalgamation property implies the uniform amalgamation property, so the second clause of the proposition is clear.

Recall that if A is a strong submodel of a generic M and A is strong in B then there is a strong embedding of B into N over A. The following definition is an approximation to this notion.

4.3 Definition. We say a model $N \in \mathbf{K}$ is *full* in the class $\mathbf{K}$ if for any finite $A \in \mathbf{K_0}$, for any finite B with $A \leq B \in \mathbf{K}_0$, any $C_1, \cdots, C_n \in \mathbf{K}_0$ with $B \subseteq C_i$ but $B \not\leq C_i$, and $A \leq (C_i - B)A$, we have

$$N \models \forall \overline{x} \left[\delta_A(\overline{x}) \rightarrow \exists \overline{y} \left(\delta_B(\overline{x}, \overline{y}) \wedge \bigwedge_{1 \leq i \leq n} \neg \exists \overline{z}_i \delta_{C_i}(\overline{x}, \overline{y}, \overline{z}_i) \right) \right].$$

where $\overline{x}$ enumerates A, $\overline{xy}$ enumerates B and $\overline{xyz}_i$ enumerates C_i.

Note that fullness is preserved under elementary equivalence. Thus each model in $\mathbf{K}$, the class of models of the theory of a generic, is full when the generic is full.

4.4 Lemma. *If* $(\mathbf{K}_0, \leq)$ *has the full amalgamation property, the* $(\mathbf{K}_0, \leq)$*-generic model* M *is full.*

Proof. Let A be an arbitrary finite subset of M. If $A \leq B \in \mathbf{K}_0$, $A \subseteq C \in \mathbf{K}_0$, then since $\mathbf{K}_0$ has the full amalgamation property, $B \bigotimes_A C \in \mathbf{K}$ and $C \leq B \bigotimes_A C$.

Let $C = \mathrm{icl_M(A)}$, then we have $C \leq B \bigotimes_A C = E$. Since M is $(\mathbf{K}_0, \leq)$-generic, and $C \leq M$, there is a $\tau : E \rightarrow_C M$ with $\tau E \leq M$. Thus, $M \models \delta_A(\overline{a}) \wedge \delta_B(\overline{a}, \tau\overline{b})$ where $\overline{a}$ enumerates A and $\overline{b}$ enumerates B.

Suppose for contradiction that there is a $C_i \subseteq M$ that contains τB, $\tau B \not\leq C_i$, and $A \leq (C_i - \tau B)A$. Then $\delta(C_i/\tau B) < 0$.

Note that $\tau E \leq M$, so $\mathrm{icl_M(\tau B)} = \mathrm{icl_{\tau E}(\tau B)}$, i.e., $\mathrm{icl_M(\tau B)} \subseteq \tau \mathrm{E}$. But $\delta(C_i/\tau B) < 0$ implies $C_i \subseteq \mathrm{icl_M(\tau B)}$. Hence $C_i \subseteq \tau E$. Let $D_i = C_i - \tau B$, then $\delta(D_i/A) = \delta(C_i/\tau B)$. Hence $\delta(D_i/A) < 0$. This contradicts the hypothesis $A \leq (C_i - \tau B)A$.

We showed in Lemma 3.19 that the intrinsic closure is contained in the algebraic closure. For classes with the full amalgamation property we have the converse. Note that the $\aleph_1$-categorical theories studied in [19] and [3] are not full and the algebraic closure is larger than the intrinsic closure.

4.5 Lemma. *Let* M *be the generic model for* $(\mathbf{K}_0, \leq)$*, which has the full amalgamation property, and* $N \models T$ *where* $T = \mathrm{Th(M)}$*. Then* $\mathrm{acl}_N(A) \subseteq \mathrm{icl_N(A)}$.

