SUPPLEMENT: Focusing a GeT course on axiomatic systems for geometry.

John T. Baldwin
Andreas Mueller

January 2, 2024

Contents

Abstract

Three major axiom systems for organizing plane geometry are used in various GeT/high school textbooks. In the light of SLO4, we describe the distinct challenges facing Euclid, Hilbert, and Birkhoff (SMSG) in this project. Building on a workshop we gave to in-service high school teachers a logician and

a high school teacher describe an amalgam of the Euclid/Hilbert system designed to avoid the technical complexities of Hilbert's system while preserving his foundation for both analytic and synthetic geometry. The supplement contains further proofs and student activities involving explorations and technology.

1 Introduction

This chapter is aimed primarily at instructors of college courses in geometry for teachers. We address the roles of the college instructor, college students, who are future high school teachers and high school students. We discuss the role of axioms in mathematics and then use a slightly non-standard set of axioms to show that verifying an easily constructed algorithm for splitting a line into equal pieces requires all the Euclidean axioms. To emphasize that axiom systems are designed by individuals to clarify the reasons for certain mathematical truths, and to provide background (alternative texts) for (future) instructors, we contrast in 2.9 to 2.13 the challenges faced by Euclid, Hilbert, Birkhoff, and the goals of this chapter.

This chapter provides both motivations for and amplifications of material in a course the two authors gave for in-service high school teachers in Fall 2012. We describe and contrast the challenges faced by Euclid, Hilbert, Birkhoff, and our own challenge in amalgamating them for a 21st century audience. We label our comments on the underlying mathematical development rather loosely as motivation, methodology, extension, or pedagogy. Hilbert's geometry book stimulated such notions of modern logic as consistency, truth (in a model), independence, and completeness; they are crucial in discussing 'models and axioms' in a college geometry course. The supplement contains both i) deeper study of logic and geometry and ii) many activities and exercises from the original course that are not appropriate for a commentary on learning standards. Our extensive bibliography aims to afford instructors opportunities to deepen their knowledge of the material. The supplement is at `https://homepages.math.uic.edu/~jbaldwin/pub/supp2Dec3.pdf`. To access activities mentioned in this supplement go to `https://homepages.math.uic.edu/~jbaldwin/CTTIgeometry/ctti` and click on the link for the named activity.

We build on the superb narrative for SLO4 and assume it below. We stress that axioms are intended to organize the study of an area of mathematics by identifying the fundamental assumptions needed to establish the results in that area and that different choices of fundamental notions (undefined terms) and axioms can provide different explanations.

We agree with the advice in the narrative that the college instructor should scale up from the earlier levels of the Van Hiele hierarchy[1]. We focus here on the development for college students of levels 3 and 4. Level 4 Rigor: At this level students understand the way how mathematical systems are established. They are able to use all types of proofs. They comprehend Euclidean and non-Euclidean geometry. They are able to describe the effect of adding or removing an axiom on a given geometric system.'

The common core demands level 3 of high school students but not level 4. As noted in SLO2 there may be students in the college course who have not fully attained level 3, while there are a number of high school students that operate comfortably at level 4 and some who appreciate non-Euclidean geometries.

The narrative defines a theorem as 'a statement that can be proved from the axioms without regard to interpretation' (i.e. holds in every interpretation that satisfy the axioms (i.e. every model). More useful for students is 'can be deduced from the axioms by the rules of logic'. The equivalence of these two characterizations of theorem is precisely Gödel's completeness theorem for first order logic. In particular Gödel's theorem makes precise the meaning of *consistent*. In first order logic, T is consistent if it satisfies one of the two equivalent conditions: i) One cannot derive a contradiction from T, ii) T has a model. We will examine

[1]Following the easily easy accessible `https://physics.mff.cuni.cz/wds/proc/pdf12/WDS12_112_m8_Vojkuvkova.pdf` the five levels are: 0 Visualization, 1 Analysis, 2 Abstraction, 3 Deduction, 4 Rigor

such rules in Extension 5.0.2 of the supplement. The crucial point is that Books I-IV of Euclid and axiom groups I-IV of Hilbert are first order; Birkhoff is not. The complexity of the continuity axioms is discussed in §10 of the supplement. They seem to require an extension of the Van Hiele hierarchy.

Motivation 1.1 (SL0 1, 3: Why axiomatics?)**.** A fundamental goal of K-12 education is to inculcate the ability to make and understand rational arguments. For over 2000 years Euclid's Elements performed this task more than any other single source. One of the standard goals for U.S. high school geometry is Common Core Standard 3 for mathematical practice: **Construct viable arguments and critique the reasoning of others**. A successful argument requires a clear statement of subject matter. The notion that reasoning skills learned in geometry transfer to e.g. political discourse raises many distinct questions. However, [IA17, CS20] find studying mathematics develops general thinking skills. Our task here is not to defend that proposition. Rather, given that it is embedded in mathematics standards, the goal here is to provide a model of reasoning in a mathematical context which is accessible to high school students – geometry is everywhere. Moreover, via Euclid et. al., geometry is precise.

We contrast three modes of persuasion: *argument:* reasoned persuasion in any subject: mathematics, law, politics, movies, *informal proof:* a typical argument in mathematics, the rules of inference are implicit and the global assumptions unstated although nominally reducible to formal set theory (e.g. Zermelo-Frankel with the axiom of choice), and *formal proof*: in a logic with strict rules for construction of sentences and deductions. This chapter concerns informal proof but clarifies the relation with formal proof, which in its most extreme form must be machine implementable [Hal08].

Methodology 1.2 (Axiom Systems)**.** The introduction to [Hil62], published in 1899, heralds a new age in the foundations of mathematics.

> The following investigation is a new attempt to choose for geometry a simple and complete set of independent axioms and to deduce from these the most important geometrical theorems in such a manner as to bring out as clearly as possible the significance of the different groups of axioms and the scope of the conclusions to be derived from the individual axioms.

The aim is to determine fundamental, 'simple and complete' reasons for 'important geometrical theorems'. Hilbert's axioms did not enter the high school curriculum because of the complexity of their use. This complexity arises from the difficult construction of the linear ordering of a line from the abstract betweenness axioms and the tedious process of transcribing such important notions as circle (Hilbert punted) into his choice of basic concepts. By merging Hilbert's framework with Euclid's we present a more accessible approach which embodies Hilbert's answer to '*What is geometry?*'

Old View: Until the 19th century it was thought that geometry demonstrated truths from *unassailable premises*. These premises were Euclid's axioms (common notions) and postulates (geometric assumptions).

New View: Geometry deduces conclusion from a specific set of geometric hypotheses. These hypotheses might be Euclidean, spherical, hyperbolic, etc. Whether these geometrical hypotheses are "true" is *not* a mathematical question. As the epigram of [HT20] puts it:

> Geometry doesn't contain the truth about how space is. Geometry is how you view space. Take charge of it – it's yours. Understand how you see things and how you imagine things. Geometry can say something about you and your universe. – David W. Henderson

But this new view leaves open the issue of how we are to understand these 'not known to be true' geometric hypotheses. What are the fundamental notions? What is true about them? What do they imply?

Motivation 1.3 (SLO1 vs SL04)**.** By contrasting axioms and models, SLO4 focuses on the roles of axiom systems for *organizing a topic* rather than particular proofs as in SLO1 and [SBM19]. We consider several alternative axiomatizations that each yield the propositions of Euclid. There is not a difference in most cases between the proof of a particular theorem; the difference is in what statements are theorems rather than axioms, or provable or not. We examine how the different problems that motivated each author affects the actual development of the geometry and accessibility to students. Since, the non-Euclidean geometries are rarely studied axiomatically, we concentrate on subsystems and differing approaches, specifically those of Euclid, Hilbert, and Birkhoff (SMSG) to the Euclidean case.

Methodology 1.4 (SLO1: Criteria for Choosing Axioms)**.** Natural criteria include that axioms should be intuitive and parsimonious. By intuitive, we mean the axioms can be easily illustrated for the students involved. An axiom system is independent if no axiom can be deduced from the others. Parsimony can be violated in two ways: i) including an independent axiom which is not needed for the intended collection of results or ii) failing to be independent. Independence may not be evident; it took two thousand years to show the parallel postulate is independent. Mathematicians were convinced that the parallel postulate was not self-evident but should be provable.

A third natural criteria is that the axioms should be, as Hilbert said in 1.4, complete. But completeness turns out to be a rather complex notion that we will explore in Section 10. For now we will say an axiom system is *descriptively [Det14] complete*[2] if it implies all the propositions it was designed to axiomatize.

2 Interpretations, Models, and Axioms

Pedagogy 2.1. *[SLO1: Synthetic and Analytic proof]* Narrative SLO2 prescribes 'understanding different types of proof such as synthetic (from axioms), analytic (using coordinates), and proofs using transformations or symmetries.' This distinction between synthetic and analytic illustrates the difference between proof from axioms in the language of geometry and proof about interpretations. A *synthetic* proof is an informal proof (Motivation 1.1) organized as sequence of statements such that each statement is either an axiom, hypothesis, previously proved theorem, follows from the earlier statements by a (perhaps vague) rule of inference. We call synthetic proof as taught in high school, 'semi-formal', reserving 'formal' for the stricter[3]., of Methodology 1.1. An *analytic* proof is an algebraic proof about the coordinatized plane, which almost always uses symbols. As such, it is a proof *about* an interpretation of the axioms.

Notation 2.2. *[Syntax/semantics/interpretation]* The crucial divide between axioms and models is between *syntax* and *semantics*. Axioms are syntactic objects, sentences (English or symbolic). The sentences are in a regimented language with a fixed vocabulary of basic terms. Interpretations (models/structures) are semantic, mathematical objects). There is a clear method (either informally or by a technical definition) to determine when a particular sentence is true in a particular structure. An *interpretation or structure* for a *vocabulary* (the basic terms) consists of a set (called e.g. world, domain, universe) and a meaning for each basic term on that domain. An interpretation is a *model* of a set of axioms if each axiom is satisfied (true) in the interpretation.

The following basic mathematical structures (possible interpretations) should be known, but perhaps not so precisely. A structure (e.g. 'the rationals') is a set with several kinds of basic terms: specified constants,

[2] More strongly it is *deductively negation complete* if every 'relevant sentence' is proved or refuted. See Definition 10.7

[3] Increasingly the term is used only for computer proof (e.g.`https://imsarchives.nus.edu.sg/files/CLThomasHales25Nov2009.pdf`

operations, and relations. The ordered field of rational numbers $\langle \mathbb{Q}, +, \times, -, ^{-1}, 0, 1, =, < \rangle$ consists of the fractions with the specified constants, operations, and relations listed. The word field indicates that both addition and multiplication are groups (satisfy associativity, commutativity with identities $0, 1$ and inverses) and that multiplication distributes over addition. 'Ordered' prescribes a linear order relation. The real numbers satisfy the same properties but also satisfy the least upper bound principle. One point of these notes is that the least upper bound principle is largely irrelevant to high school geometry.

We have given a particular interpretation of the vocabulary of fields (addition, multiplication, additive and multiplicative inverse, and identities $0, 1$, equal, less than) in symbols $\langle \mathbb{Q}, +, \times, -, ^{-1}, 0, 1, =, < \rangle$ on a particular set, the rational numbers $\mathbb{Q}$. ($\langle \mathbb{Q}, +, \times, -, ^{-1}, 0, 1, =, < \rangle$ is a *model* of the theory of fields. Since all the field axioms are satisfied[4], this interpretation is a model of the theory of fields.

The basic terms of an (incidence) geometry are points (P), lines (L) and a binary relation between points and lines I, 'A lies on ℓ'. The interpretation of the statement, 'the point A is on the line ℓ' is $\Pi(F) \models I(A, \ell)$. We need an unfamiliar symbol I because unlike fields, where we routinely work in the model, synthetic proof can be done in English with symbols only naming particular points and lines.

Definition 2.3. *For any field F, the 'coordinate plane' over F is an interpretation for the incidence geometry vocabulary. By the coordinate plane $\Pi(F)$ over a field F we mean the interpretation $\langle P, L, I \rangle$ with points being the ordered pairs in $F \times F$ and the geometry whose lines are the solutions of linear equations over F. That is, $A = (u, v)$ is on the line ℓ (determined by) $y = mx + b$ if $v = mu + b$. We say $\Pi(F)$ satisfies the statement 'A lies on ℓ' or formally $I(A, \ell)$.*

In Theorem 7.13 we show the correspondence is invertible: the field is found in the geometry. Here is a very different interpretation for the vocabulary of incidence geometries.

Exercise 2.4. *Keep $P = F \times F$ but change incidence I to I' by interpreting $I'(A, \ell)$ holds of the point A and the line ℓ (now determined by a single field element a) if $A = (u, v)$ for any v if $u = a$. Draw a picture of the lines in this plane.*

Pedagogy 2.5 (The new view and student understanding)**.** We now consider axioms for projective planes, since they are much simpler than those for Euclidean geometry. [Har99] describes the distinction between the intuitive axiomatic (Greek) and structural conception (Hilbert) of axioms in [Har99]. Moreover, he highlights that distinction as obstructing students understanding proofs and in particular to their understanding such exercises as 2.7. How can a plane be finite?

Definition 2.6 (Projective Plane)**.** *An incidence geometry is a projective plane if it satisfies the axioms: (P1) Any two distinct points lie on a unique line. (P2) Any two distinct lines meet in a unique point. (P3) There exist at least four points of which no three are collinear (i.e., are on the same line).*

Exercise 2.7.

1. **Fano Plane** *Draw a picture of the projective plane with* 3 *points on each line. (Hint: it has 7 points and 7 lines.)*

2. *Prove that in a projective plane there are four lines with no three sharing a common point.*

3. *Suppose (P, L, I) is a projective plane and there are n points on a given line ℓ. Prove each line has n points and there are $n^2 - n + 1$ points in the plane*[5].

[4] Since addition does not distribute over multiplication, if we had perversely interpreted addition as $\times$ and multiplication as $+$, we would still have an interpretation; but not a model. Note $^{-1}$ denotes the multiplicative inverse.

[5] The first author took a course in projective geometry while in college. His future wife, who had no college mathematics solved this problem.

Items 1) and 2) have very different nature; the first is a *theorem of projective geometry*; it is expressed in the vocabulary of geometry. The second is *a theorem about projective geometry*. There are no numbers in projective geometry; the result describes the models of projective geometry using concepts it cannot express.

Deductions from Euclid's five axioms include some actual gaps and others that are questionable. Many of these gaps are more apparent than real; much of the difficulty came from later mathematicians ignoring the rigorous role diagrams played in Euclidean proof (Pedagogy 5.0.1). For example, Hilbert even postulates that if B lies between A and C then B lies between C and A. For high school this is unnecessary pedantry.

Different challenges motivated the organization of geometry by different authors. To situate Birkhoff's system with the others we need some definitions.

Definition 2.8. *1. A metric on a set X is a function d from $X \times X$ into the positive elements of an ordered group (field for us) such that $d(x, x) = 0$, $x \neq y \rightarrow d(x, y) > 0$, $d(x, y) = d(y, x)$, and $d(x, z) \leq d(x, y) + d(y, z)$ (the triangle inequality).*

2. *For any ordered field F, the F-ruler postulate asserts: for each line ℓ in the plane there is a bijection f_ℓ from ℓ to F so that for $A, B \in \ell$, $d(A, B) = |f_\ell(A) - f_\ell(B)|$. So each line is an F-metric space.*

3. *the $\Re$-ruler postulate (Birkhoff/SMSG) takes F to be the real numbers $\Re$.*

Motivation 2.9. *[SL02, 7: Euclid's Challenge]* Euclid aimed to provide a unified foundation for earlier geometry, specifically the side-splitter theorem of Thales (Euclid VI.2: A line parallel to the base and intersecting both sides of a triangle creates two similar triangles) and the Pythagorean theorem. The obstacle is incommensurability[6] in each case. He has five postulates. Using a theory (now called *equi-complementability or equal content*) of area, Euclid establishes the Pythagorean theorem as the culmination of Book I. By appealing to the Axiom of Archimedes, he establishes a theory of proportion that first yields: *VI.1 the area of a triangle is proportional to its base and altitude* and then VI.2. side-splitter. While Eudoxus' method of exhaustion motivated Dedekind's construction [Ded63], the existence of continuum[7] many ($2^{\aleph_0}$) real numbers was completely foreign to Euclid.

Motivation 2.10. *[Hilbert's Challenge]* 19th century mathematicians such as Cantor, Dedekind, and Frege revolutionized the foundations of mathematics by making the natural numbers rather than Euclidean geometry fundamental. Hilbert aimed for an independent development of geometry. In doing this, he had to meet the new higher standards of rigor. He needed to develop notions of distance and proportion from geometric notions of point, line, between and congruence. Moreover, he had to build this precision into his axioms so as to avoid any reliance on diagrams (Extension 5.0.2). He deduced VI.2, side-splitter,from a geometric foundation of the theory of proportion and then VI.1 from a new theory of 'measured area'. He proved his four axiom groups are independent, although there are dependencies within the entire set. The consistency of geometries that satisfy the F-ruler postulate but are countable follows easily in § 7.

Methodology 2.11 (SLO5: Congruence vs Distance)**.** This is partly a story of the chicken (congruence) and the egg (distance). A fundamental distinction between Hilbert and Birkhoff is that Hilbert takes the congruence relation as fundamental and proves that one can define a metric (with values in a field). Birkhoff (and SMSG [SMS95] [Ced01, Appendix]) assume the $\Re$-*ruler postulate*.

The difficulty for a high school course is that limits, which *Hilbert has shown are irrelevant to the geometry of lines*, are used implicitly while basic observations are replaced by long proofs. E.g. Common notion 3 (subtraction of line segments) is 'reduced' in some texts to using the ruler postulate twice and

[6] Two line segments are commensurable if for some integers m and n, m copies of one are the same length as n copies of other.

[7] `https://en.wikipedia.org/wiki/Cardinal_number` for background.

assuming the student knows the laws of algebra well. In this chapter we take congruence of line segments or angles as fundamental, not some measure.

