THE METAMATHEMATICS OF RANDOM GRAPHS

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Abstract

We explain and summarize the use of logic to provide a uniform perspective for studying limit laws on finite probability spaces. This work connects developments in stability theory, finite model theory, abstract model theory, and probability. We conclude by linking this context with work on the Urysohn space.

Erdös pioneered the use of the probabilistic method for proving statements in finite combinatorics. In this paper we explain how logic is used to formalize and give general proofs for a large class of such arguments. We consider the role of the logic, the probability measure, and the vocabulary in formulating the problems. We report a number of results in this area, spotlighting the Baldwin-Shelah method of determined theories.

In this paper we explore some of the surprising connections between diverse areas which appeared at this conference. On the one hand we discuss the use of a specific Abstract Elementary Class as a tool for proving 0-1 laws. On the other, we conclude with a formulation of issues relating to the Urysohn space in the framework for studying random graphs developed here.

Here is a specific example of the use of the probabilistic method. A (round robin) tournament is a directed graph with an edge between every pair of points. Fix k. Is there a tournament that satisfies P_k: for each set of k-players there is another who beats each of them?

Here is a method for showing the answer is yes. Let S_n be the set of all tournaments with n players. Then, $|S_n| = 2^{\binom{n}{2}}$. Each of these is equally likely. Call a k-set X *bad* if no element dominates each member of X. If $Y(T)$ is the number of bad k-sets in a tournament T then

$$E(Y) = \binom{n}{k}(1-(1/2)^k)^{n-k}.$$

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Then $E(Y) \to 0$ and by Markov's inequality $P(Y \geq 1) \to 0$. So a.a., there is a tournament satisfying P_k. That is, as n tends to infinity, with probability 1, there is a tournament in which every set of k-players is dominated by a single player. Thus for sufficiently large n, there actually is a tournament of size n satisfying P_k.

Logic provides a formalism for identifying a large class of properties that behave like this P_k. In this paper we survey how this class of properties can depend on the choice of logic, the choice of vocabulary for expressing the property, and the probability measure chosen.

We begin by formalizing the notion that almost all members of a certain family of events are true.

Definition 1 *Let Ω_n denote the set of graphs on the vertex set $\{0, \ldots, n-1\}$. Let P_n be a probability measure assigning an element of $[0,1]$ to each subset of Ω_n. Let X be a family of sequences X_n of events in Ω_n. Then (Ω_n, P_n, X)* satisfies a zero-one law *if for each sequence $X_n \in X$,*

$$\lim_{n \to \infty} P_n(X_n) = 0$$

or

$$\lim_{n \to \infty} P_n(X_n) = 0.$$

We will consider measures that are determined by the 'edge probability' $p(n)$ of two vertices being connected.

Definition 2 *Let B be a graph with $|B| = n$ and $0 < p = p(n) < 1$.*

1. *Let* $P_n^p(B) = p^{|e(B)|} \cdot (1-p)^{\binom{n}{2} - e(B)}$.
2. *For any* $Y \subset \Omega_n$,

$$P_n^p(Y) = \sum \{P_n^p(B) : B \in Y\}.$$

The school of Erdös, Renyi, and Spencer has many results on such measures. In this note we will discuss three such probability measures.

1. $p(n)$ is constant.
2. $p(n)$ is $n^{-\alpha}$ for $0 < \alpha < 1$ and often irrational.
3. $p(n) = p_n^l$ is

$$\frac{ln(n)}{n} + \frac{l \cdot ln(ln(n))}{n} + \frac{c}{n}$$

   where l is an arbitrary fixed nonnegative integer, and c is a positive constant.

First order logic is built up from atomic formulas by Boolean operations and quantification over individuals. In first order logic k-connected (every pair of elements is connected by a path of length k) is expressible while connected (every pair of elements is connected by a path of some finite length) is not. One of the most important ways to generate a family of events X_n is by the uniform interpretation of a formula of first order logic. For each n and each formula ϕ, let $X_n = X_n^\phi$ be the graphs on n vertices which satisfy ϕ. Formally,

Definition 3 *Let B be a graph with $|B| = n$ and $0 < p = p(n) < 1$.*

1. *Let $P_n^p(B) = p^{|e(B)|} \cdot (1-p)^{\binom{n}{2} - e(B)}$.*
2. *For any formula ϕ, let*

$$P_n^p(\phi) = \sum \{P_n^p(B) : B \models \phi, |B| = n\}.$$

Now a famous theorem [Fagin [12] and (Glebski,Y. and Kogan, V. and Liogon'kii, M.I, and Taimanov, V.A.)[13]] asserts the 0-1 law for the uniform probability measure.