Proof. A routine application of the compactness theorem shows the property, $c \in \mathrm{acl}(A)$, is expressed by a single formula. Thus, the following property is true in any model of T if it is true in the generic model: If $c \in \mathrm{acl}(A)$

then $\delta(c/A) < 0$. We show now that this property holds in the generic model if $(\mathbf{K}_0, \leq)$ satisfies the full amalgamation property. If $\delta(c/A) \geq 0$, $A \leq Ac$. The amalgamation property implies that for any n, the free amalgamation of n copies of Ac over A is in $\mathbf{K}_0$. Considering $Ac_1, \ldots c_n$ as B, we deduce from the fact that the generic is full that each $Ac_1, \ldots c_n$ can be embedded in M and so c is not algebraic over A.

We now turn to the proof of free amalgamation over closed sets (Definition 3.37). For this, we need a syntactic characterisation of $N \leq M$.

4.6 Definition. i) Let (B, C) be a minimal pair for $(\mathbf{K}_0, \leq)$. Then $\phi_{B,C}(\overline{x})$ denotes the formula $\delta_B(\overline{x}) \wedge \neg\delta_C(\overline{x}, \overline{y})$.

ii) The *strong diagram* of N, denoted $\mathrm{Diag}_{\leq}(N)$ is the collection of formulas $(\forall \overline{y})\phi_{B,C}(\overline{b})$ where $\overline{b}$ enumerates a sequence from N such that $N \models \delta_B(\overline{b})$ but there is no sequence $\overline{c}$ of N such that $N \models \delta_C(\overline{b}, \overline{c})$.

Now it is immediate from the definition that

4.7 Lemma. *For $N \subseteq M \in \mathbf{K}$, $M \models Diag_{\leq}(N)$ if and only if $N \leq M$.*

4.8 Lemma. *If the class $(\mathbf{K}, \leq)$ is derived from a dimension function δ and its generic model M is full, then $T = \mathrm{Th(M)}$ has free amalgamation over closed sets.*

Proof. Suppose that N_1 and N_2 are two infinite models of T. A is a structure in $\mathbf{K}$. $f : A \longrightarrow N_1$, $g : A \longrightarrow N_2$ are strong embeddings, i.e., $fA \leq N_1$, $gA \leq N_2$. To simplify notation, we assume both f and g are the identity.

Claim 1. $Diag_e(N_1) \bigcup Diag_{\leq}(N_2) \bigcup R(N_1, A, N_2) = \emptyset$ is consistent.

Let $\phi_{B_1,C-1}(\overline{b}_1), \cdots, \phi_{B_n,C_n}(\overline{b}_n)$ be a finite set of sentences from $Diag_{\leq}(N_2)$ and let $\Gamma_0(\overline{g}_1, \overline{g}, \overline{g}_2)$ be a finite subset of $R(N_1, A, N_2) = \emptyset$. In Γ_0, let $\overline{g}_1$ enumerate a finite subset G_1 of $N_1 - A$ and $\overline{g}_2$ enumerate a finite subset G_2 of $N_2 - A$, while $\overline{g}$ enumerates a finite subset G of A and suppose that $\overline{g}_1, \overline{g}_2, \overline{g}$ are all the constants occuring in Γ_0. Let $\overline{b}_i$ enumerate B_i and $B \subseteq N_2$ be the union of the B_i along with G_2. Suppose $\overline{a}_i$ enumerates the portion of $\overline{b}_i$ in A. We will show

$$N_1 \models (\exists \overline{y}_i) \bigwedge_{i \leq n} \phi_{B_i}(\overline{a}_i, \overline{y}_i) \wedge \Gamma_0(\overline{a}_1, \overline{g}, \overline{y}). \tag{6}$$

Let t be the maximum of $|C_i - B_i|$. Fix ϵ_t according to Axiom 3.9 iii) so that if $|D| \leq t$ and $\delta(D/X) < 0$ then

$$\delta(D/X) < -\epsilon_t.$$

Let

$$A_0 = (B \cap A) \bigcup (G_1 \cap A) \bigcup G.$$

Choose A_1 with $A_0 \subseteq A_1 \subseteq \mathrm{icl_{N_1}(A_0)} = \mathrm{icl_{N_1}(A_0)} \subseteq \mathrm{A}$ such that $\delta(A_1) < d_{N_1}(A_0) + \epsilon_t$.