Motivation 2.12. *[Birkhoff's Challenge]* Birkhoff confronted the difficulty of passing from technical axioms about Hilbert's betweenness relation to the only slightly more intuitive concept of the real linear order. [Rai05] begins his discussion of geometry teaching 'new math' days in US with the side-splitter theorem (2.9). Many texts and an influential mathematics educator [Rai05, p 9] propagated an incorrect proof of this theorem by basically assuming all line segments were commensurable. Birkhoff addressed these issue with 4 postulates: the ($\Re$)-ruler postulate and an analogous protractor postulate, two points determine a line, and side-splitter[8]. The existence of the real numbers and thus the $\Re$-ruler postulate can only be stated and proved in second order logic/set theory so do not provide a *geometric* foundation.

Pedagogy 2.13. *[Our Challenge]* A prime objection to Hilbert's axioms is that their abstract nature is too hard to grasp for high school. Our aim is to amalgamate the axioms of Hilbert and Euclid to provide a more accessible account of Hilbert's foundation of both synthetic and analytic geometry on purely geometric principles culminating in a proof of VI.I and VI.2. We vary from Hilbert primarily in accepting Euclid's careful use of diagrams and taking as an axiom (Pedagogy 5.2.4) that each line has a dense linear order extending betweenness. We expound Hilbert's bi-interpretation of Euclidean geometry and ordered fields because it not only is the key step in the bi-interpretation of hyperbolic and Euclidean geometry but because *it provides a synthetic basis for high school analytic geometry*. For simplicity and succinctness, *we axiomatize only plane geometry*.

Motivation 2.14. *[Why not Birkhoff?]* We began with Hilbert's admonition to seek simple, explanatory axioms. The ruler postulate is neither. It appeals to a 'magical' notion: 'the real numbers'. Similarly, assuming the side-splitter magically connects two radically different concepts (fields and similarity) that in fact are provably (in Hilbert's system) equivalent. By magic, we mean that Hilbert's axioms identify the actual property that make the reals special, they are the largest Archimedean field. And he has proved his geometry is coordinatized by a field. But there is a reason he avoids circles. A rigorous definition of angle measure involves the exponential and trigonometric functions, using either calculus or infinite series. All this is buried by the protractor postulate. Of course, this background is obvious to Birkhoff, one of the leading analysts of the 20th century. But it isn't to a high school sophomore. Nor even to a college student who hasn't absorbed the least upper bound principle in Advanced Calculus. More practically, assuming the ruler postulate kills almost all examples of axiom independence in this chapter.

3 Common Notions vs Postulates

We now discuss Euclid's distinction between general and geometric premises and the 19th century quest for an autonomous basis for geometry.

Methodology 3.1. *[Common notions vs postulates]* Euclid's distinction between principles that are true everywhere in mathematics and those that are true only of a particular topic remains important today. But it is answered in a different way. Euclid was interested only in geometry and natural number (positive integers) arithmetic. His common notions essentially describe the properties of equality and order (among classes of 'comparable objects', i.e. magnitudes of various sorts). Length and area are incomparable magnitudes for Euclid. In modern mathematics (almost) all topics can be studied on a common basis in set theory.

[8]This trivialization seems to be followed by at least the SMSG high school text books I have looked at a Postulate AA or AAA. [Moi90] calls them theorems by what seems to be an implicit appeal to Archimedes [Moi90, p 169].

Nineteenth century geometers insisted that applicability of the common notions be explicitly based within geometry [Gio21]. Postulates describe the relations among the fundamental concepts of a particular subject. The best example for over 2000 years were Euclid's postulates for geometry. Thus the geometrical consequences of the common notions must be derived from the postulates; this required some additions (§5).

These are the common notions of Euclid. They apply equally well to geometry or numbers. Following modern usage, we call Euclid's postulates either 'axiom' and 'postulate'.

Common notion 1. Things which equal the same thing also equal one another.
Common notion 2. If equals are added to equals, then the wholes are equal.
Common notion 3. If equals are subtracted from equals, then the remainders are equal.
Common notion 4. Things which coincide with one another equal one another.
Common notion 5. The whole is greater than the part.

Methodology 3.2 (SLO5,7: Common Notion 1)**.** Euclid used 'equal' in a number of ways:, to describe congruence of segments and figures, to describe that figures have the same measure (length, area, volume). While the only *numbers* for Euclid were the positive integers > 1, he did study the comparison of what we now interpret as lengths. Following Hilbert, in Section 7 we build an 'algebra of segments' (a semi-field) and explain how to consider the segments as 'numbers' that can measure areas, a concept totally foreign to Euclid.

CN1 asserts that equality is transitive[9]. For various notions (e.g. congruence) we may need to make this property (as well as symmetry) an explicit axiom.

Methodology 3.3 (SLO5,7: Common Notion 4)**.** What Euclid means by coincide and equal is unclear ([Euc56, p 224, 248]). We adopt the view that X coincides with Y means 'one is mapped to the other by a rigid motion'; we follow the usual interpretation that in this context Euclid's equal means congruent. So, Euclid CN4 asserts any figure is congruent with itself. That is one ingredient of Hilbert's congruence axioms. We discuss other more contentious properties, symmetry and transitivity of congruence, in Axiom 5.3.1.

Methodology 3.4 (SLO1,5,8 **Definitions**)**.** Euclid begins with a list of *definitions*. Some (e.g., 'A line is breadthless length') are really just an *indicative definition*; it points to an intuition. These indicative definitions become the basic terms (vocabulary) of Definition 2.2. Others (e.g., When a straight line standing on a straight line makes the adjacent angles equal to one another, each of the equal angles is **right**.) are *stipulative definitions*. They precisely describe a new concept in terms of previous definitions. The geometric definitions in this chapter are stipulative.

Euclid and Hilbert point, line (line segment for Euclid), incidence (a point is on a line), plane, and congruence of segments as the most basic concepts., incidence (a point is on a line), plane, and congruence of segments as the most basic concepts. They regard triangles and other polygons as built from points and straight lines and facts about them follow from the axioms.

For Euclid, words in the proof refer to ideal geometric objects. But Hilbert's attitude is different. These basic concepts are named by words in the vocabulary. For him, the meaning of those words is given implicitly by the axioms [Dem94]. Blumenthal reported, 'One must be able to say at all times–instead of points, straight lines, and planes – tables, chairs, and beer mugs'.

Before giving the postulates in §5 we clarify some of the stipulative definitions in Euclid.

[9] A relation $R(x, y)$ is transitive if $R(a, b)$ and $R(b, c)$ implies $R(a, c)$. 'Descendent' is transitive; 'daughter of' is not.

Activity 3.5. SLO5, CC Standard G-C0 1. Know precise definitions[10] of angle, circle, perpendicular line, parallel line, and line segment, based on the undefined notions of point, line, distance along a line, and distance around a circular arc.

Why is distance along a circular arc given as an undefined notion? Can we define the length (congruence) of a circular arc in terms of the length (congruence of line segments)? Why is the length of the chord a less good measure than the length of the arc?

As noted [Har00, p 114], congruence of arcs can be defined by rigid motions. But in general, the length of an arc may not be the length of a straight line segment in a particular interpretation. E.g. when the interpretation is the plane over the real algebraic numbers (i.e. the field of real solutions of polynomial equations in one variable with rational coefficients) as it does not contain π. The circle of radius 1 about the origin is the set of solutions of $x^2 + y^2 = 1$. The length of the semicircle is π which is not in the interpretation.

Definition 3.6. *A metric on a set X is function d from $X \times X$ into the positive elements of an ordered group (field for us) such that $d(x, x) = 0$, $x \neq y \rightarrow d(x, y) > 0$, $d(x, y) = d(y, x)$ and triangle inequality $d(x, z) \leq d(x, y) + d(y, z)$ (a straight line is the shortest distance between two points).*

We survey here some modern postulate systems for geometry that appear in textbooks for GeT. In line with SLO4, we focus on those books that adopt an axiomatic approach and leave for other chapters those texts (e.g. [Ced01, HT20] among many) who treat other strands of geometry discussed in [Hen02]. Our categories reflect the intellectual needs of the system builders (Euclid, Hilbert, Birkhoff). We hope our discussion of the motivations of various results and argument can help the instructor respond to the intellectual needs of the students [Har13].

Methodology 3.7. *[Postulate systems classified by basic notions[11]]*

1. Hilbert[12] [Hil62, Hil71, Har15, Har00, Ser93]: congruence is fundamental; two kinds of objects: point and lines[13];

2. Birkhoff [Bir32, BB59], SMSG standards ([Ced01, SMS95]); distance is fundamental; all properties of the reals are implicitly assumed (via ruler and protractor postulates[14]);

3. Transformations are studied in two ways: i) within one of the Hilbert or Birkhoff systems [BH07, Cla12, Lib08, Mar82] ii) Viewing transformations as fundamental notions [Kin21, Wei97]. All use Birkhoff's axioms except Martin and Weinzweig[15].

Some recent approaches to high school geometry ([Edu09a, Ill19]) adopt a local approach. Rather than positing a global axiom system, they carefully state and argue from premises for particular topics.

---

[10] Activity G-C01: definition.pdf

[11] There are a number of other postulate systems for geometry with quite different choices of fundamental notion that have not been adopted for school use. See [PSS07, Szm78, Wu94].

[12] [Mil07] justifies the Euclidean use of diagrams.

[13] Tarski [Tar59, Szm78]: makes a logical but not pedagogical simplification reducing to one kind (sort) of object: a line is a set of collinear points (three points are collinear if they satisfy betweenness in some order).

[14] [Moi90, p 137] carefully distinguishes between what he calls *synthetic* and *metric* approaches. Roughly speaking, his synthetic corresponds to Hilbert (HP) and metric to Birkhoff. But Hilbert with (HP5) establishes a metric but the range is a field that depends on the model of HP5. It is only if Dedekind's axiom is assumed that this becomes a real-valued metric. From our standpoint, these are different synthetic approaches (different axioms in different logics).

[15] See Hartshorne's review [Har11] of [BH07] 'To begin with,the authors devote the first chapter to the axiomatic foundations of plane geometry. Here already, following a popular modern trend, they diverge from Euclid's purely synthetic geometry by presupposing the real numbers, and implicitly using some concepts of analysis.

Notation 3.8 (Hilbert style axiom sets for plane geometry)**.** *The relationship among the following important subsets of Hilbert's axioms for geometry that are studied in this chapter is extensively explored in [Bal18].*

1. **Neutral Geometry (HP)** *The system HP denotes (our translation of) Hilbert's first three axiom groups. A model is called a* Hilbert plane
2. **Circle free (HP5)** *The system HP5 is obtained by adding the parallel postulate to neutral geometry.*
3. **Euclidean geometry (EG)** *The system EG is HP5 plus circle-circle intersection; a model is called a Euclidean plane*
4. **Continuity axioms***: Axiom of Archimedes and Dedekind completeness (Section 10).*

4 SL02, SL08: A guiding problem

Pedagogy 4.1. *[SL02, SL08: Role of this section]* We began our workshop with the following exercise, first used with future middle school teachers, to emphasize the importance of ruler (straight-edge) and compass constructions in basic geometry and with the hope that the questions in the activity would provoke a need for the proof in Sections 5-8. While a solution using analytic geometry is fairly straight forward, the process of creating a purely geometric proof gives a deep insight into '(a) recognize and communicate the distinction between axioms, definitions, and theorems, and describe how mathematical theories arise from them, (b) construct logical arguments within the constraints of an axiomatic system' (SLO 4).

Exercise 4.2. *Each group chooses an odd number n between 2 and 10. After the number is chosen, the group will be asked to fold a string to divide it into as many equal pieces as the number they chose. Other physical models will be used. Activity - Divide a line into n equal pieces.*

Exercise 4.3. SLO8: CCSS G-C0-12 *For an arbitrary n, here is a procedure to divide a line segment into n equal segments.*

1. *Given a line segment AC.*
2. *Draw a line through A different from AC and lay off sequentially n equal segments on that line, with end points $A, A_1, A_2, \ldots$. Call the last point D.*
3. *Construct B on the opposite side of AC from D so that $AB \cong CD$ and $CB \cong AD$.*
4. *Starting at B, lay off n equal segments of length AA_1 and call the points so constructed on BC sequentially $B, B_1, B_2, ..., B_n - 1, C$.*
5. *Draw lines A_iB_i.*
6. *The points C_i where C_i is the intersection of A_iB_i with AC are the required points dividing AC into n equal segments.*

Exercise 4.4.

1. *Use the algorithm described above to divide an arbitrary line segment into 5 equal segments. (Could be done in pairs. One person draws the line; the two have to divide it up.)*

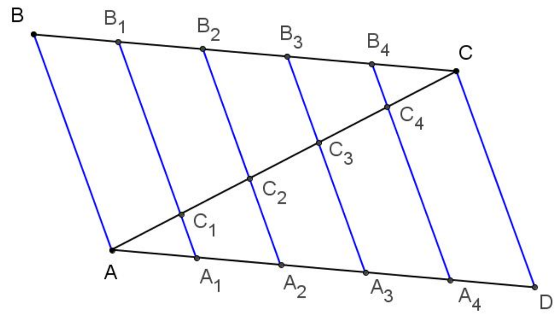


Figure 1: Dividing the line

2. *Show this construction used only Euclid's first 3 axioms, listed in Axiom 5.1.1 and 5.1.4 below.*

Pedagogy 4.5. *[SLO2: Why is this assignment made?]* We are really asking, how and why does this construction work? Working in our system we see Euclid's first three postulates suffice to make the construction. See Exercise 5.1.6. We will need SAS and more to prove it works! We start with this exercise both to give the student a reason to prove (stimulate intellectual need [Har13]) and to emphasize this distinction between rule-based construction of geometric objects and a deductive verification of geometric propositions.

5 Book I: Propositions 1-34

The construction in the guiding problem Exercise 4.3 is rather straightforward using only Euclid's first three axioms; the proof that the construction works involves much more. To prepare for this argument, we amalgamate the approaches of Euclid and Hilbert, trying to maximize both understanding and rigor. The material adapts some results from the first 34 proposition of Book I of Euclid to solving our guiding problem.

Pedagogy 5.0.1. *[SLO5, 7: Reading a diagram]* Inexact properties can be read off from the diagram: slightly moving the elements of the diagram does not alter the property. Intersections, betweenness and side of a line, inclusion of segments are inexact.

What classical diagrams don't mean

Anything about distance, congruence, size of angle (right angle!) may be deceptive. Since incidence is exact, you can't read off whether a point is on a line but you can read off that two lines intersect in a point and then name that point and then use the fact that it is on each line.

What high school diagrams mean

Classical diagrams are enhanced in modern texts. Besides the inferences allowed above, SAT instructions say 'All figures in this test are drawn to scale unless otherwise indicated', e.g., 'Figure not drawn to scale'. Students are taught tick marks for congruent segments, angle marks for congruent angles, right angle marks, parallel marks. Figures on one side of a line are assumed to be in that half-plane. Points that appear on a line(s) can be assumed to be on that (those) line(s).

Extension 5.0.2 (SLO1, SLO4, SL09: Supplemental Extension: Rules of Inference)**.** Late 19th century mathematicians banished the drawn diagram from semi-formal and even informal mathematics. The narrative defines a theorem as 'a statement that can be proved from the axioms without regard to interpretation'.

While correct in spirit, it misses an essential point; how is 'without regard to interpretation' guaranteed? The answer is to specify clear requirements on what statements are and rules for deducing one statement from earlier ones.. These can be found in any introductory logic text and many discrete math books. [BE02] includes computer software that explains 'truth in a model in a very basic way. [Lyn67] is old (My copy is stamped $3.25) but makes the distinctions immediately below very clearly.

Here is a short outline. *Propositional logic* has variables $p, q \ldots$ which stand for propositions (they are true or false). A sentence is Boolean combination of propositions (combining by and,or, not, implies).

Every tautology is an axiom of propositional logic (check by truth tables). The only rule of inference is modus ponens: from ϕ and $\phi \rightarrow \psi$ infer ψ.

Sentential Logic replaces variables $p, q \ldots$ with atomic formulas of a first order language (e.g. $I(A, \ell), B(C, A, E)$ and allows the same sorts of Boolean combinations (e.g. $I(A, \ell) \wedge B(C, A, E)$ means A lies on ℓ and is between C and A). This sentence does not choose between two contradictory extensions $I(A, \ell) \wedge B(C, A, E) \wedge I(C, \ell)$ and $I(A, \ell) \wedge B(C, A, E) \wedge \neg I(C, \ell)$. The first implies E is on ℓ and the second implies it is not. In order to continue the proof one may have to make case distinctions. See one of the many analyses online of fallacious proofs that 'all triangles are isosceles'.

We use the same rules of inference – translating a sentence into a Boolean combination of proposition by mapping each atomic formula with constants into a unique propositional variable. Then checking to see if it is an axiom by truth tables.

The *logic of geometry* is slightly more complicated. The construction postulates below have the form 'Every set of point and lines satisfying Δ_1 can be extended a set satisfying Δ_2'. Theorems (and Euclid's 4th and 5th postulate) are even easier; they have the form 'Every set of elements and lines satisfy Δ'. That is, the most complicated results can be stated in the form: for every X satisfying ϕ there exists a Y such that X and Y both ϕ and ψ.

Now there are two more rules:

1. **Existential instantiation:** Given a construction postulate and a sentence describing various points and lines some of which satisfy the hypothesis of a construction axiom. Choose a name for a witness to the construction postulate and deduce the conjunction of the given statement which the assert the conclusion of the postulate about the witness and the data which satisfies the hypothesis.

2. **Universal generalization:** From any statement ϕ about named points and lines $A, B, C, \ldots, \ell_1, \ell_2 \ldots$, we can deduce: '$\phi$ holds for all $X, Y, Z, \ldots, x_1, x_2 \ldots$'.

First order logic permits iterated use of both existential and universal quantifiers. 'There is a line with seven points' is a permissible sentence. *Second order logic* permits iterated use of both existential and universal quantifiers. The logical complexity of the continuity axioms is explored in SectionArchDed.

Extension 5.0.3. *[The fly in the ointment]*In more complicated arguments (unlikely to appear in high school), the location of the witness for a construction postulate in the existing diagram force a different proof[16]

Recent research clarifies and formalizes the ways that diagrams played an essential role in mathematical proof for 2000 years. [Man08] lays out the main issues and historical background. [ADM09] and [Mil07] provide formal systems with the diagram explicit and with methods to control the number of cases.. [ADM09] show their diagram-based system is complete for a set of sentences that include the results of Euclid. See [Bal18, §9] for a summary.