Theorem 4 *If the edge probability is given by $p(n) = 1/2$, for each formula ϕ, $\lim_{n\to\infty} P_n^p(\phi)$ is 0 or 1.*

The result works as well if 1/2 is replaced by any constant edge probability. Let T^p denote the collection of almost surely true sentences.

We will discuss various families of sequences of events that are determined by two parameters. In all cases we will be looking at the families definable in a logic. But the logic might be first order or $L_{\omega_1,\omega}$, or first order with the Ramsey quantifier: $L_{\omega,\omega}(Q_{ram,f})$. Our second parameter fixes a certain class of structures with universe n but with relations on n determined by a vocabulary. It might be just equality, or a successor function or a vector space. The crucial point here is that as we vary the logic and the vocabulary, we do not change the probability space. It is always the set of graphs on $\{0,\ldots,n-1\}$. But we change the set of events. In a different direction we may change the probability measure.

Definition 5 *Consider a family (Ω_n, P_n) and let L represent the first order sentences in a vocabulary τ.*

1. *The* almost sure *theory of (Ω_n, P_n) is the collection of ϕ such that*

$$\lim_{n\to\infty} P_n(\phi) = 1.$$

2. *A theory T is complete if for every ψ either $\psi \in T$ or $\neg\psi \in T$.*

Thus there is a first order zero-one law for (Ω_n, P_n) just if the almost sure theory is complete. To prove a 0-1 law we now have the following strategy.

Find a collection Σ of axioms that are

1. almost surely true
2. axiomatize a complete theory

The 'almost sure' will be a distinct probabilistic argument using properties of the particular measure. The completeness can be shown in a number of ways: categoricity, 'quantifier elimination', Ehrenfeucht-Games, Determined Theories

In some situations the almost sure theory is $\aleph_0$-categorical (up to isomorphism there is only one countable model); completeness follows immediately by the Łos-Vaught test. As we explain below this leads to even stronger results than a 0-1 law. Many of the completeness results were proved historically either by showing that the almost sure theory admitted quantifier elimination or by using Ehrenfeucht games to show any two models of the almost sure theory T are elementarily equivalent and concluding that T is complete. We will emphasize here a technique developed by Baldwin and Shelah [9, 1], which we call the method of determined theories. Here are a few examples.

The 0-1 law for first order logic is the classic case. The Rado random graph is the unique countable model of the following extension axioms.

Axioms ϕ_k :

$$(\forall v_0 \dots v_{k-1} w_0 \dots w_{k-1})(\exists z)\ \wedge_{i<k} (Rzv_i \bigwedge \neg Rzw_i)$$

A variant on our initial probability example shows each extension axiom has probability 1. And a back and forth argument shows the theory axiomatized by all the ϕ_k is categorical in $\aleph_0$; hence complete. The $\aleph_0$-categoricity allows the extension of this 0-1 law to a number of other logics. Hella, Kolaitis, Luosto and Vardi [15, 18] developed the nicest formalism for unifying these extensions.

Definition 6 *The logics L and L' are almost everywhere equivalent with respect to the probability measure P if there exists a collection C of finite models such that $P(C) = 1$ and for every sentence θ of L there is a sentence θ' of L' such that θ and θ' are equivalent on C (and conversely).*

With this definition we have a strong way to say that the 0-1 law extends from first order logic to the finite variable fragment (sentences which contain only finitely many variables – free or bound), $L^\omega_{\infty,\omega}$, of $L_{\infty,\omega}$.

Theorem 7 *(Hella, Kolaitis, Luosto) FO and $L^\omega_{\infty,\omega}$ are almost everywhere equivalent with respect to the uniform distribution.*

We can extend the logic in a different way by adding the *Ramsey quantifier*. For infinite models, one would think of the Ramsey quantifier as meaning $M \models Q_{Ram}\mathbf{x}\phi(\boldsymbol{a}, \mathbf{y})$ if M contained an infinite set B such that every n-tuple $\mathbf{b}$ from A satisfied $\phi(\boldsymbol{a}, \mathbf{b})$. A natural modification for finite models specifies a function f telling the size of the homogeneous set.