Let $E_1 = B \bigcup A_1$. Denote $G_1 A_1$ by A_2 and let $E_2 = G_1 \otimes_{A_1} A_2$. Clearly, we have $A_1 \leq E_1$ by Axiom A6 so $A_2 \leq E_2$. Let $F_1, \ldots, F_m$ enumerate all (isomorphism types of) structures of cardinality less than t such that each $F_k \supseteq E_2$ and $E_2 \not\leq F_k$ but $A_2 \leq A_2(F_k - E_2)$. Since N_1 is full,

$$N_1 \models \exists \overline{y} \delta_{E_2}(\overline{a}, \overline{y}) \wedge \bigwedge_k \forall \overline{z_k} \neg \delta_{F_k}(\overline{a}, \overline{y}, \overline{z_k})$$

where $\overline{a}$ enumerates A_2. Choose $\overline{b'}$ as a witness for $\overline{y}$. In particular, $N_1 \models \delta_{E_2}(\overline{a}, \overline{b'})$. Then there is an isomorphism $\alpha : E_2 \rightarrow_{A_1} E_2'$, where $\overline{b'}$ enumerates $E_2' - A_1$. The definition of E_2 guarantees that $N_1 \models \Gamma_0(\overline{g}_1, \overline{g}, \overline{b'})$.

Now we show that (6) holds. If not, for some $i \leq n$, there is a $\overline{f}_i$ enumerating $F_i' \subseteq N_1$, such that

$$N_1 \models \delta_{E_i}(\overline{a}, \overline{b}_i') \wedge \delta_{B_i}(\overline{b}_i', \overline{f}_i).$$

As $\delta(F_i/B_i) < 0$, $\delta(F_i'/E_1') < 0$.

Let $A_1(F_i' - E_1') = D$, then $\delta(D) - \delta(A_1) < 0$. Then since

$$\mid D - A_1 \mid = \mid F_k - B_1' \mid \leq \mid F_k \mid \leq t,$$

the definition of ϵ_t guarantees $\delta(D) - \delta(A_1) < -\varepsilon_t$,. Thus,

$$d_{N_1}(A_0) \leq \delta(D) = \delta(A_1) + (\delta(D) - \delta(A_1)) < (d_{N_1}(A_0) + \varepsilon_t) + (-\varepsilon_t) = d_{N_1}(A_0),$$

and this contradiction establishes Claim 1.

Claim 2. $Diag_e(N_1) \bigcup Diag_e(N_2) \bigcup R(N_1, A, N_2) = \emptyset$ is consistent.

This follows from Claim 1 by a routine alternating chains argument.

We want to express thanks to Chris Laskowski for helpful discussions and a key step in this proof and to the referee for suggesting an important further simplification which led us to the distinction between amalgamation over closed sets and free amalgamation over closed sets.

4.9 Remark. We have provided a sufficient condition for free amalgamation,which with the results of Section 3 shows stability in the unrestricted case where $\mathbf{K}_0$ contains all the structures allowed by the dimension function. The more interesting classes where there are further restrictions on the allowable finite structures are usually not full (see the end of Section 6). We provide one example of a slightly more restrictive class that is still full in Section 6.

5 Noninterpretability

The examples discussed in this paper are novel because for each of the theories T constructed, it is not possible to define an infinite group in any model of T. When $(\mathbf{K}_0, \leq)$ has the full amalgamation property this essentially reduces to the triviality of a type. But in the ω-stable cases the reason is more subtle. We will conclude the section with some refinements which dramatize the distinction between full amalgamation and amalgamation.

The distinction between i) and ii) in the following definition is well understood. We introduce the third condition to clarify the situation of this paper.

5.1 Definition. i) The structure A is *definable* in the structure B if there is an n, a definable subset of B^n and definable relations and operations on B^n yielding a structure isomorphic to A.

ii) The structure A is *interpretable* in the structure B if there is an n, a definable equivalence relation E on B^n, a definable subset X of B^n/E and definable relations and operations on X yielding a structure isomorphic to A.

iii) The structure A is *finitely-interpretable* in the structure if A is interpretable in B and all classes of the equivalence relation E have the same finite cardinality.