Pedagogy 5.0.4. We don't have space in this chapter to describe the rules of inference of propositional and sentential logic. An excellent reference for grasping these connections is [BE02], which has very helpful

[16] See the 'proof' that all triangles are isosceles [Gre93, p 48-50] and many explanations on the net.

software (Tarski's world) to explore the connections between syntax and syntax. We discuss the importance of the equivalence of an implication with its contrapositive in Definition 5.4.9 through Pedagogy 5.4.11. Understanding this equivalence and fact that such an equivalence fails for an implication and its *converse* is very important; spelling out the connection with inverse is a matter for logicians (of the 19th century).

5.1 Construction Postulates

Our vocabulary contains unary predicates P , **L**, binary I and ternary B , standing for point, line, incidence and between. We introduce further vocabulary such as predicates for congruence later. Here are Euclid's first three postulates. Axiom I and II are implied by Hilbert's betweenness axioms we don't list in detail.

Axiom 5.1.1 (Euclid's first 3 axioms in modern language)**.**

- **Axiom I** *Given any two points there is a line segment connecting them.*
- **Axiom II** *Any line segment*[17] *can be extended indefinitely (in either direction).*

  The following is a translation of Euclid's Postulate II from a rule *for a construction into a Hilbertian* assertion *that for any witness to Euclid's 'given', there are further witnesses for his conclusion.*

  For any point A and B and any C with B between A and C, there is a D such that C is between A and D.
- **Axiom III** *Given a point and any segment there is a circle with that point as center whose radius is the same length as the segment.*

Hilbert's first three axioms assert that two points determine a line and there are three non-collinear points. They follow from Euclid's first three, (Axiom 5.1.1).

Pedagogy 5.1.2. Circles Euclid chooses a fundamental notion that does not appear in Hilbert. Hence, we include I.3 which replaces [Hil71, Axiom III,1]. In addition to grounding the work students will do with circles, Axiom III is a much more tangible way to transfer distance than Hilbert's. [Har00, p 102-3] describes three of Hilbert's tools which somewhat awkwardly allow one to duplicate the constructions.

Fine historical point. Euclid does not explicitly mention that overlapping pairs of circles and circles overlapping a line actually intersect and Hilbert never mentions circles. Axiom 5.1.4 makes the assumption precise. In thinking about Exercise 5.1.3 consider why Euclid's notion of diagrams might have caused him to think no further Postulate was necessary to prove Proposition I.1.

Exercise 5.1.3. CCSS G-CO.13 *Prove Proposition I.1 of Euclid: To construct an equilateral triangle on a given finite straight line. Check with [Euc56].*

Following [Har00] we label this axiom E for Euclid as he treats circles while Hilbert doesn't.

Axiom 5.1.4 (Axiom E: Circle Intersections)**.** *If from points A and B, circles with radius AC and BD are drawn such that each circle contains points both in the interior (those points that are connected to the center of the circle by segments that don't cross the circle) and in the exterior of the other, then they intersect in two points, on opposite sides of AB.*

[17] If Euclid is being used as a supplement, emphasize to students that a line for Euclid is a line segment for us.

As Hartshorne notes, one can conclude from E a line circle axiom: If a line contains a point inside a circle it intersects the circle (twice!). In many expositions (e.g. [Gre93, p. 80]), Axiom 5.1.4 is deduced from the continuity axiom and used to prove the circle propositions from Euclid's Books III and IV. But Hartshorne [Har00, p 114, 203] shows that only the theory EG (Notation 3.8) is needed for the circle theorems.

Lemma 5.1.5 (Euclid's Proposition 2)**.** *To place a straight line (segment)*[18] *equal to a given straight line segment with one end at a given point. In modern language: Given any line segment AB and point C, one can construct a line segment of length AB and end point C.*

In straight-edge and compass constructions, we transfer segments by measuring with the compass, then copy that length to any other place on the paper (that is when we do the construction, our 'rusty compass' does not change the radius). The Rusty Compass Activity in the supplement lays out the geogebra construction (SLO6) to prove Lemma 5.1.5. See Euclid for the proof of Lemma 5.1.5 from the axioms I-III.

Exercise 4.3.1 is now easy.

Exercise 5.1.6. *Using Axioms I-III and Lemma 5.1.5 show the algorithm in Section 4 can be carried out.*

The following exercise gives the student the chance to understand satisfaction in a model in a fairly familiar example and to look at independence where the models are straightforward. While the college students have seen analytic geometry over the reals, here we note that the construction can act on *any* field.

Exercise 5.1.7. *Prove the Cartesian plane over the rationals, defined as in Definition 2.2, models Axioms I and II from Axiom 5.1.1 but not Axiom 5.1.4 (Axiom E). Thus, Axiom E is independent from axioms I-III.*

There is a close relation between these independence results and properties of fields.

Definition 5.1.8. *A field is Pythagorean if for every a, $\sqrt{(1+a^2)}$ exist and Euclidean fields if for every a, $\sqrt{a}$ exist.*

The geometric context is in e.g. [Har00, §12].

Fact 5.1.9.

1. *A field is Pythagorean iff it coordinatizes a Hilbert plane (model of HP5).*
2. *A field is Euclidean iff it coordinatizes a Euclidean plane (model of EG).*
3. *Characterizations of fields satisfying cubic equation and connections with origami can be founds in [Alp05, Mak19].*

Studying such examples integrates the geometry with elementary field theory and gives very concrete examples of independent axioms.

5.2 Betweenness, Order, and Planarity

Hilbert's 2nd group of axioms [Hil71, §I.3], labeled *Axioms of Order*, prescribe the behavior of the primitive concept: between. $B(x, y, z)$ means y is between x and z. His Theorem 6 roughly describes a linear order derived from the 'between' relation. Szmielew [Szm78, §7.1] gives ten axioms for betweenness (think of statements that are true of a symmetric (B $(A, B, C) \leftrightarrow$ B (C, B, A))) and then carefully derives the definition below of a relation $\leq$ that linearly orders the line ℓ through ABC.

[18] 'line' in Euclid means 'line segment'

Definition 5.2.1 (Linear Order)**.** *A set X is linearly ordered by $<$ if $<$ is asymmetric ($x < y$ implies $y \not< x$), irreflexive ($x \not< x$), transitive ($x < y$ and $y < z$ implies $x < z$), and satisfies trichotomy (for any x, y: $x < y$ or $x = y$ or $y < x$); it is dense if between any two points there is another.*

Definition 5.2.2. *1. Fix $\ell = \overline{ABC}$ and define $\leq$ for $P, Q \in \ell$ by*

$$P \leq Q \leftrightarrow (\boldsymbol{B}\,(P,Q,B) \wedge \boldsymbol{B}\,(P,B,C)) \vee (\boldsymbol{B}\,(P,B,C) \wedge \boldsymbol{B}\,(A,B,Q,)) \vee (\boldsymbol{B}\,(A,B,Q) \wedge \boldsymbol{B}\,(B,P,Q,)).$$

In fact, this definition can define a linear order in either direction. By a tricky argument, treating the rays in each direction separately, Szmielew proves:

Theorem 5.2.3. *[Szm78, §7.1] For any distinct A, B, C with $\boldsymbol{B}\,(A,B,C)$ the relation $\leq$ in Definition 5.2.2 is a linear order of ℓ. Assuming for all A, C there exists a B such that $\boldsymbol{B}\,(A,B,C)$ the order is dense.*

Pedagogy 5.2.4. The difficulty of the argument for Theorem 5.2.3 illustrates the intricacy of using the betweenness relation. Thus, Hilbert's axioms are not used in high school texts. However, *we will just use Theorem 5.2.3 in our development*. So an alternative axiomatization would be to replace Hilbert's order axioms with our Theorem 5.2.3 and certainly this would be a reasonable high school postulate.

Definition 5.2.5. *Given a line ℓ and points A, B on ℓ and D, E not on ℓ.*

1. *the ray $\overrightarrow{AB}$ is all points C on ℓ the same side of A as B (i.e. $\boldsymbol{B}\,(A,C,B)$ or $\boldsymbol{B}\,(A,B,C)$.*
2. *A region is* connected *if any two points can be connected by a polygonal path (a sequence of segments such that successive segments share one endpoint).*
3. *D and E are in the same half-plane determined by ℓ if the line segment between D and E does not intersect ℓ.*

Like Euclid, Hilbert develops geometry of dimension 3 with plane as a fundamental notion and so a ternary predicate P for coplanar is in his formal vocabulary and the axiom holds when P (A,B,C). We guarantee the universe is plane by requiring Pasch's axiom to hold for *for any* triplet of points; there is no predicate for plane in our system. There are two equivalent formulations of Pasch.

Axiom 5.2.6. *[Planarity Axioms]*

Pasch's Axiom: *Let A, B, C be three non-collinear points and let ℓ be any line which does not meet any of the points A, B, C. If ℓ passes through a point of the segment AB, it also passes through a point of segment AC, or through a point of segment BC.*

Separation Principle *The points of a plane not on a line ℓ are divided into two disjoint connected regions. Two points are in different regions exactly if the line connecting them intersects ℓ.*

Exercise 5.2.7. *[Betweenness and Pasch consistent] Check that for any field F, $\Pi(F)$ satisfies the betweenness and the Pasch axioms.*

We give a stipulative definition of angle, one of the indicative definitions in Euclid.

Definition 5.2.8. *An angle $\angle ABC$ is a pair of distinct rays from a point B. The rays BA and BC split the plane into two connected regions. The region such that any two points are connected by a segment entirely in the region is called the* interior *of the angle. Two angles are* adjacent *if they share a ray but no interior points.*

Activity 5.2.9. *What are at least three different units for measuring the size of an angle? (Answers include, degree, radian, turn, grad, house (astrology), Furman.)*

Activity 5.2.10. *Measure, don't calculate, the circumference of a convenient cylinder. Compare the result if you measure the radius or the diameter and then calculate the circumference. We have found this a useful exercise for college freshman; we urge future teachers to clarify this distinction for their students.*

Remark 5.2.11. We differ from Euclid here in allowing straight angles. Thus, we avoid the awkward locution of the 'two right angles' for 'straight angle'. To define 'right angle', we must consider congruence.

5.3 Congruence Axioms

This section fills what is generally agreed to be a true gap in Euclid. In Proposition I.4, he purports to prove SAS. His argument implicitly relies on the superposition principle (Remark 5.3.17). As in Euclid, we take the notions of segment congruence ($AB \cong A'B'$) and angle congruence ($\angle ABC \cong \angle A'B'C'$) as primitive. We follow Hilbert [Hil62, §6] and assert:

Axiom 5.3.1 (Congruence Axioms)**.** *Congruence* is an equivalence relation on undirected line segments (or angles) (reflexive, symmetric, transitive and the sum (difference) of congruent (line segments, angles) is congruent.

Methodology 5.3.2. *[On congruence axioms]* The symmetry of angle congruence arises because, following Euclid and Hilbert we are comparing angles not measuring rotation. We stated this axiom in English. Formally, for angles we would add a 6-ary predicate (4-ary for segments) and write C (A, B, C, D, E, F) to translate the axiom for two angles ABC and DEF. Euclid uses 'equal' for our 'congruent' for segments and angles.

Lemma 5.3.3. *The congruence axioms and SSS hold in $\Pi(F)$ for any Pythagorean field F ($c \in F \Rightarrow \sqrt{(1 + c^2)} \in F$.*

Proof: See [Har00, §16, §17] for SAS. $\square_{5.3.3}$.

Methodology 5.3.4. *[Failure of Protractor postulate] The F-protractor postulate fails for an F that doesn't contain π. The smallest model of the F-protractor postulate is $\Pi(K(\pi))$ where is K is the field of constructible numbers described at [Har00, 16.4.1].*

As in Euclid, we take the notions of segment congruence ($AB \cong A'B'$) and angle congruence ($\angle ABC \cong \angle A'B'C'$) as primitive. We follow Hilbert [Hil62, §6] and assert:

Definition 5.3.5. *A* rigid motion *is a bijection from points to points that preserves betweenness, collinearity (so it induces a bijection on lines), and congruence of segments and angles.*

A rigid motion is a reflection about ℓ if it fixes ℓ pointwise and sends a point A not on ℓ to an A' such that $AA' \perp \ell$ and AA' is bisected at the point that it intersects ℓ.

Methodology 5.3.6. *[Labeled triangle congruence]* Some mathematicians and some high school texts treat congruence as a property of labeled triangles (But then under some permutations of the names of the vertices of a scalene triangle the resulting labeled triangles may not be congruent). By looking at the statement of I.4, it is clear this is not Euclid's intent.He specifies 'some correspondence'; in particular, reflected triangles are congruent. Since rigid motions preserve congruence, under labeling reflections are no longer rigid motions.

Hilbert treats a weakening of SAS, [Hil71, Appendix II] to act only on oriented triangles (so rigid motions must preserve orientation).

While congruence is a property of triangles not of labeled triangles it is a useful convention to require that $\triangle ABC \cong A'B'C'$ implies that the primes indicate the correspondence. Often, in describing polygon ABCDE... any consecutive letters in the name are consecutive (connected by a side) vertices in the polygon.

Methodology 5.3.7 (Axiom Choice)**.** Just as we had a choice of which concepts to specify as basic, we have choices to make for axioms. Euclid's Theorem I.4 (SAS) has been known since antiquity to rely on an implicit 'principle of superposition'. In modern language we express this by saying the group of rigid motions (below) acts transitively on congruent angles. Hilbert chose to do this by simply making SAS an axiom. Euclid uses superposition (unnecessariy) again to prove I.8 SSS and proved without any hidden assumptions that SAS implies ASA and AAS. We chose SSS and prove SSS implies SAS. Here are two reasons for choosing SSS. 1) It is very practical: any three sticks that can form a triangle will always form the same triangle. It is minimalistic: SSS only uses segments in its statement, all others use segments and angles, and defining angles is not trivial.

Pedagogy 5.3.8. *[Too many axioms]* A major weakness of many high school texts is to think the equivalence proofs of the congruence propositions are too hard for high school. Some high school geometry texts list many of the congruence theorems (SSS, SAS, ASA, HL etc.) as separate axioms. This destroys one of the main features of axiomatics: the search for a small number of (ideally independent) assumptions from which the theory can be deduced. The cost is that students think mathematics is about memorization. This objection is not mere pedantry; calling a known theorem a postulate destroys the concept of axiom system. If to cover certain material (for reasons of time or perceived difficulty) one has to skip proofs, announce that. Don't pretend a new hypothesis has to be introduced.

Axiom 5.3.9. *[The triangle congruence postulate: SSS]* **CCSS G-C0-8** *Let ABC and $A'B'C'$ be triangles with $AB \cong A'B'$ and $AC \cong A'C'$ and $BC \cong B'C'$ then $\triangle ABC \cong \triangle A'B'C'$*

Methodology 5.3.10. *[Independence of SAS/SSS]* With the interpretation of the congruence predicate as in Lemma 5.3.3, we clearly verify SSS; so it is consistent with the previous axioms. To show SSS is independent from the earlier axioms, we must show the negation of SSS is consistent. For this, following [Moi90, 112] we show the negation of SAS is consistent. By Theorem 5.3.14, this suffices. Namely, use $\Pi(F)$ with the usual protractor and angle measure indicating congruence except on one line where $d'(f_{\ell'}(A), f_{\ell'}(B))$ equals $2|f_{\ell'}(A) - f_{\ell'}(B)|$. Now if ABC and DEF are congruent triangles with congruence defined normally, under d' we have $BC \cong EF$, $AC \cong DE$, and $\angle ACB \cong DEF$, so the triangles satisfy SAS in the new interpretation but they are not congruent in the new interpretation since $d'(AB) = 2d'(DB)$.

Here is Moise's diagram.

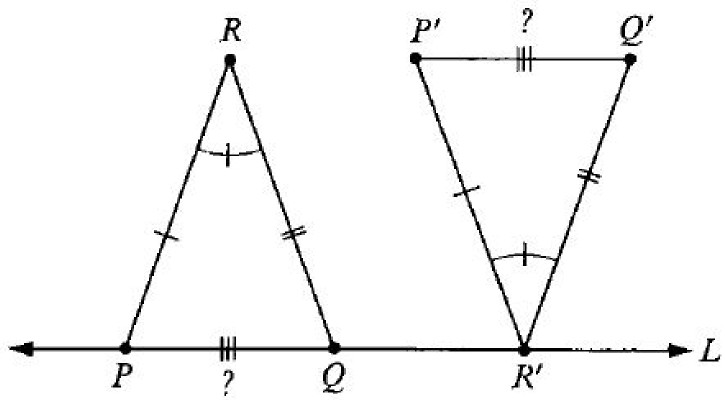


Figure 6.17

Definition 5.3.11 (Right Angle)**. CCSS G-C0-1** *When a straight line standing on a straight line makes the adjacent angles equal to one another, each of the equal angles is* right*, and the straight line standing on the other is called a* perpendicular *to that on which it stands.*

Activity 5.3.12. *Fold paper to make a right angle.*

Extension 5.3.13 (All right angles are equal)**.** *The 4th postulate of Euclid becomes a theorem of Hilbert ([Hil62, Theorem 15][Har00, 9.6].*

Theorem 5.3.14 (SAS)**. CCSS G-C0-8, G-C0-10** *Let ABC and $A'B'C'$ be triangles with $AB \cong A'B'$ and $AC \cong A'C'$ and $\angle CAB \cong \angle C'A'B'$ then $\triangle ABC \cong \triangle A'B'C'$*

Pedagogy 5.3.15. *We prove Theorem 5.3.14 twice to illustrate the close connections between two styles of presenting proofs. The paragraph style allows the use of English to smooth and emphasize the particular inferences. The 'two-column' style regiments giving a reason for each step.*

Proof. We must show $\triangle ABC \cong \triangle A'B'C'$. Draw circles with radius AC from A' and with radius BC from B' using Axiom 3. Let them intersect at a point D on the same side of $A'B'$ as C'. Note that triangle $A'DB' \cong ACB$ by SSS. ($AB \cong A'B'$, $BC \cong B'D$ and $AC \cong A'D$). So $\angle CAB \cong \angle DA'B'$. But then by transitivity of congruence, $\angle C'A'B' \cong \angle DA'B'$. But then D lies on $A'C'$ and in fact D must be C'. So we have proved the theorem. $\square_{5.3.14}$

1	$AB \cong A'B', AC \cong A'C', \angle CAB \cong C'A'B'$	$given$
2	Draw circle with radius AC from A'	$Axiom$ 5.1.1.III
3	Draw circle with radius BC from B'	$Axiom$ 5.1.1.III
4	Choose the point of intersection D of the circles on the same side $A'B'$ as C'.	$Axiom$ 5.1.4
5	$AD \cong AC$	Def circle, 2, 3
6	$\triangle A'DB' \cong \triangle ACB$	$SSS, 5$
7	$\angle CAB \cong \angle DA'B'$	$DefofCongruence$
8	$\angle C'A'B' \cong \angle DA'B'$	$Axiom$ 5.3.1
9	D lies on A'C'	Def: Congruence
10	$D = C'$	$DA' \cong CA'$
11	$C'B' \cong CB$	6, 10
12	$\triangle ABC \cong \triangle A'B'C'$	$SSS, 1, 10$

Definition 5.3.16. *[ERM] A plane Π has enough rigid motions if*

1. *For any $A, A' \in \Pi$, there is a rigid motion ϕ with $\Phi(A) = A'$.*
2. *For any three points $O, A, A' \in \Pi$, there is a rigid motion ϕ that fixes O and sends the ray $\overrightarrow{OA}$ to $\overrightarrow{OA'}$ and*
3. *for any line ℓ there is a rigid motion ϕ that reflects Π over ℓ.*

Note that preserving the first three implies preserving congruence of angles by use of SSS.