Definition 8 *The Ramsey quantifier ($Q_{ram,f}$) is defined by $Q^n_f\mathbf{x}\phi(\mathbf{x}, \mathbf{y})$ holds in a finite model $|A|$ if there is a homogeneous subset for ϕ of cardinality at least $f(|A|)$.*

The next result unites probability on finite graphs with two areas of model theory. The finite cover property is a very strong compactness property for formulas.

Definition 9 *The first order theory T does not have the* finite cover property *if and only if for every formula $\phi(x; y)$ there exists an integer n depending on ϕ such that for every A contained in a model of T and every subset p of $\{\phi(x, a), \neg\phi(x, a); a \in A\}$ the following implication holds: if every $q \subseteq p$ with cardinality less than n is consistent then p is consistent.*

The finite cover property was introduced by Keisler in the late 60's [17] to produce unsaturated ultrapowers. It has since played a varied role in model theoretic topics ranging from $\aleph_1$-categoricity through the expansions of models by naming submodels [21] to the effect of naming automorphisms [11] to providing a further hierarchy of theories [5]. Here we note its connection with 0-1 laws.

Theorem 10 *[3] If f is unbounded, the logic $L_{\omega,\omega}(Q_{ram,f})$ is almost everywhere equivalent to first order logic on graphs with respect to either the uniform distribution or edge probability $n^{-\alpha}$.*

We sketch the proof to show the interaction of a wide range of logical tools.

1. Baldwin and Kueker [7] proved that the Ramsey quantifier is eliminable from T in the $\aleph_0$-interpretation if T is either $\aleph_0$-categorical or does not have the finite cover property.

A little fiddling derives the result for the uniform probability from the $\aleph_0$-categoricity of the random graph.

2. As we discuss below, T^α is the almost sure theory for finite graphs with edge probability $n^{-\alpha}$ and irrational α. Baldwin and Shelah [10] proved the almost sure theory T^α does not have the finite cover property. Combining observation 1) with this result gives the theorem.

In the late 80's two major results were published that I later discovered depend on the same fundamental ideas.

Theorem 11 (Spencer-Shelah-1988) *If α is irrational, for each formula ϕ, $\lim_{n\to\infty} P^\alpha_n(\phi)$ is 0 or 1.*

Theorem 12 *[Hrushovski late 80's]*

1. *There is an $\aleph_0$ categorical strictly stable theory.*
2. *There is a strongly minimal set which is neither 'trivial', nor 'vector-space like' nor 'field-like'.*

We next try to explain the connection between these different ideas. We list the two parts of Theorem 12 together as they are both examples of the use of predimensions described below. However, only Theorem 12.1 is directly relevant to 0-1 laws. Working out this connection lead to the 'method of determined theories' for proving 0-1 laws.

On the one hand, we have seen that the Rado random graph is the unique countable model of T^p. Here is another description.

Definition 13 *Let $\boldsymbol{K}_0$ be the collection of all finite graphs (including the empty graph) and write $A \prec_{\boldsymbol{K}} B$ if A is subgraph of B.*

Definition 14 *The class $(\boldsymbol{K}, \prec_{\boldsymbol{K}})$ satisfies the* amalgamation property *(***AP***) if for any situation:*

$$\begin{array}{ccc} & & B \\ & \nearrow & \\ A & & \\ & \searrow & \\ & & C \end{array}$$

there exist a $D \in \boldsymbol{K}$ and strong embeddings, such that the following diagram is commutative.

$$\begin{array}{ccccc} & & B & & \\ & \nearrow & & \searrow & \\ A & & & & D. \\ & \searrow & & \nearrow & \\ & & C & & \end{array}$$

Definition 15 *If $\boldsymbol{K}$ is a collection of finite models, the countable model M is* $(\boldsymbol{K}, \prec_{\boldsymbol{K}})$-generic *if*

1. *If $A \leq M, A \leq B \in \boldsymbol{K}$, then there exists $B' \leq M$ such that $B \cong_A B'$,*
2. *For every finite $A \subseteq M$ there is a finite B with $A \subseteq B \prec_{\boldsymbol{K}} N$.*