It is immediate that if T admits elimination of imaginaries then definability and interpretability coalesce. It is not too difficult to see that some of these examples do not admit elimination of imaginaries, although the situation for the $\aleph_1$-categorical examples is not clear. However, we can always establish the following weaker condition and it motivates the concept of finite interpretability.

5.2 Definition. The theory T is said to admit *weak elimination of imaginaries* if for every model M of T, for every $A \subseteq M$ and for every n, every definable equivalence relation $\simeq$ on A^n is given by $\overline{x} \simeq \overline{y}$ if and only if $f(\overline{x}) = f(\overline{y})$ where for some m, f is a definable map from A^n to A^m/E and E is a definable equivalence relation on A^m with finite classes.

The following result is proved in [13].

5.3 Proposition. *If T is stable and for every $M \models T$, any elementary extension N of M, every n, and every $\overline{a} \in N^n$, there is a unique minimal algebraically closed $B \subseteq M$ such that an automorphism α of M preserves $p = \mathrm{tp}(\overline{\mathrm{a}}/\mathrm{M})$ if and only if α fixes B setwise, then $T = \mathrm{Th}(\mathrm{A})$ admits weak elimination of imaginaries.*

Using the fact that intrinsic closure is contained in algebraic closure we now establish that our theories satisfy weak elimination of imaginaries. (We can apply Lemma 5.3 to the algebraic closure of the intrinsically closed base we determine below.)

5.4 Lemma. *Let T be the theory of a generic model derived from a dimension function δ satisfying Axiom 3.9. For every $N \models T$, every n and every $\overline{a} \in N^n$, there is a unique minimal intrinsically closed $B \subseteq N$ such that an automorphism α of N preserves $p = \mathrm{tp}(\mathrm{a}/\mathrm{N})$ if and only if α fixes B setwise. Thus, T admits weak elimination of imaginaries.*

We deduce this result immediately from the following lemma. We continue to work in the monster model $\mathcal{M}$.

5.5 Lemma. *If $B_1, B_2 \subseteq N$ are intrinsically closed, $A \downarrow^d_{B_1} N$, and $A \downarrow^d_{B_2} N$ then $A \downarrow^d_{B_1 \cap B_2} N$.*

Proof. The closure condition is immediate; we must show $d(A/(B_1 \cap B_2) = d(A/N)$. By Lemma 3.33 applied to $\overline{AB_1}$ and N ($\overline{AB_2}$ and N), we have $R(A, B_1, B_2) = \emptyset$ ($R(A, B_2, B_1) = \emptyset$). This implies that $R(A, B_1B_2) \subseteq R(A, B_1 \cap B_2)$. Thus,

$$d(A/B_1) = d_{\overline{AB_1}}(A/B_1) = d_{\overline{AB_1}}(A/B_1 \cap B_2) = d(A/B_1 \cap B_2).$$

Since the intersection of intrinsically closed sets is intrinsically closed, the existence of a unique minimal 'base' is now evident.

The following easy result now is seen to apply to the theory of any generic structure.

5.6 Lemma. *If the theory T admits weak elimination of imaginaries and the structure N is interpretable in a model M of T, then N is finitely interpretable in M.*

Thus our task is to show that no infinite group is finitely interpretable in a generic structure. The following definition extends an idea of Hrushovski for geometries to the more general situation of a structures with a rank satisfying Axiom 3.9.

5.7 Definition. Suppose $(\mathbf{K}_0, \leq)$ satisfies Context 3.13 and suppose M is the generic model.

i) If $E_i (i \in I)$ is a collection of finite subsets of M and s is a nonempty subset of I, let E_s denote $\bigcap_{i \in s} E_i$ and $E_\emptyset = \bigcup_{i \in I} E_i$.

ii) $(\mathbf{K}_0, \leq)$ is *flat* if for any finite set of closed subsets of M,

$$\sum_{s \subseteq I} d(E_s) \leq 0.$$

5.8 Lemma. *Suppose $(\mathbf{K}_0, \leq)$ is derived from a dimension function $\delta_{1,\alpha}$ for some α with $0 < \alpha \leq 1$. Then $(\mathbf{K}_0, \leq)$ is flat.*

Proof. The linearity of δ makes this a straightforward computation as in Lemma 15 of [19].