Methodology 5.3.17. As we noted in Methodology 5.3.7 rigid motions are defined to clarify the concept of superposition: if a rigid motion takes one figure to another, then they are congruent. This makes Euclid's argument rigorous. [Har00, §17] shows 'enough rigid motions' (ERM) in any Hilbert plane with SAS and conversely that from the axioms for a Hilbert plane without SAS, ERM implies SAS. This is essentially Euclid's proof of Proposition I.4. Thus the problem of superposition can be solved by adding any one of SAS, ERM, SSS to Hilbert planes without SAS.

The most immediate formalization of rigid motions is to add second order quantifiers over arbitrary permutations of the set of points. But one can add a new sort $\mathbf{M}$ for motions and a ternary relation $\mathbf{R}$ on P $\times$ P $\times\, \mathbf{M}$ that for each f in $\mathbf{M}$ the pairs $\langle a, b\rangle$ such that $\mathbf{R}(\mathbf{a}, \mathbf{b}, \mathbf{f})$ is the graph of a rigid motion.

Theorem 5.3.18. *Every rigid motion is a composition of reflections, translations and rotations.*

Proof. A rigid motion ϕ falls into one of four disjoint classes according to the number of points they fix.

1. ϕ fixes all points; $\phi = \psi^2$ where ψ is a reflection.
2. ϕ fixes at least two points A, B but not all. In that case ϕ fixes the line ℓ through AB setwise. So under ϕ each X on ℓ remains the same distance from A and B; thus ℓ is pointwise fixed.

   Suppose $C \notin \ell$ and $\phi(C) = C''$ with $C'' \neq C$ is on the same side of ℓ as C. As ϕ takes the segment AC to AC''. But one is congruent to a proper subset of the other. So $C \notin \ell$ implies $\phi(C) = C'$ is on the opposite side of ℓ from C. Then for any $X \in \ell$, $XC \cong \phi(X)C'$ and $\phi(X) \in \ell$. In particular $AC \cong A\phi(C)$ and $BC \cong BC'$

   Let ℓ' be the line extending CC'. It is distinct from ℓ, so intersects ℓ only in one point D. But since ϕ fixes all lines setwise $\phi(D)$ is on $\ell \cap \ell'$, i.e, $\phi(D) = D$. So $DA \cong DB$ and $DC \cong DC'$. Thus

$\triangle DBC \cong \triangle DBC'$ and $\triangle DAC \cong \triangle DAC'$. So $\angle CDB$ is a right angle and $\ell \perp \ell'$. Now we can see that ϕ is a reflection in ℓ.

Let ℓ'' denote the image of ℓ under ϕ.

3. ϕ fixes a single point A. Then since ϕ preserves lines, it must be a rotation around A (not equal to a full turn).

4. ϕ fixes no point. Since ϕ sends lines to lines and no points are fixed; if for any ℓ, $\ell \parallel \phi(\ell)$; ϕ is a translation, if not it is a glide reflection [CK17, p 82].

$\square$

Pedagogy 5.3.19. *[SLO1: Van Hiele level of Transformational geometry]* Taking into account the necessity for a deep understanding of the notion of abstract function[19], one might posit a further 'Van Hiele' level (thought not geometric): Ability to work with abstract functions. This may not be an issue for college students but additional work on functions might be helpful (and appear in the supplement).

The HS teacher testifies against this, 'At the HS level we successfully work with transformations without using functions. Working in the coordinate system, given two possibly congruent shapes, visually draw a series of transformations of that shape to find out if the two coincide after the transformations.'

The method of proving the following important exercise is embedded in the proof of Theorem 5.3.14.

Exercise 5.3.20 (Move Angle)**.** *Let ABC be an angle. For any segment DE, choose a point F so that $\angle ABC \cong \angle DEF$.*

Construction 5.3.21 (Constructing perpendiculars)**. CCSS G-C0-12** *Given a line AD there is a line perpendicular to the line through AD at D.*

Proof. Extend AD and let B be the intersection of that line with the circle of radius AD centered at D. Now construct an equilateral triangle with base AB by using Axiom 5.1.1 twice to construct the vertex C. Draw CD. SSS implies $\triangle ACD \cong \triangle BCD$; so $\angle CDA \cong \angle CDB$ and therefore $CD \perp AB$. $\square_{5.3.21}$

Extension 5.3.22 (Independence of Congruence Axioms)**.** In the proof we constructed an equilateral triangle using only the first three postulates. We seem to need SSS to finish. [Hil71, p 39] shows by varying the distance formula in the real plane, that the congruence axioms are independent from first two groups.

Definition 5.3.23 (Straight Angle)**.** *An angle $\angle ABC$ is called a* straight *angle if A, B, C lie on a straight line and B is between A and C.*

Since Euclid does not introduce a measure for angles, he has names for the most important, straight and right, and a rough indications of size such as acute and obtuse.

Note a perpendicular creates two right angles on each side of a line. Constructing a perpendicular at the vertex of a straight angle and applying Euclid's fourth postulate yields:

Theorem 5.3.24. CCSS G-C0-9 *All straight angles are equal (congruent).*

[19] See [Har14] for an argument against the use of transformation-based systems in high school; the unfamilarity of sophomores with functions is a key point.

Proof. Let $\angle ABC$ and $\angle A'B'C'$ be straight angles. Construct lines BD and $B'D'$ perpendicular to AC and $A'C'$, respectively. Now $\angle ABD + \angle DBC = \angle ABC$ and $\angle A'B'D' + \angle D'B'C' = \angle A'BC'$. By Axiom 5.3.1, $\angle ABD = \angle A'B'D'$ and $\angle DBC = \angle D'B'C'$. $\square_{5.3.24}$

Theorem 5.3.24 is statement about the uniformity of the plane. In terms of transformations, it says any point and a line through it can be moved by a rigid motion to any other point and any line through it.

Definition 5.3.25. *If two distinct lines intersect, non-adjacent (Definition 5.2.8) angles that have only the vertex in common are called vertical angles.*

Exercise 5.3.26 (CCSS G-C0-9)**.** *Deduce from Theorem 5.3.24 that vertical angles are equal.*

Definition 5.3.27 (Isosceles)**.** *A triangle is* isosceles *if at least two sides have the same length. The angles opposite the equal sides are called the* base angles*. (Note some textbooks require exactly two sides have the same length).*

Activity 5.3.28. *[SLO8, 10: G-CO 11,12] Make two Geogebra constructions using transformations so that a) one takes always yields an isosceles triangle but it may not be equilateral and b) the other also yields an equilateral triangle.*

Activity 5.3.29. G-CO 10 *Activity: Prove the isosceles triangle and exterior angle theorems. Compare 'paragraph' and 'two column' proof.*

Theorem 5.3.30. CCSS G-C0-10 *The base angles of an isosceles triangle are equal (congruent).*

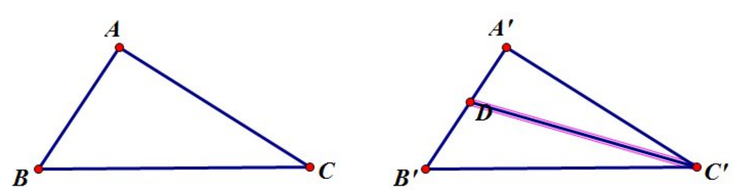
Proof. Let ABC be an isosceles triangle with $AC \cong BC$. We will prove $\angle CAB \cong \angle CBA$. The trick is to prove $\triangle ABC \cong \triangle BAC$. ($\triangle BAC$ is obtained from $\triangle ABC$ by flipping the triangle over its altitude.) We have two ways to prove the congruence. We know $BC \cong AC$ and $AC \cong BC$. We can also note $AB \cong BA$ and use SSS or $\angle ACB \cong \angle BCA$ and use SAS. In any case, since the triangles are congruent $\angle CAB \cong \angle CBA$. $\square_{5.3.30}$

Activity 5.3.31. *Prove the angles of an equilateral triangle are equal. (Note that there are two proofs, using either SSS or SAS, and they are distinguished by which correspondences are made in defining the congruence. Explain this by considering the theorem in terms of rotational or reflective symmetry.)*

We include the proof of the following result to show a typical use of *proof by contradiction*.

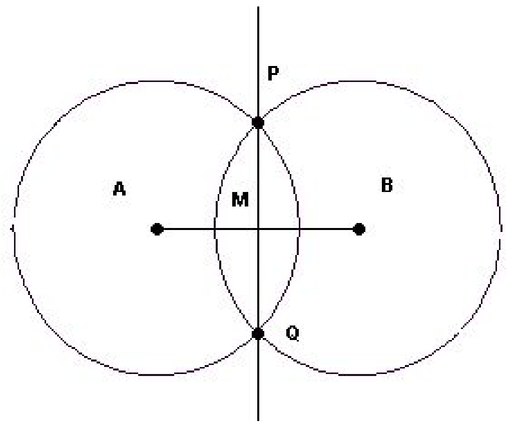
Theorem 5.3.32. *[ASA]* **CCSS G-C0-8, G-C0-10** *If two triangles have two angles and the included side congruent, then the triangles are congruent.*

Proof. Suppose ABC and $A'B'C'$ satisfy $\angle ABC = \angle A'B'C'$, $\angle ACB = \angle A'C'B'$ and $BC = B'C'$. We will show the triangles are congruent.



Choose D on $A'B'$ so that $AB \cong DB'$ (We'll assume D is between A' and B' for contradiction. If A' is between B' and D, there is a similar proof.) Now, $AB \cong DB'$, $BC \cong B'C'$ and $\angle ABC \cong \angle A'B'C'$ so by SAS, $\triangle ABC \cong \triangle DB'C'$. Since the angles correspond, $\angle DC'B' \cong \angle ACB$ and so by Common Notion 1, $\angle DC'B' \cong \angle A'C'B'$. But this is absurd since $\angle DC'B'$ is a proper subangle of $\angle A'C'B'$. $\square_{5.3.32}$

Theorem 5.3.33 (Constructing Perpendicular Bisectors)**. CCSS G-C0-12** *For any line segment AB there is a line PM perpendicular to AB such that M is the midpoint of AB.*



Proof. Set a compass at any length at least that of AB and draw two equal circles centered at A and B respectively. Let the two circles intersect at P and Q on opposite sides of AB and let M be the intersection of AB and PQ.

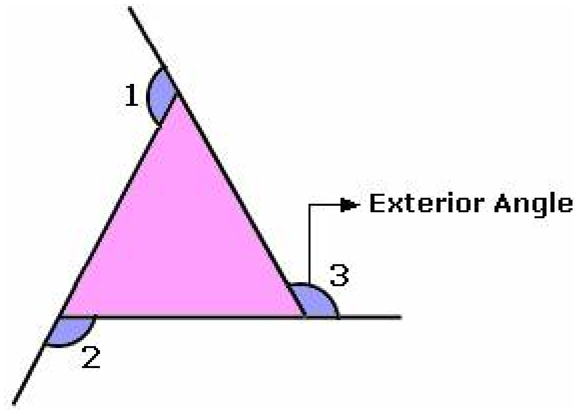
To show PQ perpendicular to AB, note first that $\triangle APQ \cong \triangle BPQ$ by SSS. So $\angle APM \cong \angle BPM$. Then by SAS, $\triangle APM \cong \triangle BPM$. Thus $\angle AMP \cong \angle BMP$. And therefore these are each right angles by Definition 5.3.11. But $\triangle APM \cong \triangle BPM$ also implies $AM \cong BM$ so M bisects AB. $\square_{5.3.33}$

Remark 5.3.34. *Note we could be more prescriptive and just as correct by requiring in the proof of Theorem 5.3.33 that the circle have radius AB. But this is an unnecessary additional requirement.*

Definition 5.3.35. *If D is in the interior of angle $\angle ACB$, line CD bisects the angle $\angle ACB$ if $\angle ACD \cong \angle BCD$.*

Theorem 5.3.36. *[Exterior Angle Theorem, Euclid I.16] An exterior angle of a triangle is greater than either of the interior and opposite angles*

Some modern texts write remote interior angles for interior opposite.



Angle1.jpg

Proof.

Here is Euclid's proof. `http://aleph0.clarku.edu/~djoyce/java/elements/bookI/propI16.html` But there is a subtle dependence on betweenness. See the treatment in [Har00] on page 36.

5.4 The Parallel Postulate

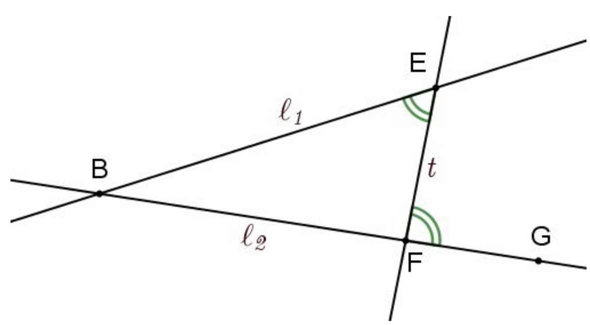
Of course, the change in viewpoint of what axioms mean (Methodology 1.2) stems from the proof of the independence of the parallel postulate. We do not rehearse here the well-known history but do discuss a subtle shift in meaning of the phrase 'the parallel postulate'.

Definition 5.4.1. *Two lines are parallel if they do not intersect.*

The difference between several statements which are close to the parallel postulate provides interesting historical and pedagogical background. The most succinct statement is: For a line ℓ and point A not on ℓ, there is at most one line parallel to ℓ through A. Observe that Euclid proved the existence of parallel lines (Remark 5.4.3). So spherical geometry, which was studied by the Greeks, could not have been seen as example to show the independence. Playfair and Hilbert rephrased the postulate as the existence of unique parallel lines which was confused even by prominent mathematicians [HT05].

The definitions of corresponding, interior, and exterior angles can be found in any geometry text.

Theorem 5.4.2. *[Euclid I.27] If two lines are crossed by a third and alternate interior angles are equal, the lines are parallel.*

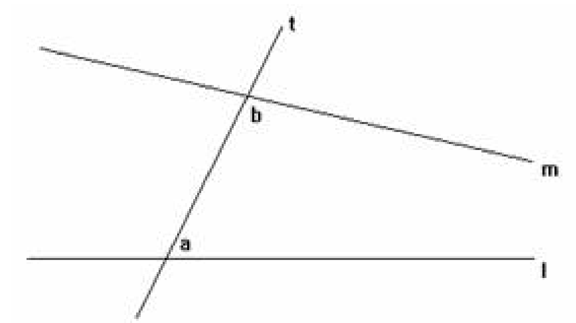


Proof. Assume the lines are not parallel and intersect in point B. The hypothesis says the exterior angle EFG to triangle BFE is equal to the interior angle FEB. That contradicts the exterior angle theorem 5.3.36. So our assumption is wrong.

Remark 5.4.3 (Parallels Exist)**.** *Since the hypothesis of Theorem 5.4.2 is easily constructed, Euclid has proved the existence of parallel lines.*

Axiom 5.4.4. *Heath's statement of Euclid's 5th postulate:*

If a straight line crosses two straight lines in such a way that the interior angles of the same side are less than two right angles, then, if the two straight lines are extended, they will meet on the side on which the interior angles are less than two right angles.



Theorem 5.4.5. *Axiom 5.4.4 implies there is most one parallel to ℓ through P.*

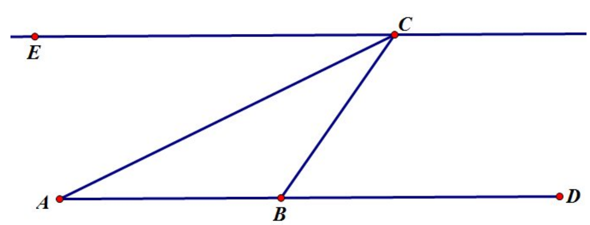
Proof. Suppose two distinct lines ℓ_1, ℓ_2 through P are parallel to ℓ. Fix a transversal m that intersects ℓ with P on m. Since they are distinct the sum of the interior angles for the two lines must be different and so for one of ℓ_1, ℓ_2, say ℓ_1, and for one side of m, the sum of the interior angles must be less than a straight angle. Then by Axiom 5.4.4, ℓ is not parallel to ℓ_1, as required. □parun

This establishes the distinguishing feature of HP5. For any ℓ and P, there is a unique line parallel to ℓ through P. In Theorem 5.5.9 of the supplement, we prove that in HP5 the sum of the angles of a triangle is 180 degrees (a straight angle).