Theorem 16 *Any two countable $(\boldsymbol{K}, \prec_{\boldsymbol{K}})$-generic structures are isomorphic.*

The Rado graph can also be seen as the $(\boldsymbol{K}_0, \prec_{\boldsymbol{K}})$-generic. In order to explain the advances by Hrushovski, Shelah and Spencer, we introduce the notion of predimension. In the following we work with graphs– symmetric binary relations. The restriction to binary relations is solely for ease of exposition. The proofs go through *mutatis mutandis* for any finite collection of *symmetric* finitary relations. (By symmetric I mean $R(\boldsymbol{a})$ holds if and only if $R(\boldsymbol{a}')$ holds, where $\boldsymbol{a}'$ is any permutation of $\boldsymbol{a}$.)

Fix a base language L and expand it by a new binary relation, R. Call the new language L^+. R is symmetric and irreflexive. For any finite B, $e(B)$ is number of 'edges' of B, the number of (unordered) pairs in B that satisfy R.

Definition 17 *Define predimensions on finite structures as follows.*

1. *Fix an real number α, $0 < \alpha < 1$ and let*

$$\delta_\alpha(B) = |B| - \alpha e(B).$$

2. *Let $\boldsymbol{K}_\alpha$ be all finite graphs B such that for all $A \subseteq B$, $\delta_\alpha(A) \geq 0$.*

3. *For any M, and finite $A \subseteq M$, $d_M(A) = \inf(\delta_\alpha(B))$ for $A \subseteq B \subseteq_\omega M$.*

Definition 18 *For $M \subseteq N$, we say that M is* **strong** *in N, and write $M \leq N$, if for all finite $X \subseteq M$,*

$$d_N(X) = d_M(X).$$

Now we introduce a very particular example of an abstract elementary class. We are concerned only with finite and countable structures. Recall [14, 23, 2] that an AEC is a class of structures $\boldsymbol{K}$ along with a notion of strong submodel that satisfies A1-A3 and the consequence A4' of A4 below and has the property that every subset X of a model in $\boldsymbol{K}$ is contained in a strong submodel N with $|N| \leq |X| + \kappa$, where κ is fixed as the Löwenheim-Skolem number of $\boldsymbol{K}$. Here we ignore the Löwenheim-Skolem number but as we simply take the closure of the class described below under countable unions we obtain an AEC with Löwenheim-Skolem number $\aleph_0$.

Assumption 19 (Axiom Group A) *Let $A, B, C \in \boldsymbol{K}$.*

- *A1 $A \leq A$.*
- *A2 If $A \leq B$ then $A \subseteq B$.*
- *A3 If $A, B, C \in \boldsymbol{K}$, then*

$$A \leq B \leq C \Longrightarrow A \leq C.$$

- *A4 If $A, B, C \in \boldsymbol{K}$, $A \prec_{\boldsymbol{K}} C$, $B \prec_{\boldsymbol{K}} C$ and $A \subseteq B$ then $A \prec_{\boldsymbol{K}} B$.*
- *A4' If $A \subset B \prec_{\boldsymbol{K}} C$ and $A \prec_{\boldsymbol{K}} C$ then $A \prec_{\boldsymbol{K}} B$.*
- *A5 $\emptyset \in \boldsymbol{K}$ and $\emptyset \leq A$ for all $A \in \boldsymbol{K}$.*

A first approach to connecting the Shelah-Spencer random graph and the Hrushovski construction is given by the following theorem.

Theorem 20 *If* $(\boldsymbol{K}_0, \prec_{\boldsymbol{K}})$ *is a collection of finite relational structures that satisfies A1-A5 and has the amalgamation property then there is a countable* $\boldsymbol{K}_0$*-generic model* M*.*

Lemma 21 *The class* $\boldsymbol{K}_\alpha$ *satisfies* **A1-A5** *and has the amalgamation property.*

1. *If* $\alpha = .5$ *the generic model is an* $\aleph_1$*-categorical non-Desarguesian projective plane (Baldwin).*
2. *If* α *is irrational the theory* T_α *of the generic model is a strictly stable first order theory (Baldwin-Shi).*