5.9 Lemma. *Suppose M is generic for $(\mathbf{K}_0, \leq)$ which is flat and suppose that for $A \subseteq M$ and $a \in \mathrm{acl}(A)$, $d(a/A) = 0$. Then no infinite group is finitely interpretable in M.*

Proof. Again, the argument follows that set forth by Hrushovski for the case of a geometry (Lemma 14 of [19]). Consider the group configuration formed by three independent generics a_1, a_2, a_3 of a group finitely interpretable in M. The key points which rely on the condition relating dimension and algebraic closure are that the dimension of each line in the diagram is the same as the dimension of two point on the line (needed for Hrushovski's argument) and that the points in the diagram can be replaced by the entire equivalence class in which they lay (using *finite* interpretability).

Combining Lemma 5.6 and Lemma 5.9 we have:

5.10 Corollary. *Suppose* $(\mathbf{K}_0, \leq)$ *is derived from dimension function* $\delta_{1,\alpha}$ *for some* α *with* $0 < \alpha \leq 1$ *and* M *is generic for* $(\mathbf{K}_0, \leq)$ *then then there is no infinite group interpretable in* M*.*

When $(\mathbf{K}_0, \leq)$ has full amalgamation, we can strengthen this result. The following defines a weaker condition than the existence of an infinite group.

5.11 Definition. A binary function f is cancellable if $f(x, y) = f(x, y')$ implies $y = y'$ and $f(x, y) = f(x', y)$ implies $x = x'$.

5.12 Corollary. *If* T *is the theory of a generic of* $(\mathbf{K}_0, \leq)$*,* $\leq$ *is derived from a* δ *satisfying Axioms 3.9, and the full amalgamation property, or more specifically, such that for each* $N \in \mathbf{K}$*,* $\mathrm{acl}_N = \mathrm{icl}_{\mathrm{N}}$*, then no cancellable function with a pair of independent points in its domain is finitely interpretable in any model of* T*.*

Proof. Fix a model M of T and suppose with respect to some equivalence relation E with finite classes, f is cancellable. Choose b and c independent over M with $\langle b/E, c/E\rangle$ in the domain of f. Suppose $a/E = f(b/E, c/E)$. Since the classes of E are finite, $c \in \mathrm{acl}(a, b)$ and so if $a \in \mathrm{acl}(b)$ then $c \in \mathrm{acl}(b)$. Thus, $b \downarrow_M c$ implies $a \notin \mathrm{icl}(\mathrm{Mb})$ and $a \notin \mathrm{icl}(\mathrm{Mc})$. Since $\mathrm{acl}_N = \mathrm{icl}_{\mathrm{N}}$, $a \in \mathrm{icl}(\mathrm{Mbc})$. But this contradicts Lemma 3.31 (the triviality of intrinsic closure).

Applying Lemma 4.5, we obtain:

5.13 Corollary. *If* T_α *is the theory of the generic of* $\mathbf{K}_\alpha$ *there is no infinite group interpretable in* T*.*

The arguments given here generalize and clarify the hypotheses of the more special cases considered in [3] and [15], and in [19].

The assumption that f is cancellable may be much stronger than is actually needed here. However, the requirement that there be full amalgamation is essential for Lemma 4.5 and Theorem 5.12. We show in Section 6 the existence of an ω-stable projective plane (which therefore has definable cancellable binary functions); but $\mathbf{K}_0$ is restricted to omit squares. Moreover, the construction of a generic $\aleph_1$-categorical projective plane [3] provides an example where the algebraic closure of A is precisely the c with $d(c/A) = 0$ but there are definable cancellable functions. However, Theorem 5.10 (taking $\alpha = 1/2$) showed that even in these cases no infinite group is interpretable. We discuss in Section 6 the case of irrational α and omitting squares.