A key equivalent to the parallel postulate is that the measures of the angles in a triangle sum to 90^o. In fact, the simplest definition of a degree is $\frac{1}{90}$ of a right angle. Non-Euclidean geometries can be classified by whether that sum is more (semi-elliptic) or less (semi-hyperbolic[20]) than a straight angle [Har00, p 311].

Theorem 5.4.6. *[HP5]* **CCSS G-C0-10** *The sum of the angles of a triangle is* 180^o.

Proof. That is, we must show the sum of the angles of a triangle is a straight angle.



[20]Elliptic is used when any two lines intersect and Hartshorne reserves hyperbolic for semi-hyperbolic satisfying the limiting parallels axiom.

Draw EC so that $\angle BCE \cong \angle DBC$. (Exercise 5.3.20) Then $EC \parallel AD$. (Theorem 5.4.2) So $\angle BAC \cong \angle ACE$ (Axiom 5.4.7) So $\angle BAC + \angle ACB = \angle DBC$ But $\angle ABC + \angle DBC$ is a straight angle. So $\angle ABC + \angle BAC + \angle ACB$ is a straight angle. $\square_{5.4.6}$

Now consider the converse of Theorem 5.4.2.

Proposition 5.4.7. *If two parallel lines are cut by a transversal then each pair of alternate interior angles contains two equal angles.*

The rest of this subsection illustrates the use of contraposition with an intriguing result: in HP, Proposition 5.4.7 is equivalent to Axiom 5.4.4. Thus the crucial theorems showing the independence of the parallel postulate, incidentally give an example of a sentence and its converse that are not equivalent. We begin with by observing the relation between alternate and adjacient interior angles that depends only on the fact that all straight angles are equal.

Lemma 5.4.8. *For any pair of lines and a transversal, the following are equivalent:*

1. *One pair of alternate interior angles are equal iff the other is.*
2. *One pair of adjacent interior angles sum to a straight angle iff the other does*
3. *There is a pair of equal alternate interior angles*
4. *There is a pair of adjacent interior angles that sum to a straight angle*

The next argument uses the logical notion of contraposition twice.

Definition 5.4.9 (Contraposition)**.** *Let A and B be mathematical statements. The* contrapositive *of 'A implies B' is '$\neg B$ implies $\neg A$'*

Fact 5.4.10 (Logical fact)**.** *Any implication is* equivalent *to its contrapositive.*

Pedagogy 5.4.11. SLO1 This is easily checked by truth tables. High school geometry texts sometimes ask students to memorize the names of the four variants on a conditional (if-then) statement. One is the inverse that I know only from such books. This is counter-productive; only the conditional, converse and contrapositive are used frequently. A frequent difficulty is to understand why 'A implies B' is declared true when both A and B are false. The first author found it useful in undergraduate logic courses to emphasize that we are formalizing English. The ambiguity between inclusive or (either one or both) and exclusive or (but not both) or is easy to illustrate. Logicians decided use $\vee$ to mean inclusive or. A similar decision was made for implication $\rightarrow$. Of course if the instructor finds explanations that convince students that's even better.

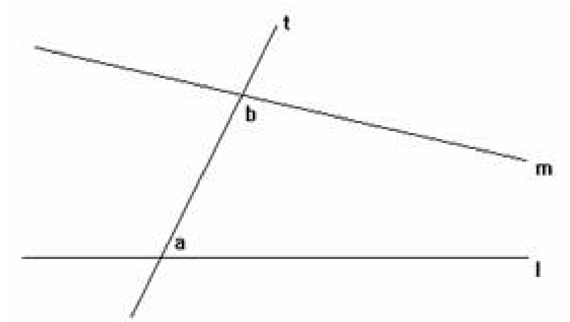
The first direction of the next proof uses the contrapositive of Proposition 5.4.7, which is easy to read off. For the other direction we have to untangle the contrapositive of Axiom 5.4.4.

Theorem 5.4.12. *Axiom 5.4.4 is equivalent to Proposition 5.4.7.*

Proof. Proposition 5.4.7 implies Axiom 5.4.4: Now suppose Proposition 5.4.7 and the hypothesis of Axiom 5.4.4. We have two lines and the sum of the interior angles on one side of a transversal sum to less than a straight angle. By Lemma 5.4.8 neither pair of alternate interior angles equal. Now, by the contrapositive of Theorem 5.4.2, the lines are not parallel; we have shown the conclusion of Axiom 5.4.4

The contrapositive (and so equivalent) to Heath's version of Euclid's 5th postulate reads: If the two lines crossed by a straight line do not meet on one side of that straight line, then the interior angles on that side are not less than two right angles.

Axiom 5.4.4 implies Proposition 5.4.7: Suppose Axiom 5.4.4 and two lines are parallel, satisfying the hypothesis of Proposition 5.4.7. By the contrapositive to Axiom 5.4.4 each pair of adjacent interior angles sum to at least a straight angle and so both pairs sum to a straight angle since the sum of all four is two straight angles. By Lemma 5.4.8 the alternate interior angles are equal yielding Proposition 5.4.7.



6 Proof that the division of a line into n equal parts succeeds

We began this excursion into axiomatic geometry by trying to prove that for any n we could divide a line into n equal segments. The construction (Figure 1) used only Euclid's first 3 axioms. *We need to show the segments cut off by the C_i are actually equal.* We use the methods of Section 5 to *almost* prove the procedure in Exercise 4.3 works. We will discover that entirely different methods are needed for the last step in the proof – the side-splitter theorem 8.5.

Looking at the diagram from our guiding problem, since a quadrilateral whose opposite sides are equal is a parallelogram (Theorem 6.0.3), $ABCD$ is a parallelogram. We DO NOT know that A_4B_4CD is a parallelogram. In order to establish that it is, we need some more information about parallelograms.

Pedagogy 6.0.1. The classification of quadrilaterals is a major topic in high school geometry. It is essential to first clarify the notion of 'classify'; it does not help to say 'a square is a rectangle just as a parallelogram is a quadrilateral' (heard from a high school teacher). The analogy the student needs is 'dogs and cats are animals'.

Classifications may be 'exclusive' or 'inclusive'. Euclid requires an isosceles triangle to have *exactly* two equal sides while modern texts include classifications that are inclusive; equilateral triangles are isosceles.

Definition 6.0.2. *A parallelogram is a quadrilateral such that the opposite sides are parallel.*

Theorem 6.0.3. CCSS G-CO.11 *If the opposite sides of a quadrilateral are equal, the quadrilateral is a parallelogram.*

Proof. Suppose $ABCD$ is the parallelogram; draw diagonal AC. Then $\triangle ABC$ and $\triangle ACD$ are congruent by SSS. Therefore $\angle BAC \cong \angle ACD$. Now since alternate interior angles are equal, $AB \parallel DC$. Similarly, $BC \parallel AD$. $\square_{6.0.3}$

The argument also shows:

Theorem 6.0.4 (Euclid I.34). **CCSS G-CO.11** *In any parallelogram the opposite sides and angles are equal. Moreover, each diagonal splits the parallelogram into two congruent triangles.*

Lemma 6.0.5. CCSS G-CO.11 *If one pair of opposite sides of a quadrilateral $ABCD$, labeled as in Figure 1, are equal and parallel, the figure is a parallelogram.*

Proof. Draw the diagonal AC. By alternate interior angles $\angle BCA \cong DAC$. The triangles ACB and ACD are congruent by SAS, using the hypothesis and that they share a side. So $\angle BAC \cong \angle ACD$. Now viewing AC as a transversal of BA and CD, they are parallel and we finish.

Lemma 6.0.6. *If $ABCD$ is a parallelogram, labeled as in Figure 1, and two points X, Y are chosen on the opposite sides BC and AD so that $XC \cong YD$ then $XCDY$ is a parallelogram.*

Proof. Apply Lemma 6.0.5 taking X for A and Y for C.

Motivation 6.0.7. *[SLO1, SLO2]* We are giving the proof of our guiding problem in reverse to show how the abstract side-splitter theorem is needed to solve a concrete problem. The proof of it requires a new central idea - proportionality. The next two sections are devoted to providing a firm foundation for proportion. By using Hilbert's proof rather than Euclid's we avoid reliance on the Archimedean axiom.

To finish the proof we need a very strong result:

Theorem 6.0.8. *Euclid VI.2: Side-splitter* **CCSS G-SRT.4** *If a line is drawn parallel to the base of a triangle the corresponding sides of the two resulting triangles are proportional and conversely.*

Proof of the guiding problem assuming sidesplitter: By repeating the argument for Lemma 6.0.6, we show all the lines $A_iC_iB_i$ are parallel. In particular the line C_4B_4 is parallel to the base B_3C_3. Applying Theorem 8.5, we complete our proof as follows:

$$\frac{CB_4}{CB_3} = \frac{CC_4}{CC_3}.$$

But we constructed $B_4C \cong B_3B_4$, so $C_4C \cong C_3C_4$, which is what we are trying to prove. Now move along AC, successively applying this argument to each triangle. $\square_{??}$

To finish the proof we need a very strong result:

Theorem 6.0.9. *Euclid VI.2: Side-splitter* **CCSS G-SRT.4** *If a line is drawn parallel to the base of a triangle the corresponding sides of the two resulting triangles are proportional and conversely.*

Proof of the guiding problem assuming sidesplitter: By repeating the argument for Lemma 6.0.6, we show all the lines $A_iC_iB_i$ are parallel. In particular the line C_4B_4 cuts the triangle B_3C_3C and is parallel to the base B_3C_3. Applying Theorem 8.5, we complete our proof as follows:

$$\frac{CB_4}{CB_3} = \frac{CC_4}{CC_3}.$$

But we constructed $B_4C \cong B_3B_4$, so $C_4C \cong C_3C_4$, which is what we are trying to prove. Now move along AC, successively applying this argument to each triangle. $\square_{4.3}$

7 Finding the underlying field

We reduced our cutting the line problem to the side-splitter theorem VI.2; that is, to the fundamental result about the similarity of triangles. Hilbert defines a (semi)-field of segments (addition and multiplication on the positive elements of an ordered field). He thus has the modern algebraic theory of proportion and VI.2 follows easily (Section 8). Then (Section 9) he defines a measure of area function which recovers Euclid's theory of area and connects it with numerical measures of area.

Motivation 7.1. *[SLO8* **Irrationality: the Pythagorean scandal***]* The geometry course is an excellent place to organize historically and conceptually the college students understanding of irrational and transcendental numbers (Section 10). Two or more magnitudes are *commensurable* if they share a common measure. Two feet and three feet are commensurable, each being a multiple of a foot; but the diagonal and side of a square are incommensurable. Thus, the irrationality of $\sqrt{2}$ is usually attributed to 5th century BCE Pythagoreans. A solution to comparing irrationals was developed by Eudoxus in the 4th century BCE and expounded in Euclid Book V on proportion, perhaps a century later. Crucially, this was a study of 'magnitudes' of various dimensions. The notion of ascribing a number to a measure of area was only adopted in geometry during the 19th century AD and put on a firm footing by Stolz and Pasch as expounded in [Hil62]. A beauty of Hilbert's approach is that he shows that (a suitable translation) of the (first order) axioms of Euclidean geometry allow the measure of area in any Euclidean plane (Notation 3.8) by interpreting a field into the plane. In Section 10, we will note how the real numbers provide the most commonly used example. For further background on Greek study of irrational numbers see [Smo08].

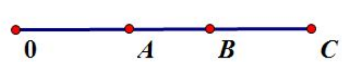
The proof of the side-splitter theorem (Theorem 8.5.) is difficult because the meaning of ratio between two incommensurable sides is obscure at best. To solve this problem, Hilbert defines *geometrically* a multiplication of line segments. Identify the collection of all congruent line segments and choose a representative segment OA for this class. There are then three distinct historical steps. (For SLO7, see [GG09] and Heath's notes to Euclid VI.12 (`http://aleph0.clarku.edu/~djoyce/java/elements/bookVI/propVI12.html`.) In Greek mathematics numbers (i.e. 1, 2, 3 ...) and magnitudes (what we would call length of line segments) were distinct kinds of entities and areas were still another kind. Numbers simply count the number of some unit; the unit varies from situation to situation. For them the notion of a assigning a number as the length of the diagonal of a unit square is incomprehensible.

We first introduce an addition and multiplication on line segments and then prove the geometric theorems to show that these operations satisfy the field axioms except for the existence of an additive inverse.

Notation 7.2. *Note that congruence forms an equivalence relation on line segments. We fix* a ray $\overrightarrow{\ell}$ *with end point* 0. *For each equivalence class of segments, we consider the unique segment* $0A$ *on* $\overrightarrow{\ell}$ *in that class as the representative of that class. We will often denote the class (i.e. the segment* $0A$*) by* a. *We say a segment (on any line)* CD *has length* a *if* $CD \cong 0A$.

Definition 7.3 (Segment Addition)**.** *Consider two segment classes* a *and* b. *Fix representatives of* a *and* b *as* OA *and* OB *in this manner: Extend* OB *to a straight line, and choose* C *on* OB *extended (on the other side of B from A) so that so that* $BC \cong OA$. OC *is the sum of* OA *and* OB.

Diagram for adding segments



Activity 7.4. *Prove that this addition is associative and commutative.*

Of course there is no additive inverse if our 'numbers' are the lengths of segments which must be positive. We discuss finding an additive inverse after Definition 7.12. Following Hartshorne [Har00], here is our official definition of segment multiplication.

Definition 7.5. *[Multiplication] Fix a unit segment class* 1*. Consider two segment classes* a *and* b*. To define their product, define a right triangle*[21] *with legs of length* 1 *and* a*. Denote the angle between the hypotenuse and the side of length* 1 *by* α*.*

Now construct another right triangle with base of length b *with the angle between the hypotenuse and the side of length* b *congruent to* α*. The length of the leg opposite* α *is* ab*.*

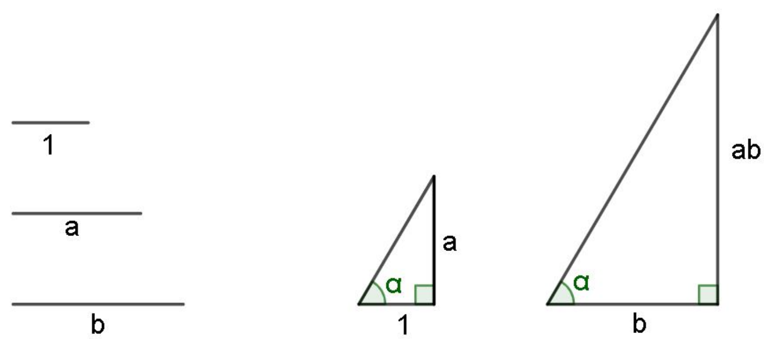


Figure 2: Multiplication

Exercise 7.6. *We now have two ways in which we can think of the product* $3a$*. On the one hand, we can think of laying 3 segments of length* a *end to end. On the other, we can perform the segment multiplication of a segment of length* 3 *(i.e. 3 segments of length* 1 *laid end to end) by the segment of length* a*. Prove these are the same.*

Before we can prove the field laws hold for these operations, we introduce a few more geometric facts. The crux of the argument is to prove that the multiplication is associative and commutative. Hilbert and many successors give this argument as arising from the Desargues and Pappus theorems which hold in HP5 (neutral geometry plus the parallel postulate). Because the techniques of its proof are more similar to standard high school material, we rely on the cyclic quadrilateral theorem.

Theorem 7.7. *[Euclid III.20]* **CCSS G-C.2** *In a circle, if a central angle and an inscribed angle cut off the same arc, the inscribed angle is congruent to half the central angle.*

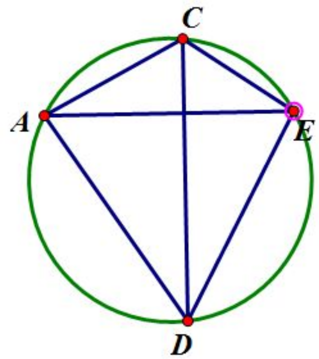
Exercise 7.8. *Do the activity: Determining a curve (determinecircle.pdf).*

Activity 7.9. *Prove a central angle is twice an inscribed angle that inscribes the same arc. How many diagrams (cases) must you consider? This activity is on the website in both java and geoalgebra.*

We need (Corollary 7.10) of [Har00, Proposition 5.8] (Corollary 7.10), which is a routine (if sufficiently scaffolded) high school problem.

[21] The right triangle is just for simplicity; we really just need to make the two triangles similar.

Corollary 7.10. *[***CCSS G-C.3: Cyclic Quadrilateral Theorem***] Let $ACED$ be a quadrilateral. The vertices of $ACED$ lie on a circle (the ordering of the name of the quadrilateral implies A and E are on the opposite sides of CD) if and only if $\angle EAC \cong \angle CDE$.*



Proof. Given the conditions on the angle draw the circle determined by ABE. Observe from Lemma 7.7 that D must lie on it. Conversely, given the circle, apply Lemma 7.7 to get the equality of angles. $\square_{7.10}$

Theorem 7.11. *The multiplication defined in Definition 7.5 satisfies:*

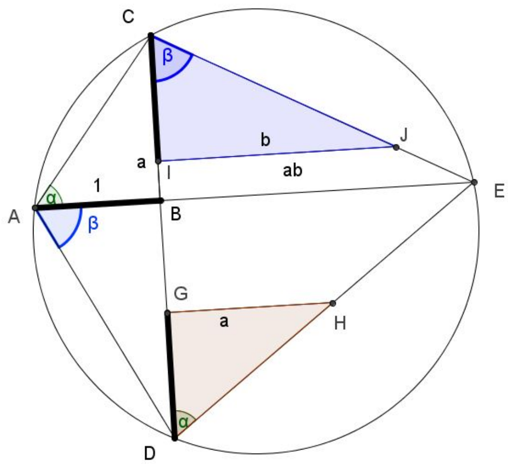
1. *For any a,*
$$a \cdot 1 = a$$
2. *For any a, b*
$$ab = ba.$$
3. *For any a, b, c*
$$(ab)c = a(bc).$$
4. *For any a there is a b with $ab = 1$.*
5. *$a(b + c) = ab + ac$.*

Proof. We prove item 2 (Figure below), since that requires some work. The slight variants for associativity and distributivity are in [Har00, 19.2].