But emphasis on the 'generic' model is misplaced. In analogy to the uniform probability case, one would like to identify the generic model as the 'the random graph'. But T_α is not $\aleph_0$-categorical; in fact it has $2^{\aleph_0}$ countable models. In order to prove 0-1 laws we must use another strategy to identify T_α as an almost sure theory. We call this new strategy the *method of determined theories.*

Definition 22 *The theory* T *is* determined *if there is a family of functions* F^n_M *with the following property. For any formula* $\phi(x_1 \ldots x_r)$ *there is an integer* ℓ_ϕ*, such that for any* $M, M' \models T$ *and any* r*-tuples* $\boldsymbol{a} \in M$ *and* $\boldsymbol{a}' \in M'$ *if* $F^{\ell_\phi}_M(\boldsymbol{a}) \sim F^{\ell_\phi}_{M'}(\boldsymbol{a}')$ *by an isomorphism taking* $\boldsymbol{a}$ *to* $\boldsymbol{a}'$*, then* $M \models \phi(\boldsymbol{a})$ *if and only if* $M' \models \phi(\boldsymbol{a}')$*.*

Theorem 23 (Baldwin-Shelah[9]) *If* T *is determined and for each* $M, M' \models T$ *and each* n*,* $F^n_M(\emptyset) \sim F^n_{M'}(\emptyset)$ *then* T *is complete.*

The following theories (explained below) are determined:

1. [9] The semigeneric structures with respect to the class $\boldsymbol{K}_\alpha$. (Expansions of equality)
2. [4] The semigeneric structures with respect to the class $\boldsymbol{K}^S_\alpha$. (Expansions of successor)
3. [4] The semigeneric structures with respect to the class $\boldsymbol{K}^V_\alpha$. (Expansions of vector spaces over finite fields)
4. [8] The theory T^ℓ of Spencer and Thoma.

The axioms of 1,2, and 4 can be proved to be almost surely true (for the appropriate probability measure). But this requires some further machinery, which we now describe.

Definition 24 *For* $A, B \in S(\boldsymbol{K}_0)$*, we say* B *is an* intrinsic *extension of* A *and write* $A \leq_i B$ *if* $\delta(B/A') < 0$ *for any* $A \subseteq A' \subset B$*.*

Definition 25 *For any $M \in \boldsymbol{K}$, any $m \in \omega$, and any $A \subseteq M$,*

$$\mathrm{cl}^m_M(A) = \bigcup\{B : A \leq_i B \subseteq M \& |B - A| < m\}.$$

Definition 26 *If $B \cap C = A$ we write $B \otimes_A C$ for the structure with universe $B \cup C$ and no relations other than those on B or C.*

And now we weaken the notion of genericity to semigenericity.

Definition 27 *The countable model M is $(\boldsymbol{K}_0, \prec_{\boldsymbol{K}})$-semigeneric, or just semigeneric, if*

1. *$M \in \boldsymbol{K}$*
2. *If $A \prec_{\boldsymbol{K}} B \in \boldsymbol{K}_0$ and $g : A \mapsto M$, then for each finite m there exists an embedding $\hat{g}$ of B into M which extends g such that*
   (a) $\mathrm{cl}^m_M(\hat{g}B) = \hat{g}B \cup \mathrm{cl}^m(A)$
   (b) $M|\mathrm{cl}^m_M(gA)\hat{g}B = \mathrm{cl}^m_M(gA) \otimes_A \hat{g}B$

Crucially, it is possible to axiomatize the class of semigeneric structures [9].

Lemma 28 *There exist formulas $\phi^m_{A,B,C}$ (indexed by relevant configurations of finite structures A, B, C) such that the structure $N \in \boldsymbol{K}$ is semigeneric, if and only if for each $A \prec_{\boldsymbol{K}} B$ and $C \in \mathcal{D}_A$ and each $m < \omega$, $N \models \phi^m_{A,B,C}$*

Thus the key step in proving the 0-1-law is to establish.

Theorem 29 *If $A \prec_{\boldsymbol{K}} B$ and $A \leq_i C$ with $|\hat{C}| < m$ then*

$$\lim_{n\to\infty} P_n(\phi^m_{A,B,C}) = 1.$$

Under appropriate hypotheses we can prove all the semigeneric models are elementarily equivalent.