6 Stable Pseudoplanes

In this section we restrict to the study of classes of graphs. Hrushovski constructed a strictly stable $\aleph_0$-categorical theory by constructing such a stable $\aleph_0$-categorical pseudoplane and remarking that [9] had shown no such could be superstable. The pseudoplane is derived from a graph that omits squares by regarding each vertex of the graph as both a point and a line of the plane. In this section we determine which α give rise to projective and pseudoplanes and the stability class of the associated structures.

6.1 Definition. A pseudoplane is a structure of points, lines, and an incidence relation satisfying the following axioms.

i) Each pair of points is on only finitely many lines.

ii) Each pair of lines intersect in only finitely many points.

iii) Each line is infinite.

In a projective plane, two points lie on exactly one line and every pair of lines intersect nontrivially.

6.2 Definition. Let $\mathbf{K}'_\alpha$ be the class of finite graphs A which omit squares and satisfy $|X| - \alpha \cdot e(X) \geq 0$ for all $X \subseteq A$.

The lemma below was pointed out by D.Marker.

6.3 Lemma. *If the class $(\mathbf{K}'_\alpha, \leq)$ has AP, then the coefficient $\alpha \geq 1/2$.*

Proof. Suppose not, i.e. $\alpha < 1/2$.

Consider the following structure. $A = \{a_1, ..., a_{m+1}\}$ is a discrete graph. $B_1 = A \bigcup \{b_1, ..., b_m\}$ with relations $R(a_i, b_i)$, $i = 1, \cdots, m$, $R(b_i, b_{i+1})$, $i = 1, \cdots, m-1$, and $R(b_m, a_{m+1})$. $B_2 = A \bigcup \{b'_1, ..., b'_m\}$ with $R(b'_i, b'_{i+1})$ for $i < m$ and $R(a_i, b'_i)$ exactly when $R(a_i, b_i)$ holds in B_1 except that in addition $R(b'_1, a_2)$ holds.

Since $\alpha < 1/2$, it is easy to check $A \leq B_1$. It is also easy to check that for all X,

$$A \subseteq X \subseteq B_2 \Rightarrow \delta(X/A) \geq m(1 - 2\alpha) - \alpha.$$

Since the expression on the right is positive for all sufficiently large m, we have $A \leq B_2$ for all sufficiently large m.

Although $A \leq B_1$ and $A \leq B_2$ hold, if $\mathbf{K}'_\alpha$ omits squares we can not map B_1 to B_2 fixing A. For, $\mathbf{K}'_\alpha$ omits squares, so to amalgamate we have to map b_m to b'_m since we must fix a_m and a_{m+1} under mapping. Thus we have to map b_{m-1} to b'_{m-1}, b_{m-2} to b'_{m-2},... until we map b_1 to b'_1. But the last step is impossible because $R(b'_1, a_2)$ holds in B_2 but $R(b_1, a_2)$ does not hold in B_1. Thus AP fails.

6.4 Lemma. *If $\alpha > 1/2$, $A, B, C \in \mathbf{K}'_\alpha$ and $A \leq B$, $A \leq C$, then $B \bigotimes_A C$ omits squares.*

Proof. $A \leq B$, $A \leq C$ imply that each point in $B - A$ or $C - A$ can only have one edge connecting it to a point in A. Thus if B and C omit squares, so does $B \bigotimes_A C$.

6.5 Corollary. *If $\alpha > 1/2$ then $(\mathbf{K}'_\alpha, \leq)$ has the full amalgamation property.*

6.6 Lemma. *If $\alpha \geq 1/2$, then $(\mathbf{K}'_\alpha, \leq)$ has the sharp amalgamation property.*

Proof. Fix an integer m. Let finite $A, B, C \in \mathbf{K}'_\alpha$, satisfy $A = B \bigcap C$, $A \leq B$, B is a primitive extension of A and $A \leq_{|B|-|A|} C$.

Case 1. $\alpha > 1/2$. $A \leq B$, $A \leq_{|B|-|A|} C$ imply that each point in $B - A$ or $C - A$ can only have one edge connecting it to a point in A. Thus if B and C omit squares, so does $B \bigotimes_A C$. Thus, $B \bigotimes_A C \in \mathbf{K}'_\alpha$.