Given a, b, first make a right triangle $\triangle ABC$ with legs 1 for AB and a for BC. Let α denote $\angle BAC$. Extend BC to D so that BD has length b. Construct DE so that $\angle BDE \cong \angle BAC$ and E lies on AB extended on the other side of B from A. The segment BE has length ab by the definition of multiplication.

Since $\angle CAB \cong \angle EDB$ by Corollary 7.10, $ACED$ lie on a circle. Now apply the other direction of Corollary 7.10 to conclude $\angle DEA \cong \angle DCA$ (as they both cut off arc AD). Now consider the multiplication beginning with triangle $\triangle DAB$ with one leg of length 1 and the other of length b. Then since $\angle DAB \cong \angle BCE$ and the leg adjacent to $\angle BCE$ has length a, the length of BE is ba. Thus, $ab = ba$.

The key point for proportionality is 4): the ability to find inverses. This is done by noting that in Figure 2, if multiplication by a is given by the angle α, multiplication by a^{-1} comes from β, the other acute angle in the triangle.



$\square_{7.11}$

We have a *semi-field* because the addition does not form a group because there are no additive inverses (negative segments). This is important for Hilbert because he is giving an entirely geometric proof. The above semi-field can be modified to become a field by taking points on a line rather than a ray, and then have both positive and negative numbers, therefore getting additive inverses. With this geometrically based field we give in the next section an algebraic basis for the theory of proportion which allows us to prove side-splitter.

Definition 7.12 (Adding points)**.** *Recall that a line is a set of points. Fix a line ℓ and a point 0 on ℓ. We define an operations $+$ on ℓ. Recall that we identify a with the (directed length of) the segment 0a.*

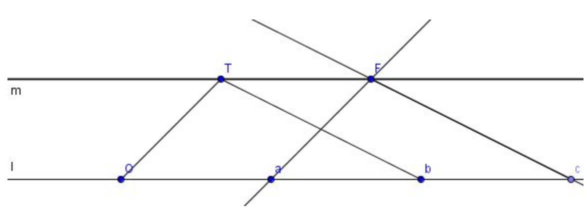
For any points a, b on ℓ, we define the operation $+$ on ℓ:

$$a + b = c$$

if c is constructed as follows.

1. *Choose T not on ℓ and m parallel to ℓ through T.*
2. *Draw $0T$ and bT.*
3. *Draw a line parallel to $0T$ through a and let it intersect m in F.*
4. *Draw a line parallel to bT through F and let it intersect ℓ in c.*

Diagram for point addition



$$0b \cong ac$$

That is, $0a + ac = 0c$ which means $a + b = c$.

After extending multiplication to the whole line by requiring that multiplication by a negative reverses orientation we have proved:

Theorem 7.13. *If Π is a model of $HP5$, then, fixing any two points in Π as $0, 1$, there are first order formulas defining $<, +, \times$ such that $\langle \ell, <, +, \times \rangle$ is an ordered field.*

Methodology 7.14. Definition 2.3 showed we could define a coordinate plane in any field. Combined with Theorem 7.13, we have a bi-interpretation of fields and planes, described informally in Methodology 11.1 and formally in the Appendix to the supplement. This means that the algebraic proofs in high school analytic geometry can (but not easily) be converted to synthetic proofs in first order geometry.

Problem 7.15. *Add a and b (i.e. construct c) when a is to the left of 0 on ℓ. What is the inverse of a? The additive inverse of a is a' provided that $A'0 \cong 0A$ where a' is on ℓ but on the opposite side of 0 from A. That is, if a is the directed segment $\overrightarrow{0A}$, $-a$ is the directed segment $\overleftarrow{A'0}$ that is congruent to $0A$.*

8 Similarity, Proportion, and Side-splitter

In this section, we define proportion using Section 7 and then prove the side-splitter theorem. We need a couple of definitions. Recall that in Section 7 we defined a field whose elements were line segments on a fixed line $\overline{01}$. So we make the following definitions using a, b etc. to range segments $(O, A), (O, B)$ etc. Most texts will have identified these segments with real numbers. We emphasize that the results are much more general than that.

Definition 8.1. *Let a, b, a', b' be segments on a fixed line $\overleftrightarrow{01}$. Then we say the ratios $a : b$ and $a' : b'$* satisfy the proportion $a\colon b = a'\colon b'$ *(also written $a\colon b :: a'\colon b'$ or $\frac{a}{b} = \frac{a'}{b'}$) if $ab' = ba'$.*

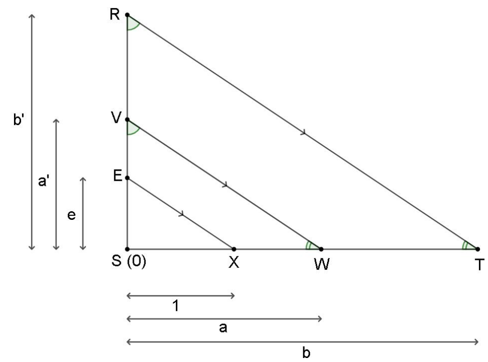
Definition 8.2. *Two triangles $\triangle ABC$ and $\triangle A'B'C'$ are similar if under some correspondence of angles, corresponding angles are congruent; e.g. $\angle A' \cong \angle A$, $\angle B' \cong \angle B$, $\angle C' \cong \angle C$.*

Activity 8.3. *Various texts define 'similar' as we did, or as corresponding sides are proportional or require both. Discuss the advantages of the different definitions. Why are all permissible?*

Theorem 8.4. *Similar triangles have proportional sides.*

Proof. Suppose SVW and SRT are similar triangles as displayed in the diagram below we show

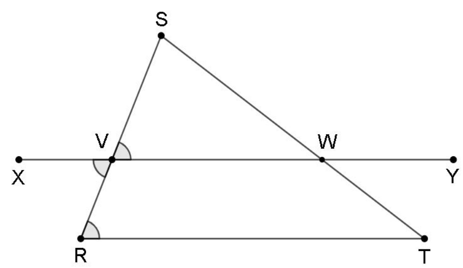
$$\frac{SV}{SR} = \frac{SW}{ST}.$$



Consider the special case that $\angle RST$ is a right angle. Label SW as a, ST as b, SV as a', SR as b', Then think of S as 0 and pick a point X of ST with $SX \cong 01$. Now using segment multiplication the diagram shows $ab' = ba'$. So by definition $a : b = a' : b'$ or $\frac{SW}{ST} = \frac{SV}{SR}$. [Hil71, p. 56] gives the half page argument that the restriction to a right angle is unnecessary. $\square_{8.5}$

Theorem 8.5. *Euclid VI.2: Side-splitter* **CCSS G-SRT.4** *If a straight line is drawn parallel to one of the sides of a triangle, then it cuts the sides of the triangle proportionally; and, if the sides of the triangle are cut proportionally, then the line joining the dividing points is parallel to the remaining side of the triangle.*

Proof. On $\triangle SRT$ draw VW parallel to RT. As in the following diagrams, extend VW to a line and pick points X and Y on VW on opposite sides of the triangle as shown.

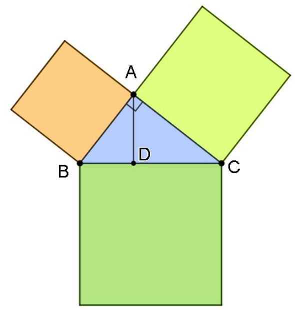


Now $\angle XVR$ and $\angle VRT$ are alternate interior angles for the transversal RS crossing the two lines XY and RT. So $\angle XVR \cong \angle VRT$ if and only if $VW \parallel RT$. But $\angle XVR \cong \angle SVW$ since they are vertical angles. So $\angle SVW \cong \angle VRT$ if and only if $VW \parallel RT$. So $\triangle SRT$ and $\triangle SVW$ are similar and we finish by Lemma 8.4.

Suppose we only know VW cuts each side proportionally. Choose W' with $VW' \parallel RT$. The parallelism and our definition of multiplication imply VW' is to ST as SV is to ST. But we know SW' satisfies the same proportion so $W = W'$. Thus, $VW \parallel RT$ as required. $\square_{8.5}$

As we will sketch in Section 9, Euclid developed the notion of area (he says equal figure.) in I.35-I.48, *Commentators agree that this was specifically to avoid the use of proportion in the proof of Pythagoras.* In particular, Euclid needed the Archimedean axiom for his theory of proportion and so to prove the side-splitter. Hilbert grounds the theory of proportion *purely geometrically* without assuming Archimedes' axiom.

Exercise 8.6 (Euclid VI.31 CCSS G-SRT4)**.** *Prove the Pythagorean theorem using similarity.*



The supplement suggests several other proofs of Pythagoras including one due to President Garfield.

9 Area of Polygons

Pedagogy 9.1. SL01, 02,05,07: Experience with college students in precalculus and calculus who react to min-max problems by saying 'I know the formula is lw or $2l + 2w$ but I don't know which' motivates this section. The connection between (equi)-decomposition and area needs to be made in high (if not middle school). While the argument in argument in the supplement is too technical for high school, it provides future high school teachers with a necessary perspective.

This section has both methodological and pedagogic content. We reserve the details of the methodological concerns to the supplement.

Methodology 9.2 (SLO4: What is area?)**.** This section expounds the differences among three methods of computing area that are frequently conflated in high school texts. Euclid begins by (implicitly) defining what it means for two figures to have same area (Euclid-equal). By this means, he is able to prove the Pythagorean theorem without invoking the notion of proportion -showing it is a fully *geometric* result. In contrast, calculus based notions of measuring area rely fundamentally on approximating figures by infinitely many smaller figures and taking limits.

Using the field defined in Section7, Hilbert defines 'equal area' by a slightly different notion (Hilbert-equal) and introduces a finite procedure of assign a numeric value as the area of a polygon. In fact, these three notions of equality are the same. However, they cannot be proved the same as equi-decomposable (scissors-congruent) without the use of the Archimedean axiom.

We established a linear order on (congruence classes) of segments by $[AB] < [CD]$ if $AB \cong A'B'$ for some proper subsegment $A'B'$ of CD. This is not so easy in two dimensions; a long skinny rectangle might or might not 'be bigger' than a short fat one. What sorts of objects we can assign area to and when two 'figures' have the same area?

Definition 9.3.

1. *A* (rectilineal) figure *is a subset of the plane that can be expressed as a finite union of disjoint triangles (Sides may overlap; interiors can't.).*
2. *A* polygon *is a closed figure whose sides are line segments that intersect only at their endpoints and each endpoint is shared by exactly two segments. Closed means you can trace the outer edges and come back to where you started without any repetition.*

The term figure is introduce to allow for various types of decomposition.

However, there are at least two ways to implement Method 2.

Definition 9.4. *[Two ways to measure]*

1: 'equal' area Define an equivalence relation[22] $E(P, P')$ on figures and define $[P_1] < [P_2]$ if some representative of $[P_1]$ is congruent to a proper subset of a representative of $[P_2]$.

We give three different equivalence relations of this sort in Definition 9.8, 9.9, 5.3.1. We see below that the first and third are the same for HP5; Scissors congruence becomes equivalent in Archimedean geometries.

2: equal numerical measure

Analytic measure Fix a unit of area; say, a square; tile the plane with congruent squares. Then to measure a figure, continually (perhaps infinitely often) refine the measure by cutting the squares in quarters and counting only those (possibly fractional) squares which are contained in the figure.

Geometric measure (Hilbert) Decompose the figure into finitely many disjoint triangles, which are each assigned area $\frac{bh}{2}$, and add those areas.

We call the last geometric area because the multiplication is the geometric multiplication of Section 7. We consider now the three ways to implement Method 1. Before giving the formal definition we see how two of these methods are abstracted from the proof of Euclid I.35.

Theorem 9.5. *Proposition I.35 Parallelograms which are on the same base and in the same parallels equal one another.*

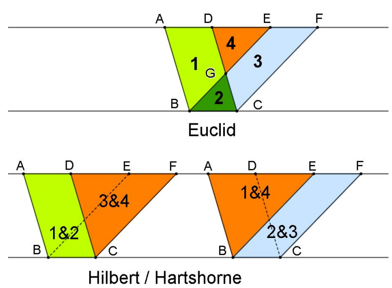


Figure 3: Euclid I.35

Proof. There are two ways of understanding the proof. The terms 'Euclid equal' and 'Hilbert equal' are defined below (generalized from this argument. Euclid says triangle 1 + 4 is congruent to triangle 3 + 4 . Subtract 4 from the first to get trapezoid 1 and from the second to get trapezoid 3. So 1 and 3 have the same area. Add 2 to each to see the two parallelograms, 1 + 2 and 3 + 2, have the same area.

Hilbert says adding triangle 3&4 to parallelogram 1&2 gives the same as adding triangle 1&4 to parallelogram 2&3, and 1&4 and 3&4 are equidecomposable (in this case congruent) so we can conclude the two parallelograms have equal area. The distinction is that the weaker condition 'equidecomposablity' on the triangles 1&4 and 3&4 allows him to build scissors decomposition into his notion. $\square_{9.5}$

[22] In the supplement we define two such equivalence relations, Euclid-equal and Hilbert-equal, and prove they each agree with geometric measure.

Both understandings of the proof required *both adding and subtracting area* rather than scissors congruent. One way of expressing the Archimedean postulate is to say 'every line segment is finite'. We show in Theorem 10.4 of the supplement there are non-Archimedean planes that satisfy HP5. Neither understanding of the proof of Theorem 9.5 requires finite line segments. But we now see that scissors congruence does.

Example 9.6. *Suppose the lines BE and CF are infinitely long. The parallelograms $ABCD$ and $EBCF$ are not equidecomposable.*

Proof. The sides of $ABCD$ are all finite and so a decomposition must be into finitely many triangles, each with all side finite. But a finite decomposition of $EBCF$ requires infinitely many finite triangles because the entire line EB must be covered by edges of the decomposing triangles. However, $ABCD$ and $EBCF$ have the same altitude and same base so they have the same geometric measure. $\square$9.6

Pedagogy 9.7. The distinction described in this section is not high school material. But it is background to avoid fallacious assertions. It is natural in K-12 education to describe equal area in terms of scissors congruence and certainly scissors congruent figures have the same area. But Example 9.6 shows that in some models of $HP5$ there are parallelograms of equal area that are not scissors congruent. Thus their putative equivalence is still another example of an independent proposition. This is not a topic for high school. But teachers can remember to say 'scissors congruent figures have the same area' while not saying 'figures with the same area are scissors congruent'.

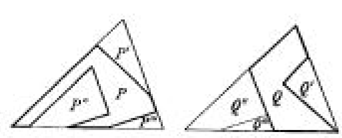
Definition 9.8. *[Euclid-equal polygons] For figures A and B:*

1. *A has 1-equal area with B there is a figure C such that $A + C$ is congruent to $B + C$ or there is a C such that $A - C$ congruent to $B - C$.*
2. Euclid-equal *(provably same as equi-complementable) is the transitive closure of the symmetric and reflexive relation* 1-*Equal content.*

Definition 9.9 (Scissor Congruence)**.** *Two polygons are* scissor-congruent *or* equidecomposable *if one can be cut up into a finite number of triangles which can be rearranged to make the second.*

SLO7: It is a sign of Euclid's genius that he realized that a type of refinement of scissors-congruent, dubbed *equal content* by Hilbert around 1900, allowed the proof of proportionality of area to base and height without appeal to Archimedes axiom.

Definition 9.10 (Equal content)**.** *Two figures P, Q have* equal content aka equicomplementable *or* Hilbert equal[23] *if there are figures $P'_1 \ldots P'_n$, $Q'_1 \ldots Q'_n$ such that none of the figures overlap, each P'_i and Q'_i are scissors congruent and $P \cup P'_1 \ldots \cup P'_n$ is scissors congruent with $Q \cup Q'_1 \ldots \cup Q'_n$.*



Here is Hilbert's finite scheme for measuring area.

Lemma 9.11. *[Har00, §23] For any n and any triangulation of a figure by n triangles with base b_i and height h_i the sum $\Sigma_n \frac{b_i h_i}{2}$ is the same. That value is the geometric measure of the area of a polygon. So, the equivalence relation imposed by 'same geometric measure' is well-defined.*

[23] The diagram is taken from [Hil71].

While Euclid-equality is transitive by definition, it is considerably more difficult [Har00, p 199-201] to prove that Hilbert-equality is transitive.

Theorem 9.12. *[Har00, §23] In any plane satisfying HP5, figures that have equal area under either Hilbert's or Euclid's notion of equal area if and only if they have the same geometric measure.*

However the analytic method of Definition 9.4 is an outlier.

Definition 9.13. *Two figures are* analytically equivalent *if they have the same analytic measure.*

The supplement contains an example showing the Archimedean hypothesis is essential for the next result.

Fact 9.14 (Wallace-Bolyai-Gerwien Theorem[24])**.** *Two polygons* in an Archimedean plane *are equidecomposable (scissors congruent) if and only if they have the same analytic measure.*

Note that the Archimedean hypothesis is essential. If the line BE in Figure 3 is infinite (Invert the segment $\overline{AB}$ created in Remark 12.2.), while all lines in $ABCD$ are finite then the parallelograms $ABCD$ and $EBCF$ are not equidecomposable even though they are Hilbert and Euclid equal. This equivalence often appears in high school text books without making it clear that it requires a vastly stronger hypothesis than any of the other results on polygons.

Interestingly, not all polyhedra (3D figures with plane surfaces) are scissors congruent, for example a regular tetrahedron cannot be cut up into polyhedral and rearranged into a cube.

Fact 9.15 (Dehn-Sydler Theorem)**.** *Two polyhedra in $\Re^3$ are scissors congruent iff they have the same volume and the same Dehn invariant.*

Dehn [D] proved in 1901 that equality of the Dehn invariant is necessary for scissors congruence. Sydler proved the converse forty years later. We return to one of our original targets.