Definition 30 *We denote by Σ_α the conjunction of a) the sentences axiomatizing $(\boldsymbol{K}_0, \leq)$-semigenericity and b) the sentences asserting that if $\boldsymbol{a} \in \mathrm{icl_M}(\emptyset)$ then $\neg R(\boldsymbol{a})$ (for any $R \in L$-L') and describing the L'-structure of $\mathrm{icl_M}(\emptyset)$.*

Theorem 31 *If T_α is the theory of the semigeneric models of Σ_α then T_α is a complete theory, axiomatized by Σ_α. Moreover, T_α is nearly model complete and stable. And T_α is* not *finitely axiomatizable.*

There are two major applications of this method.

1. L' has only equality.

2. L' has successor.

If L' has only equality, we get the 0-1 law for edge probability $n^{-\alpha}$ when α irrational. [9] gave the first complete proof of [22] and in addition concluded that T^{α} was stable. If L' has successor, we get the 0-1 law for the random graph over successor for edge probability $n^{-\alpha}$ when α irrational [4]. It was a surprise to graph theorists that the arguments of [9] worked as well for any finite symmetric relational language as for graphs.

A first order theory is said to be *nearly model complete* if every formula is equivalent to a Boolean combination of existential formulas. This generalizes the notion that a theory is *model complete* if every formula is equivalent to an existential formula. It is easy to see that any model complete theory is $\forall\exists$-axiomatizable. Lindström [20] showed if a theory is categorical in some infinite cardinal then converse holds: If T is also $\forall\exists$-axiomatizable then T is model complete. He gave a rather contrived example showing the categoricity hypothesis was necessary. In [6], we show a variant on the Hrushovski construction for the expansion of fields is a more natural example.

As given, the axioms for semigenericity are $\forall\exists\forall$. (For every finite A and finite extensions B and C satisfying specified conditions, for every embedding of AC there is an extension to an embedding of B which cannot be extended to any of a finite list of models extending the free union of B and C over A). This raise the question. Is the theory T_{α} $\forall\exists$-axiomatizable? We showed in [9] that it is not model complete. At the conference I thought we had proved it was not. However, Laskowski [19] building on a related analysis by Ikeda [16] has recently shown:

Theorem 32 (Laskowski) *For any (symmetric) language the theory T_{α} is $\forall\exists$-axiomatizable.*

Laskowski's arguments are both model theoretic and combinatorial. By some nice combinatorial arguments building on Ikeda, he shows that a certain set S_{α} of $\forall\exists$-sentences axiomatize a nearly model complete theory, which is then easily seen to be complete. These axioms can be seen to almost surely true (for $n^{-\alpha}$) by earlier arguments and the near model completeness yields stability directly. These arguments work for arbitrary finite relational (symmetric) languages thus extending to these languages the results of [10] showing dop and nfcp for the Shelah-Spencer graph. The key is use combinatorics to replace the axioms for semi-genericity by axioms which are simpler to state.

The fundamental connection between the 'generic model' context and the 0-1-law context is given by the following observation. We will work in the context: L' has the ambient vocabulary: successor. Let L extend L' to include the graph relation R. $\delta(B)$ is the number of components of $(B, S) - \alpha e$ where e is the number of edges in the graph.

Definition 33 *Let $A \subseteq B$ be L-structures. Fix an L'-isomorphism f from A into the L'-structure (n, S, I, F), and $M \in \Omega_n$, i.e. M is an L-structure expanding (n, S, I, F). Let N_f be a random variable such that $N_f(M)$ is the number of extensions of f to (L-L')-homomorphism over A mapping B onto M.*

Lemma 34 *For all sufficiently large n and all $f : A \to n$, the expectation*

$$\mu_f = E(N_f) \sim n^{\delta(B/A)}.$$

The crux is to prove the following lemma.

Theorem 35 *Fix L-structures $A \subseteq B$ with $A \leq B$. Let V be the event (which depends on c_1): for every L'-isomorphism $f : A \to n$,*

$$n^{v-r}(\ln n)^{-(v+1)} < N_f < c_1 n^{v-r}. \tag{1}$$

Then, for some choice of c_1

$$\lim_{n\to\infty} P_n(V) = 1.$$

The upper bound is proved exactly as in Spencer-Shelah; the lower bound is a new argument in [4] avoiding the second moment method. From these computations it is fairly straight-forward [9, 4] to prove that the axioms $\phi_{A,B,C}$ are almost surely true and to conclude the intrinsic closure of the empty-set is empty. Completeness and the 0-1 law follows.