Case 2. $\alpha = 1/2$. Suppose B is a primitive extension of A and $A \leq_{|B|-|A|} C$. Either $B \bigotimes_A C \in \mathbf{K}'_\alpha$ or there are two points d_1, d_2 in A and $b \in B - A$,

$c \in C - A$ with $R(d_1, b), R(d_2, b), R(d_1, c)$ and $R(d_2, c)$. But then B must equal Ab and identifying b and c defines an embedding of B into C. So $\mathbf{K}'_\alpha$ has the sharp amalgamation property.

6.7 Corollary. *i) Let $0 < \alpha \leq 1$. The generic graph determined by $(\mathbf{K}_\alpha, \leq)$ is stable, ω-stable if α is rational, and a pseudoplane if $\alpha > 1/2$.*

ii) Let $1/2 \leq \alpha < 1$. The generic graph determined by $(\mathbf{K}'_\alpha, \leq)$ is a stable pseudoplane, which is ω-stable if α is rational, and a projective plane if $\alpha = 1/2$.

Proof. i) $\mathbf{K}_\alpha$ is full by Lemma 4.2 and so the associated generic structure is stable (ω-stable) by Lemmas 3.34 and 4.8. When α is greater than 1/2 if c is connected to both a and b, the pair $\langle \{a, b\}, \{c\} \rangle$ is a minimal pair. Thus, applying Lemma 3.19, there are a bounded number of lines through any pair of points and we have a pseudoplane.

ii) When $\alpha > 1/2$, the $(\mathbf{K}'_\alpha, \leq)$-generic model M is full and so the associated generic structure is stable (ω-stable) by Lemmas 3.34 and 4.8. Since $(\mathbf{K}'_\alpha, \leq)$ omits squares the associated structure is a pseudoplane for $\alpha \geq 1/2$. When $\alpha = 1/2$ then there is a line through each two points and the structure is a projective plane. By Lemma 6.6, for $\alpha \geq 1/2$, $(\mathbf{K}'_\alpha, \leq)$ satisfies the sharp amalgamation property and thus by Lemma 2.32 the uniform amalgamation property. By Theorem 2.28, when α is rational $(\mathbf{K}'_\alpha, \leq)$ has finite closures. By Lemma 2.21, $(\mathbf{K}'_\alpha, \leq)$ has amalgamation over finite closed sets. For rational α, using the finite closures again, $(\mathbf{K}'_\alpha, \leq)$ has amalgamation over closed sets. Now ω-stability follows as before.

6.8 Limitations. i) To obtain a projective plane by this kind of construction we need that adding a single point connected to two points in the ground structure is a strong extension. This requires that $\alpha \leq 1/2$. But Lemma 6.3 shows that when $\alpha < 1/2$, we cannot even require that two points lie on at most one line. Thus, a projective plane can be constructed by these methods only by choosing $\alpha = 1/2$. Geometrically different projective planes can be obtained by omitting further graphs from $\mathbf{K}'_\alpha$ [4]. In particular this shows that construction cannot be modified to yield a stable $\aleph_0$-categorical projective plane. For $\alpha = 1/2$, implies any such attempt would yield an ω-stable structure and it is well known that there are no $\aleph_0$-categorical, ω-stable projective planes.

ii) Hrushovski's construction yields an $\aleph_0$-categorical pseudoplane by restricting $\mathbf{K}'_\alpha$ even further for selected α of 'infinite index'. This is expounded well in the opening sections of [17]. When $\alpha > 1/2$, we saw that $\mathbf{K}'_\alpha = \mathbf{K}_\alpha$ so acl is the same as icl and the argument of Theorem 5.12 applies and no infinite group c an be interpreted in T_α.

iii) If α is $1/2$ the graph gives rise to an ω-stable (but not $\aleph_0$-categorical) pseudoplane. The construction in [3] converts this to an $\aleph_1$-categorical structure. The proof of ω-stability in this case proceeded by showing that $\mathbf{K}'_\alpha$ satisfies the uniform amalgamation property. We cannot even conclude it is stable by the means that work for $\alpha > 1/2$ since