Theorem 9.16 (Euclid VI.I)**.** *If two triangles have the same height, the ratio of their areas equals the ratio of the length of their corresponding bases.*

Proof. Definition 9.4 gave the geometric measure of a triangle to be $\frac{bh}{2}$ and Theorem 9.12 showed geometric measure is equivalent to Euclid equal. So the result follows from realizing that $A = \frac{bh}{2}$ can be read as 'the area is jointly proportional to the base and the height. $\square_{9.16}$

In Euclid this result holds for irrationals only by the method of Eudoxus, which is a precursor of the modern theory of limits, but did not envision the existence of arbitrary real numbers. He deduces side-splitter from the proportionality while Hilbert goes in the other direction[25]. The development here shows that for any triangles which occur in a geometry satisfying the axioms here[26] the areas and their ratios are represented by line segments in the field.

[24] The forward direction was proven in the 19th century by William Wallace (not one of the progenitors of calculus), Farkas Bolyai (his son discovered non-Euclidean geometry) and P. Gerwien. See wikipedia.

[25] [Edu09b] shows the area of one of two similar figures is r^2 times the area of the other, where r is the constant of proportionality between lengths. They deduce this from side-splitter. It was Al Cuoco of the CME team who alerted the first author to Euclid going in the other direction.

[26] Crucially, neither Archimedean, nor Dedekind complete, is assumed.

10 Archimedes, Dedekind, and Completeness

We quoted in Methodology 1.2, Hilbert's desire 'to choose for geometry a simple and complete set of independent axioms'. In this section we first discuss Hilbert's continuity axioms in the context introduced in Methodology 1.4: T is *descriptively complete* [Det14] if T implies all the statements in our preexisting list of 'true geometrical statements'. Then we consider more formal notions of 'complete' which were developed in the first third of the 20th century.

A main theme of the preceding sections is that Hilbert (1899) showed descriptive completeness of his first four groups of axioms (not only for Euclid's plane geometry but establishing Descartes' analytic geometry [Har00, §20-23]). Hilbert's continuity axioms (Group V) aimed at establishing

1. a *geometric* basis for what is variously called Cartesian/coordinate/analytic geometry
2. categoricity –T is categorical if it has a unique model (up to isomorphism).

At the turn of the 20th century the only rigorous basis for the real plane was the construction of the real numbers from the natural numbers by [Ded63] (1888) and then constructing the Cartesian plane over the reals. But Section 7 works from a plane satisfying geometric axioms and defines the field in it. By adding an axiom implying the plane and the field are unique both goals are reached. The rather complicated story for completeness is told in Methodology 10.10.

Hilbert's Group V (continuity axioms) contains two axioms. The Archimedean axiom is usually taken as a property of an ordered group (or field). However for geometry it says for any pair of line segments AB and CD there is a natural number n such that n copies of AB cover CD. Since the n is unbounded, this axiom is not first order but rather in a logic called[27] $L_{\omega_1,\omega}$. Note that the statement of the Archimedean axiom involves some notion of 'addition of lengths'.

Euclid uses the Archimedean axiom in Book V on proportion and then to prove VI.2, the side-splitter theorem. As we have seen Hilbert establishes VI.2 on the basis of axiom groups I-IV which are all first-order.

Although expressed in an unusual way[28], Hilbert's *completeness axiom* can be regarded as asserting Dedekind completeness (equivalently, the least upper bound axiom) in the theory of ordered fields. This is his only use of these axioms to prove geometric theorems in [Hil62]; The other applications are to proving metamathematical (independence/consistency) results.

A standard result in advanced calculus courses shows every Dedekind complete field is Archimedean. So the Archimedean axiom is redundant in Hilbert's system. He singles it out to show that the 'completeness' is not needed for such important results as Theorem 9.14 showing the equivalence between decomposition and measure for determining area.

Pedagogy 10.1 (Student background)**.** In particular, for Hilbert to show that his results do not depend on Archimedes, he must show that non-Archimedean fields exist. Hilbert gives a concrete proof (involving the study of rational function fields) of the existence of non-Archimedean fields, taking t to be infinite in an ordering of the rational function field $\Re(t)$. This is not usually taught in an undergraduate algebra course. We give now a proof using the 'compactness' theorem for first order logic– a standard topic in an undergraduate course in mathematical logic.

The example of the plane over the real algebraic numbers given after Activity 3.5 shows:

[27] Quantification is allowed only over individuals but infinite conjunctions and disjunctions are allowed. The Archimedean axiom asserts an infinite disjunction: $\bigvee_n \phi_n(A, B, C, D)$ where ϕ_n says n copies of AB cover CD.

[28] The axiom asserted 'a maximal Archimedean geometry', hence unique. Currently, 'categoricity' means uniqueness. And, complete means negation complete 10.7. Strictly speaking, Hilbert's 'maximality' axiom is only expressed in the arcane 'sort logic' [Vää14].

Theorem 10.2. *The ruler postulate and Hilbert's 'completeness axiom are independent from all the other axioms'.*

Pedagogy 10.3. *[Impact on other courses] This observation is important for teaching precalculus and calculus as it emphasizes the gap between transcendental and algebraic numbers. In fact, there are only countably many algebraic numbers.*

Theorem 10.4. *[Proof of Existence of non-Archimedean fields] There exists a non-archimedean field.*

Proof. We note in Methodology 10.8 that Tarski's negation-complete extension of Euclidean geometry is the theory of $\Pi(M)$ where $M \models T_{rcf}$, the set of all first order sentences in the vocabulary of fields true in the real field. It has models of arbitrary cardinality and most are non-Archimedean. Consider the set Σ of sentences: $\{n \times \overline{AB} < \overline{01}\}$ for $n \in \mathbb{N}$. Clearly every finite subset of Σ is satisfiable. By the compactness Theorem[29], they are simultaneously satisfiable in some model M of T_{rcf}. Such an $\overline{AB} \in M$ is an infinitessimal. Moreover, no complete first order extension of EG (Euclidean Geometry; Notation 3.8) is finitely axiomatizable [Zie82]. There are uncountably many first order complete theories extending EG. $\square_{10.4}$

Pedagogy 10.5. *[Continuity used]* Theorem 9.14 is one of two places where the continuity axioms are *necessary* for a topic that may occur in a high school geometry course. The other actually is one instance of Dedekind completeness; formulas like $C = 2\pi r$ and $A = \pi r^2$ can be true only if π is in the coordinatizing field.

Methodology 10.6. A key question is whether it is the same π is both of the equations in Pedagogy 10.5. Archimedes argues they give the same ratio, which is not a number for him. In [Bal14] we outline arguments of [Apo67, Spi80] using calculus and [Wei20] clarifying Archimedes.

We return to the issue of making the notion of a complete theory rigorous. Given a collection of statements Φ about possible systems for geometry, there are several ways in which a subset Ψ of Φ can be thought complete for a collection of axioms T. Of course, each $\psi \in \Psi$ must be satisfied in each model of T. And the most natural notion of complete is is:

Definition 10.7. *[Categoricity and Completeness]*

1. *A consistent theory in a logic $\mathcal{L}$ is* negation complete *if for any $\mathcal{L}$-sentence ϕ, $T \vdash \phi$ or $T \vdash \neg\phi$.*
2. *A theory T is* categorical *if it has* exactly one model.

It may seem obvious that if a theory T is categorical then T is negation complete. However, Methodology 10.10 explains the truth of that claim depends on the choice of logic $\mathcal{L}$.

Methodology 10.8. *[First order completeness]* The first order theory T_{rcf} of the Cartesian plane over real numbers is negation complete; one adds to EG the infinitely many axioms that say of the coordinatizing field that every odd degree polynomial has a root [Tar59]. Alternatively, analogously to the Peano axioms for arithmetic, Dedekind cuts are formalized to hold only for first order definable cuts [TG99, p 185].

Independence of the parallel postulate shows the axioms HP for neutral geometry are not complete. Even the descriptively complete theory EG is far from negation complete. In fact, [Bee, Zie82] (first in English) proves that if T is finitely axiomatized geometry consistent with T_{rcf} there is no algorithm to decide which sentences are consistent with T.

[29] In first order logic, if every finite subset of a set Σ of first order sentences is satisfiable so is Σ.

Methodology 10.9. *[logicS]* Hilbert wrote [Hil62] in German, not in a formal language. So he had no precise way of expressing negation completeness. What makes a German sentence 'geometric'? Roughly 20 years after the publication of [Hil62], Hilbert developed his notion of formal logic. In his general formulation quantification is allowed over individuals (x), sets of individuals (X), sets of sets of individuals and so on (This corresponds to Russell-Whitehead's theory of types.) He later observed [HA38] that groups I-IV are what we now call first order (for him, the restricted predicate calculus): quantification is only over individuals and only finite conjunctions and disjunctions are allowed in combining statements. Now the key distinction arises from Gödel's completeness theorem: For first order logic, there is a system of inference rules so that θ can be derived from T if and only if θ is true for every model of T. So for first order logic, negation completeness implies the stronger *deductive negation completeness*: for $\phi \in \Phi$, either ϕ or $\neg\phi$ is provable from the axioms of T. But we explain in the next two paragraphs that this is impossible in 2nd order logic.

Methodology 10.10. *[Completeness, categoricity and the choice of logic]* Categoricity was confused with negation completeness (Notation 10.7) until the late 1920's. It seems obvious that categoricity implies completeness. If each ϕ and $\neg\phi$ are consistent with T they both hold in the unique model of T, which is clearly impossible. For first order theories, this argument is almost correct[30]. The difficulty is with the 'clearly impossible'. It is true for first order sentences since no change in the axioms of set theory will change the truth value of ϕ in M. But the truth of a second order sentence about the real field may depend on the set theory in which you work.

Methodology 10.11. *[Incompleteness of 2nd order geometry despite categoricity]* Write the statement in pure second order logic expressing the continuum hypothesis[31]. By the celebrated work of Gödel and Cohen, the continuum hypothesis is independent from the Zermelo-Frankel axiom for set theory (even with the axiom of choice). Thus Hilbert's axiomatization is not negation complete for 2nd order logic.

As we noted in Remark 3.7 Birkhoff's axioms are phrased as in set theory as a complicated description of the geometry over the real field ('Real field' is defined in set theory). With Hilbert's definition of the field, we can make this into a legitimate second order axiomatization of a theory that is categorical in any particular model M of ZFC. But the second order theory will depend not just on the given axioms but only what set theoretic statements are true in M (as in 10.11).

Theorem 10.12. *Fix two points* $0, 1$ *on a Hilbert plane* M *and the line* ℓ *through them. Let* $<, +, \times$ *be the ordering relation and field operations defined on* ℓ *by Theorem 7.14. Add the least upper bound axiom:*

$$(\forall X)(\exists y)(\forall x \in X)[x < y \land [(\forall w)(\forall x \in X)x < w) \rightarrow y \leq w]].$$

The field on ℓ *is a complete ordered field and so is isomorphic to the reals.*

Proof. So clearly the ruler postulate holds on ℓ. But we know by Theorem 5.3.18 that the group of rigid motions acts transitively on lines so the ruler postulate holds on every line and so on M. □

11 Non-Euclidean Geometry

We showed in Section 7, specifically Methodology 7.14, that the theories of Euclidean geometry and fields were bi-interpretable. The same is true of Euclidean and hyperbolic geometry. In particular, Poincaré showed

[30] One has to replace unique model with there is a cardinal κ with unique model M of cardinality κ. As the Löwenheim-Skolem theorem say that any first order sentence with an infinite model has a model of every infinite cardinality. Thus, if each of ϕ and $\neg\phi$ is consistent with T then they must both hold in M.

[31] Consider the sentence: $(\exists X)(\exists f) f$ is an injective function from X onto a proper subset of $X \land (\exists Y)(\exists g) g$ is an injective function from Y onto X

that one could interpret hyperbolic geometry in a disc on the Euclidean plane. A geometric argument analogous to that in Section 7 appears in [Har00, §39], showing that an ordered field is definable in hyperbolic geometry.

The switch from the old to the new view of geometry (Comment 1.2) stemmed from the proof of the independence of the parallel postulate. Most of the modern work on non-Euclidean geometry assumes the existence of a real-valued metric (distance function) and is *not* done synthetically. However, [Har00] elaborates on some axiomatic non-Euclidean geometry. In neutral geometry, he proves there is a rectangle if and only if the sum of the angles of a triangle is two right angles and introduces an axiomatic trichotomy of semi-Euclidean, semi-hyperbolic, and semi-elliptic geometries depending on the order between the sum of the angles of a triangle and two right angles. Further, he proves that a semi-hyperbolic plane satisfying Hilbert's 'limiting parallel axiom' (*hyberbolic geometry*) defines a field.

Methodology 11.1. *[Informal description of Interpretation]* It is easy to confuse two meanings of interpretation i) (somewhat archaic for logicians but used above for consistency with the SLO) as a witness to truth: 'a model of ϕ or T' is called 'an interpretation of ϕ or T' and ii) a relation between two (languages, theories, models). We mean the second.

Two theories are bi-interpretable if there are interpreting maps F, G from each to the other such that $F \circ G$ is the identity.

One way to prove the consistency of, say, hyperbolic geometry, is to interpret it in a Euclidean model; redefine the undefined terms of geometry (point, line, between, congruent, etc.) by formulas of Euclidean geometry and prove that *with this interpretation* the axioms of Hyperbolic geometry are satisfied in each model of Euclidean geometry. This yields that hyperbolic geometry is *relatively consistent* with Euclidean geometry. We give a full definition in the supplement –Definition 12.4. Nice introductions to interpretation for those familiar with modern algebra is in [BN94, §3: Interpretability] *and for the more logically oriented [Ena2x].*

We define some theories of geometries and indicate interpretability relations.

Definition 11.2. *[Limiting Parallels]*

1. *Two rays are* coterminal *if they eventually coincide.*
2. *A plane has* limiting parallels *if there are rays a through A and b through B that are either coterminal or they lie on distinct lines not equal to AB and every ray in the interior of the angle BAb meets the ray Bb. [Har00, p 312]*

Theorem 11.3.

1. *The theory of ordered fields is bi-interpretable with HP5 (neutral geometry + parallel postulate. (Hilbert coordinatization and analysis of the cartesian plane over an ordered field.)*

   Whence, the theory of ordered fields is interpretable in EG (neutral geometry + Euclid's 5th + E (circle-circle intersection)
2. *The theory of Euclidean fields is bi-interpretable with EG. (See e.g. [Har00, Mak19].*
3. *Call Hyperbolic geometry (HL) neutral geometry + limiting parallels). The theory of Euclidean ordered fields is interpretable in HL [Har00, §41].*
4. *Call semihyperbolic geometry (neutral geometry + sum of the angles of a triangle $< 2RA$.*

   Exercise [Har00, 39.24] shows there is a semihyperbolic plane which is not hyperbolic. It is unclear to me whether either semi-hyperbolic plane discussed in this exercise interprets a Euclidean ordered field.

5. *Clearly EG is not interpretable into HP5. If the coordinatizing field $\Phi(M)$ of a model of $HP5$ is not Euclidean (Some positive number doesn't have square root.), $\Pi(\Phi(M))$ is not Euclidean (There are two overlapping circles that don't intersect.).*

While he doesn't state it quite this way, [Har00, §40] proves

Theorem 11.4. *The theory of hyperbolic geometry with limiting parallels (HL) is bi-interpretable with EG.*

Rather surprisingly, since both hyberbolic geometry and Euclidean geometry are bi-interpretable with the real field, they are themselves bi-interpretable. That is, it is possible to define a model of each in any model of the other. This emphasizes that interpretation preserves not meaning but consistency.

12 Appendix: Formal Systems

This section is background for instructors who want more details on the logical notions that are sketched in the text. The aim is to give a precise notion of truth in a mathematical structure and give more a more precise account of the interpretation of theories for non-euclidean case, which are much more complicated than the examples given in the chapter. Two accessible sourcs are [BE02] and [LK15]. This material is in any introductory course in mathematical logic – and much more fully explained.

Definition 12.1. *A formal system of first order logic consists of*

1. *vocabulary*
   (a) *Logical vocabulary:* $(,), \wedge, \wedge, \neg, = \forall, \exists$ *variables* $v_1, v_2, \ldots$.
   (b) *non-logical vocabulary* τ[32]*: a list of relations, function, and constant symbols of prescribed arity.*
2. *Terms (expressions) are defined by induction.*
   (a) *A variable or a constant is a term.*
   (b) *If f is an n-ary function symbol and $t_1, \ldots t_n$ are terms then $f(t_1, \ldots t_n)$ is a term.*
3. *well-formed formulas (wff) are defined by induction.*
   (a) *Atomic formulas*
      i. *If t_1, t_2 are terms then $(t_1 = t_2)$ is an atomic formula.*
      ii. *If $t_1, \ldots t_n$ are terms and R is an n-ary relation symbol then $R(t_1, \ldots t_n)$ is an atomic formula.*
      For example, if $<$ is a binary relation symbol, 0 a constant and x a variable, $x < 0$ is an atomic formula.
   (b) *If ϕ and ψ are wffs*
      i. *$\neg\phi$ is a wff;*
      ii. *$(\phi \wedge \psi)$ is a wff;*
      iii. *$(\phi \wedge \psi)$ is a wff;*

[32] If there are no function symbols, the vocabulary is called *relational*; if there are no relations it is called *algebraic*.

iv. $(\exists v_i)\phi$ *and* $(\forall v_i)\phi$ *are wffs.*

4. *A* τ*-structure*[33] *is a set* A *and for each*
   (a) *constant symbol* $c \in \tau$, *an element* c^A *of* A;
   (b) $n - ary$ *relation symbol* $R \in \tau$ *a relation* $R^A \subset A^n$;
   (c) $n - ary$ *function symbol* $f \in \tau$ *a function* $f : A^n \to A$.
   For example, the rational field $(\mathbb{Q}, +, \times, 0, 1)$ *is* τ *structure for the vocabulary* $+, \times, 0, 1$.
5. *To define truth of* τ*-sentences in an* τ*-structure* A:
   (a) *Expand* τ *to* τ^* *by adding a constant symbol* c_a *for each* $a \in A$. *(That is,* $c_a^A = a$.*)*
   (b) *The denotation* t^A *of terms* t *is defined by induction.*
      i. *The denotation of a constant* c *is* c^A.
      ii. *The denotation of a term* $t = f(t_1, \ldots t_n)$ *is* $t^A = f^A(t_1^A, \ldots t_n^A)$.
   (c) *Now truth of a formula* $\phi(t_1, \ldots, t_n)$ *(where the* t_i *are* τ^**-terms) is defined by induction:*
      i. *If* ϕ *is* $t_1 = t_2$, $A \models \phi$ *if* $t_1^A = t_2^A$.
      ii. *If* ϕ *is* $R(t_1, \ldots t_n)$, $A \models \phi$ *if* $\langle t_1^A, \ldots t_n^A \rangle \in R^A$.
      iii. *If* $\phi(t_1, \ldots, t_n)$ *is* $\psi(t_1, \ldots, t_n) \wedge \chi(t_1, \ldots, t_n)$ *then* $A \models \phi(t_1, \ldots, t_n)$ *if* $A \models \psi(t_1, \ldots, t_n)$ *and* $A \models \chi(t_1, \ldots, t_n)$.
      iv. *If* $\phi(t_1, \ldots, t_n)$ *is* $\neg\psi(t_1, \ldots, t_n)$ *then* $A \models \phi(t_1, \ldots, t_n)$ *if it is not the case that* $A \models \phi(t_1, \ldots, t_n)$.
      v. *If* $\phi(t_1, \ldots, t_n)$ *is* $(\exists v_i)\psi(t_1, \ldots, t_n, v_i)$ *then* $A \models \phi(t_1, \ldots, t_n)$ *if for some* $a \in A$, $A \models \psi(t_1, \ldots, t_n, c_a)$.
      For example, the sentence $(\exists x)(x^2 = 1 + 1)$ *is false in the structure* $(\mathbb{Q}, +, \times, 0, 1)$ *and true in the structure* $\mathbb{C}, +, \times, 0, 1)$ *(where the QQ and* $\mathbb{C}$ *indicate we are to interpret as the rational and complex field respectively.*
   (d) *The sentence* ϕ *is valid if it is true in every structure. For every* M, $M \models \phi$.
   (e) *The sentence* ϕ *is a* logical consequence *of the sentence* ψ *if for every* M, *if* $M \models \psi$ *then* $M \models \phi$.