Let us summarize two situations.

THE RANDOM GRAPH –uniform distribution

1. unstable; prototypical theory with independence property
2. $\aleph_0$-categorical
3. has the finite cover property
4. elimination of quantifiers
5. $L^{\omega}_{\infty,\omega}$ almost equivalent to first order.
6. $\forall\exists$-axiomatizable

THE RANDOM GRAPH –edge probability $n^{-\alpha}$, α irrational.

1. stable
2. not $\aleph_0$-categorical; not small
3. does not have the finite cover property

4. nearly model complete, not model complete
5. $L^{\omega}_{\infty,\omega}$ is not almost equivalent to first order (McArthur-Spencer).
6. $\forall\exists$ axiomatizable (Laskowski).

The method of determined theories extends to prove 'limit laws'.

Definition 36 *Consider a family* (Ω_n, P_n) *and let* L *represent the first order sentences in a vocabulary* τ*.* (Ω_n, P_n, L) *has a limit law if for each* L*-sentences* ϕ

$$\lim_{n\to\infty} P_n(\phi)$$

exists.

Spencer and Thoma consider the probability measures:

$$p_n^l = \frac{ln(n)}{n} + \frac{l \cdot ln(ln(n))}{n} + \frac{c}{n}$$

where l is an arbitrary fixed nonnegative integer, and c is a positive constant.

They prove limit laws for this probability by Ehrenfeucht games. Baldwin and Mazzucco [8] prove that for each ℓ and c, the almost sure theory for p_n^l is determined for an appropriate notion of closure. In contrast to the T_α case the closure of the empty set is not empty. Using determined theories we obtain:

Theorem 37 *There are a family of easily described sentences* σ_s^l*. Let* $lim_{n\to\infty} p_n^l(\sigma_s^l) = q_s^l$*. For any* L*-sentence* θ*, there exists a finite set* I *of nonnegative integers such that* $lim_{n\to\infty} p_n^l(\theta) = \Sigma_{i\in I} q_i^l$ *or* $lim_{n\to\infty} p_n^l(\theta) = 1 - \Sigma_{i\in I} q_i^l$*.*

In this situation, the various completions of the incomplete almost sure theory are determined by possibilities for the intrinsic closure of the empty set. By combining the probabilities of these finite graphs one obtains expressions for the limit probability of each first order sentence.

Many talks in this conference discussed the Urysohn space. Vershik [24] considers the notion of a random metric space. But his measures are on infinite sets. We develop here an account in the spirit of this article: the Urysohn space arises as the limit of finite spaces with probability measures. Here is a problem concerning this space which I posed during the conference.

Let $\boldsymbol{K}_0$ be the set of finite metric spaces in the language containing binary relations R_q for each positive rational q. Cameron pointed out that if $\mathbb{Q}$ is the homogeneous universal (i.e Fraïssé limit) for $\boldsymbol{K}_0$ then the completion of $\mathbb{Q}$ is the Urysohn space.

Vershik's version replaces the relations symbols for rational distances specifies a set of constant a_i and the distances between a_i and a_j. But under either formalism, we need the *prime* model of the theory of the generic. So the infinitary logic of the model theory talks enters again – by omitting all non-principal types.

Here is a probability model of the kind considered earlier in this paper but which applies to the Urysohn space.

Fix a slow growing function $f(n)$ and let L_n contain the R_q with the denominator of q less than $f(n)$ and $0 \leq q \leq 1$.

Let Ω_n be the set of L_n structures with universe n that satisfy the universal axioms of metric spaces. Let P_n be the uniform measure on Ω_n.

Let $\boldsymbol{K}_0$ be the class of substructures of models in $\bigcup \Omega_n$. It is easy to see:

Lemma 38 $\mathbb{Q}$ *is the Fraïssé limit of* $\boldsymbol{K}_0$ *under substructure.*

Blass suggested that making f sufficiently slow growing would aid in proving the following conjecture (which remains open).

Conjecture 39 *The extension axioms for finite metric spaces are almost surely true with respect to* (Ω_n, P_n).

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