6.9 Remark. If $\alpha = 1/2$, then the generic model M is not full in $\mathbf{K}'_\alpha$.

First note that a simple consequence of fullness (by taking $n = m_1 = 1$) in the definition is that if M is full then for any $A \leq B \subseteq C \in \mathbf{K}'_\alpha$ with $B \not\leq C$, and $A \leq (C - B)A$, for any $\sigma : A \longrightarrow M$ there is $\tau : B \longrightarrow M$ with $\sigma \subseteq \tau$ such that there is no $\theta : C \longrightarrow M$ with $\tau \subseteq \theta$.

Thus to show M is not full, it is suffices to show that for such σ and τ above there is $\theta \supseteq \tau$ embedding C into M. Let $A = \{a, b, c, d\}$ be a discrete graph. Let $B = Aefg$ with $R(a, e), R(b, f), R(c, g), R(d, g), R(e, f)$ and $R(f, g)$. Let $C = Bh$ with $R(a, h)$, $R(d, h)$ and $R(e, h)$. Thus $A \leq B \subseteq C$, $B \not\leq C$ and $A \leq (C - B)A$. Since the generic model M is $(\mathbf{K}'_\alpha, \leq)$-universal, there are $a', b', c', d', e', f', g', h' \in M$ with $R(a', e'), R(b', f'), R(c', g'), R(d', g'), R(e', f'), R(f', g'), R(a', h'), R(d', h')$ and $R(e', h')$.

Let $\sigma : A \longrightarrow M$ by $a \to a', b \to b', c \to c', d \to d'$. Thus to avoid forming a square, any $\tau \supseteq \sigma$ has to map g to g', f to f', e to e'. But let $\theta \supseteq \tau$ map h to h', then θ embeds C to M. Hence M is not full.

7 Concluding Remarks

The axiomatization laid out in this paper is more general than just codifying the arguments for proving the stability class of the most important examples. Here is a brief survey of applications. The first three have been extensively investigated and motivated the axiomatization here; the later ones are new.

i) $(\mathbf{K}_\alpha, \leq)$ where $\leq$ is induced by $\delta_{1,\alpha}$.

ii) Subclasses of $(\mathbf{K}_\alpha, \leq)$ where $\leq$ is induced by $\delta_{1,\alpha}$ where α is rational. In [19], [3], and [4] the theory of the generic is $\aleph_1$-categorical. [5] shows that even a very minor variation yields infinite Morley rank.

iii) Subclasses of $(\mathbf{K}_\alpha, \leq)$ where $\leq$ is induced by $\delta_{1,\alpha}$ where α is irrational. Classes which are finitely closed are studied in [18] and a certain limit of these is studied by [17].

iv) Subclasses $\mathbf{P_0}$ of $(\mathbf{K}_\alpha, \leq)$ where $\leq$ is induced by $\delta_{1,\alpha}$ but where $\mathbf{P_0}$ is a class of finite graphs,defined by graph theoretic conditions, that has the full amalgamation property. Examples are the class of graphs which omit finite cycles [10] and the class of graphs which omit a specified finite sum of complete graphs [11].

v) Finally, one could modify the choice of δ in a more radical way. In the usual context $e(A)$ counts the number of connected two element subgraphs (edges). Fix e.g. a particular complete graph such as a triangle and let $e(A)$ be the number of triangles embedded in A plus the number of edges of A. Let $\delta(A) = |A| - \alpha \cdot e(A)$ and induce $\leq$ from δ as usual. Now if $\mathbf{P_0}$ is the collection of finite graphs A with $\delta(B) \geq 0$ for every subgraph B, it is easy to check that Axioms 3.9 hold and so does the full amalgamation property. So we get a new stable theory.

There may be other classes with more subtle methods of amalgamation that allow for example the construction of a strictly superstable generic structure.

In another direction, Kitty Holland (in preparation) has shown how to modify this rubric to encompass Hrushovski's construction of the fusion of strongly minimal sets.

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