If a sentence ϕ *is true in a structure* M, *we say* M *is a* model *of* ϕ. *If* M *satisfies all axioms of a theory* T, M *is a* model *of* ϕ.

Theorem 12.2. *[Completeness and Compactness]*

Gödel's Completeness theorem *For any sentence of first order logic and any* T:

$$T \models \phi \leftrightarrow T \vdash \phi.$$

Compactness theorem *For any and constants* $\mathbf{a}$ $\phi_n(\mathbf{a})$ *and collection of sentences* $\phi_n(\mathbf{a})$.

If there is a model M *for each* $N < \omega$, *there is an* $\mathbf{a}_N$, *and* M_N *such that* $M \models \bigwedge_{n<N} \phi_n(\mathbf{a}_N)$ *then there is a model* M_ω *and tuple* $\mathbf{a}_\omega$ *such that* $M_\omega \models \bigwedge_{n<\omega} \phi_n(\mathbf{a}_N)$.

[33] A structure for a relational vocabulary is called a *relational structure*; a structure for an algebraic vocabulary is called an *algebra*.

Definition 12.3. *Proof system We now specify a proof system for first order logic. However, we not recommend proofs in such a formal system in a GeT course.*

The key point is that the arguments in Euclid generally follow a simple form. A configuration of a finite number points is given and one must show that there exist further points satisfying a further configuration. That is the theorem can be expressed by formula $(\forall x_1, \dots x_n)\theta(x_1, \dots x_n) \rightarrow \exists(y_1, \dots y_m)\psi(x_1, \dots x_n, y_1, \dots y_m)$. For more detail see [ADM09, §2.4] or [Mue06, p 11-14].

Logical Axioms

1. *Any sentence that is true in every τ structure (a tautology);*
2. *The equality axioms;*
3. *$(\forall x)\phi \rightarrow \phi^x_t$ (if t is substitutable for x in ϕ);*
4. *$(\forall x)(\phi \rightarrow \psi) \rightarrow [(\forall x)\phi \rightarrow (\forall x)\psi]$;*
5. *$\phi \rightarrow (\forall x)\phi(x)$ (if x not free in ϕ).*

Inference rule
(Modus Ponens): From ϕ and $\phi \rightarrow \psi$, infer ψ.

This is slightly altered version of the definition of interpretation in [Sho67]

Definition 12.4. *[Formal definition of interpretation]*

1. *We say that I is an interpretation of L in L', where L and L' are first-order languages, if I is a function such that*
   - i *there is a universe for the image of I, represented by a unary predicate symbol U_I (or formula) of L' ;*
   - ii *for each n-ary function symbol f of L , a corresponding symbol f_I of L';*
   - iii *for each n-ary predicate symbol P of L (with the exception of $=$, which is generally taken to be a logical symbol), a corresponding symbol P_I of L'.*
2. *Moreover, we say that I is an interpretation of L in a theory T' if I is an interpretation of L in the language L' of T' and also:*
   - i *$T' \vdash (\exists x)U_I(x)$ (it proves that the domain is non-empty);*
   - ii *for each $f \in L$, $T' \vdash (U_I(x_1) \wedge \dots \wedge U_I(x_n) \rightarrow U_I(f_I(I(x_1 \dots , x_n)))$ (it proves that the domain is closed under functions).*
3. *Now, if ϕ is a formula of L and I an interpretation of L in L', then we can define for ϕ its interpretation in L' , ϕ_I . We start by defining a formula ϕ_I of L' which is obtained by starting with ϕ and replacing each symbol of the original language by its interpretation in L' (e.g., if ϕ is $f(x) = y$, then we replace f by f_I to obtain $f_I(x) = y$), and then relativizing the existential quantifiers to the domain (i.e. replace every $(\exists x)\psi$ by $(\exists x)(U_I(x) \wedge \psi)$. As the last step, if $x_1 \dots , x_n$ are the free variables of ϕ , set ϕ_I to be $(U_I(x_1) \wedge \dots \wedge U_I(x_n) \rightarrow \phi_I$.*
4. *Finally, an interpretation of a theory T in a theory T' is an interpretation I of the language of T in T' such that $T' \vdash \phi_I$ for every nonlogical axiom (i.e. an L-sentence ϕ that is not universally valid that has been taken as axiom of T).*

[Interpretation of theories and structures] We have noticed that there are first order formulas defining the Cartesian plane over a field and, more surprisingly, conversely if the plane satisfies HP5. We say that the theory of fields and the theory of Hilbert planes satisfying the 5th postulate are *mutually interpretable*. As emphasized in [Mak19] a stronger connection is more usefel: the defining maps are inverses of each other; the theories are said to be *bi-interpretable* if the defining maps are inverses of each other. In particular, bi-interpretation preserves decidability while mutual interpretability may not. A basic exposition is at `https://gup.ub.gu.se/file/167690` and the original source [TMR68] is quite readable.

References

[ADM09] J. Avigad, Edward Dean, and John Mumma. A formal system for Euclid's elements. *Review of Symbolic Logic*, 2:700–768, 2009.

[Alp05] Roger C. Alperin. Trisections and totally real origami. *The American Mathematical Monthly*, 112:200–211, 2005.

[Apo67] T. Apostol. *Calculus*. Blaisdell, New York, 1967.

[Bal14] John T. Baldwin. First order justification of $C = 2\pi r$. 12 pages, submitted, `http://homepages.math.uic.edu/~jbaldwin/pub/pisub.pdf`, 2014.

[Bal18] John T. Baldwin. *Model Theory and the Philosophy of Mathematical Practice: Formalization without Foundationalism*. Cambridge University Press, 2018.

[BB59] George David Birkhoff and Ralph Beatley. *Basic Geometry*. 3rd ed. Chelsea Publishing Co., 1959. 1st edition 1941: Reprint: American Mathematical Society, 2000. ISBN 978-0-8218-2101-5; free online at `https://kupdf.net/download/birkhoff-amp-beatley-basic-geometry_58b4448e6454a79179b1e939_pdf`.

[BE02] J. Barwise and J. Etchemendy. *Language, proof, and logic*. University of Chicago Press, 2002.

[Bee] M. Beeson. Some undecidable field constructions. translation of [Zie82]`http://www.michaelbeeson.com/research/papers/Ziegler.pdf`.

[BH07] William Barker and Roger Howe. *Continuous Symmetry: From Euclid to Klein*. American Mathematical Society., 2007.

[Bir32] Birkhoff, George. A set of postulates for plane geometry. *Annals of Mathematics*, 33:329–343, 1932.

[BN94] A. Borovik and A. Nesin. *Groups of Finite Morley Rank*. Oxford University Press, 1994.

[Ced01] J. Cederberg. *A Course in Modern Geometries*. Springer, 2001.

[CK17] V Climenhaga and A Katok. *From Groups to Geometry and Back*. AMS, Providence, 2017.

[Cla12] David M. Clark. *Euclidean Geometry: A guided inquiry approach*. Mathematical Circles Library. Mathematical Sciences Research Institute, 2012.

[CS20] Clio Cresswell and Craig P. Speelman. Does mathematics training lead to better logical thinking and reasoning? a cross-sectional assessment from students to professors. *Plus One*, 2020. open access: https://doi.org/10.1371/journal.pone.0236153.

[Ded63] Richard Dedekind. *Essays on the Theory of Numbers*. Dover, Mineola, New York, 1888/1963. first published by Open Court publications 1901: first German edition 1888.

[Dem94] William Demopoulos. Frege, Hilbert, and the conceptual structure of model theory. *History and philosophy of logic*, 15:211–225, 1994.

[Det14] M. Detlefsen. Completeness and the ends of axiomatization. In J. Kennedy, editor, *Interpreting Gödel*, pages 59–77. Cambridge University Press, 2014.

[Edu09a] Educational Development Center: . *CME Algebra I*. Pearson, 2009. EDC: Educational Development Center.

[Edu09b] Educational Development Center: Benson, Cuoco, et al. *CME Geometry*. Pearson, 2009.

[Ena2x] A. Enayat. Interpretations and mathematical logic: A tutorial. 202x. accessed Dec. 2023:https://gup.ub.gu.se/file/167690.

[Euc56] Euclid. *Euclid's Elements*. Dover, New York, New York, 1956. In 3 volumes, translated by T.L. Heath; first edition 1908; online at http://aleph0.clarku.edu/~djoyce/java/elements/.

[GG09] I. Grattan-Guinness. Numbers, magnitudes, ratios, and proportions in Euclid's *elements*: How did he handle them? In *Routes of Learning*, pages 171–195. Johns Hopkins University Press, Baltimore, 2009. first appeared Historia Mathematica 1996.

[Gio21] E. Giovannini. David Hilbert and the foundations of the theory of plane area. *Archive for History of Exact Sciences*, 2021. https://doi.org/10.1007/s00407-021-00278-z.

[Gre93] M. Greenberg. *Euclidean and non-Euclidean geometries: development and History*. W. H. Freeman and Company, 1993.

[HA38] D. Hilbert and W. Ackermann. *Grundzüge der Theoretischen Logik, 2nd edn*. Springer, Berlin, 1938. first edition, 1928.

[Hal08] T. Hales. Formal proof. *Notices of the AMS*, 55:1370–1380, 2008.

[Har99] G. Harel. Students understanding of proofs: A historical analysis and implications for the teaching of geometry and linear algebra. *Linear Algebra and its applications*, 302-303:601–613, 1999.

[Har00] Robin Hartshorne. *Geometry: Euclid and Beyond*. Springer-Verlag, 2000.

[Har11] Robin Hartshorne. Review of Barker and Howe. *The American Mathematical Monthly*, pages 565–568, 2011. https://www.jstor.org/stable/10.4169/amer.math.monthly.118.06.565.

[Har13] Harel, G. Intellectual need. In K. Leatham, editor, *Vital Direction for Mathematics Education Research*, pages 119–151. Springer, 2013.

[Har14] G. Harel. Common core state standards for geometry: An alternative approach. *Notices of the AMS*, pages 24–35, 2014.

[Har15] Matthew Harvey. *Geometry Illuminated; An Illustrated Introduction to Euclidean and Hyperbolic Plane Geometry*. Mathematical Association of America, 2015.

[Hen02] D. Henderson. Geometry: Euclid and Beyond by Robin Hartshorne. *Bulletin of the A.M.S.*, 39:563–571, 2002. https://math.berkeley.edu/~kpmann/henderson%20review.pdf.

[Hil62] David Hilbert. *Foundations of geometry*. Open Court Publishers, LaSalle, Illinois, 1962. original German publication 1899: reprinted with additions in E.J. Townsend translation (with additions) 1902: Gutenberg e-book #17384 http://www.gutenberg.org/ebooks/17384.

[Hil71] David Hilbert. *Foundations of Geometry*. Open Court Publishers, 1971. translation from 10th German edition by Harry Gosheen, edited by Bernays 1968.

[HT05] D. Henderson and D. Taimina. How to use history to clarify common confusions in geometry. In A. Shell-Gellasch and D. Jardine, editors, *From Calculus to Computers*, volume 68 of *MAA Math Notes*, pages 57–74. Mathematical Association of America, 2005.

[HT20] D. Henderson and D. Taimina. *Experiencing Geometry: Euclidean and Non-Euclidean with history*. Pearson, 2020. https://projecteuclid.org/ebooks/books-by-independent-authors/Experiencing-Geometry/toc/10.3792/euclid/9781429799850.

[IA17] M. Inglis and N. Attridge. *Does Mathematical Study Develop Logical Thinking*. World Scientific, 2017.

[Ill19] Illustrative Math. Illustrative mathematics: Geometry. https://im.kendallhunt.com/HS/index.html, 2019. Accessed: 2023-09-27.

[Kin21] J. King. *Geometry Transformed: Euclidean Plane Geometry Based on Rigid Motions*. Pure and Applied Undergraduate Texts. American Mathematical Society, 2021.

[Lib08] S. Libeskind. *Euclidean and Transformation Geometry: A Deductive Inquiry*. Jones and Bartlett, 2008.

[LK15] C. Leary and L Kristiansen. *A Friendly Introduction to Mathematical Logic*. Milne Library Publishers, 2015. Geneseo Authors. 6. https://knightscholar.geneseo.edu/geneseo-authors/6.

[Lyn67] Roger Lyndon. *Notes on Logic*. Van Nostrand, New York, 1967. available on Abe books for less than $5.

[Mak19] J. A. Makowsky. Can one design a geometry engine? On the (un)decidability of affine Euclidean geometries. *Annals of Mathematics and Artificial Intelligence*, 85:259–291, 2019. online: https://doi.org/10.1007/s10472-018-9610-1.

[Man08] K. Manders. Diagram-based geometric practice. In P. Mancosu, editor, *The Philosophy of Mathematical Practice*, pages 65–79. Oxford University Press, 2008.

[Mar82] George E. Martin. *Transformation geometry, an introduction to symmetry*. Springer-Verlag, 1982.

[Mil07] N. Miller. *Euclid and his Twentieth Century Rivals: Diagrams in the Logic of Euclidean Geometry*. CSLI Publications, 2007.

[Moi90] Edwin Moise. *Elementary Geometry from an Advanced Standpoint*. Addison-Wesley, 1990. 3rd edition.

[Mue06] Ian Mueller. *Philosophy of Mathematics and Deductive Structure in Euclid's Elements*. Dover Books in Mathematics. Dover Publications, Mineola N.Y., 2006. First published by MIT press in 1981.

[PSS07] Victor Pambuccian, Horst Struve, and Rolf Struve. Metric geometries in an axiomatic perspective. In L. et al. Ji, editor, *From Riemann to Differential Geometry and Relativity*, volume 48, pages 399–409. Springer International Publishing AG, 2007.

[Rai05] Ralph Raimi. Ignorance and innocence in the teaching of mathematics. `https://homepages.math.uic.edu/~jbaldwin/pub/Raimi.pdf`, 2005.

[SBM19] A.J. Stylianides, K.N. Bieda, and F. Morselli. Proof and argumentation in mathematics education research. In Á Gutiérrez, G Leder, and P. Boero, editors, *The Second Handbook of Research on the Psychology of Mathematics Education: The Journey Continues*, pages 315–351. SensePublishers Rotterdam, Rotterdam, 2019.

[Ser93] Michael Serra. *Discovering Geometry: An inductive Approach*. Key Curriculum Press, 1993.

[Sho67] Joseph Shoenfield. *Mathematical Logic*. Addison-Wesley, 1967.

[Smo08] C. Smorynski. *History of Mathematics: A supplement*. Springer-Verlag, 2008.

[SMS95] SMSG. The SMSG Postulates for Euclidean Geometry, 1995. Search for: Geometry/The SMSG Postulates for Euclidean Geometry, e.g. `https://faculty.winthrop.edu/pullanof/MATH{%}20393/The{%}20SMSG{%}20Postulates.pdf`.

[Spi80] M. Spivak. *Calculus*. Publish or Perish Press, Houston, TX, 1980.

[Szm78] W. Szmielew. *From Affine to Euclidean Geometry: An Axiomatic Approach*. D. Reidel, Dordrecht, 1978. edited by Moszyńska, M.

[Tar59] A. Tarski. What is elementary geometry? In Henkin, Suppes, and Tarski, editors, *Symposium on the Axiomatic method*, pages 16–29. North Holland Publishing Co., Amsterdam, 1959.

[TG99] A. Tarski and S. Givant. Tarski's system of geometry. *Bulletin of Symbolic Logic*, 5:175–214, 1999.

[TMR68] Alfred Tarski, Andrezej Mostowski, and Raphael Robinson. *Undecidable Theories*. North Holland, 1968. First edition: Oxford 1953.

[Vää14] Jouko Väänänen. Sort logic and foundations of mathematics. *Lecture Note Series, IMS, NUS*, pages 171–186, 2014.

[Wei97] A.I. Weinzweig. Geometry through transformations. privately published, 1997.

[Wei20] D. Weisbart. Modernizing Archimedes' construction of π. Σ *Mathematics*, page 23 pages, 2020. online open access: doi:10.3390/math8122204.

[Wu94] Wen-Tsü Wu. *Mechanical Theorem Proving in Geometry*. Texts and Monographs in Symbolic Computation. Springer-Verlag, New York, 1994. Chinese original 1984.

[Zie82] M. Ziegler. Einige unentscheidbare Körpertheorien. *Enseignement Mathmetique*, 28:269–280, 1982. Michael Beeson has an English translation.