SUPPLEMENT: SLO 4: Understand the relationships between axioms, theorems, and different geometric models in which they hold.

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Contents

Abstract

In this chapter a logician and a teacher a) provide historical and logical background on the concepts of proof and models that reorganizes a 'short geometry' written for a workshop that focused on motivating and exploring those topics. We emphasize the global goals of Euclid and Hilbert that are often missed in high school geometry.

1 Introduction

{intro}

To access activities mentioned in this supplement go to https://homepages.math.uic.edu/~jbaldwin/CTTIgeometry/ctti and click on the link for the named activity.

This chapter is aimed primarily at instructors of college courses in geometry for teachers. We will discuss the roles of the college instructor, college students, who are future high school teachers and high school students. We discuss the role of axioms in mathematics and then use a slightly non-standard set of axioms to show how an easily constructed algorithm for splitting a line into equal pieces requires all the Euclidean axioms.

We build on the superb narrative for SLO4 and assume it below. Later in the paper we label various remarks with other learning outcomes. We begin by elaborating on certain points of the narrative and indicating some minor changes in terminology.

We stress that axioms are intended to organize the study of an area of mathematics by identifying the fundamental assumptions needed to establish the results in that area and that various choices of fundamental notions and axioms can provide different explanations. Thus, rather than 'undefined terms', we speak of a 'vocabulary τ' and call the collection of possible 'interpretations' the τ-structures. As in the narrative, a model is a structure where the axioms hold. This emphasizes that without the axioms, there may be no connection of the interpretation with the intuition the axioms are trying to catch.

We agree with the advice in the narrative that the college instructor should scale up from the earlier levels of the Van Hiele hierarchy[1]. We focus here on the development for college students of levels 3 and 4.

Level 4 Rigor: At this level students understand the way how mathematical systems are established. They are able to use all types of proofs. They comprehend Euclidean and non-Euclidean geometry. They are able to describe the effect of adding or removing an axiom on a given geometric system.

This indicates a difference in expected Van Hiele level between high school and college students. The common core demands level 3 but not level 4. As noted in SLO2 there may be students in the college course who have not fully attained level 3, while there are a number of high school students that operate comfortable at level 4 and some who appreciate non-Eucliean geometries.

The narrative defines a theorem as 'a statement than can be proved from the axioms without regard to interpretation'. We prefer 'can be deduced from the axioms by the rules of logic'. The equivalence of these two statements is precisely Gödel's completeness theorem for first order logic[2]. We will examine such rules in XXXX/. Understanding this distinction is perhaps stronger than Van Hiele level 4.

▌ commented out original description of sections

{whyax}

Motivation 1.1 (SL0 1, 3: Why axiomatics?)**.** A fundamental goal of K-12 education is to inculcate the ability to make and understand rational arguments. For over 2000 years Euclid's Elements performed this

[1]Following the easily easy accessible https://physics.mff.cuni.cz/wds/proc/pdf12/WDS12_112_m8_Vojkuvkova.pdf the five levels are: 0 Visualization, 1 Analysis, 2 Abstraction, 3 Deduction, 4 Rigor

[2]The equivalence fails for Birkhoff's system.

task more than any other single source. One of the standard goals for U.S. high school geometry is Common Core Standard 3 for mathematical practice: **Construct viable arguments and critique the reasoning of others**. A successful argument requires a clear statement of subject matter. The notion that reasoning skills learned in geometry transfer to e.g. political discourse raises many distinct questions. However, [IA17, CS20] find studying mathematics develops general thinking skill'. Our task here is not to defend that proposition. Rather, given that it is embedded in mathematics standards, the goal here is to provide a model of reasoning in a mathematical context which is accessible to high school students – geometry is everywhere. Moreover, via Euclid et. al., geometry is precise.

We contrast three modes of persuasion: *argument:* reasoned persuasion in any subject: mathematics, law, politics, movies *informal proof:* a typical argument in mathematics, the rules of inference are implicit and the global assumptions unstated although nominally reducible to formal set theory (e.g. Zermelo-Frankel with the axiom of choice), and *formal proof*: in a logic with strict rules for construction of sentences and deductions. This chapter concerns informal proof but clarifies the relation with formal proof.

{axsys}

Methodology 1.2 (Axiom Systems)**.** The introduction to [Hil62] heralds a new age in the foundations of mathematics.

> The following investigation is a new attempt to choose for geometry a simple and complete set of independent axioms and to deduce from these the most important geometrical theorems in such a manner as to bring out as clearly as possible the significance of the different groups of axioms and the scope of the conclusions to be derived from the individual axioms.

The aim is to determine fundamental, 'simple and complete', reasons for 'important geometrical theorem'. Hilbert's axioms have not entered the high school curriculum because of the complexity of their use. This complexity arises from the difficult construction of the linear ordering of a line from the abstract betweenness axioms and the tedious process of transcribing such important notions as circle (Hilbert punted) into his choice of basic concepts. By merging Hilbert's framework with Euclid's we present a more accessible approach which gives Hilbert's answer to *What is geometry?*'

Old View: Until the 19th century it was thought that geometry demonstrated truths from *unassailable premises*. These premises were Euclid's Axioms (common notions) and Postulates (geometric assumptions).

New View: Geometry deduces conclusion from a specific set of geometric hypotheses. These hypotheses might be Euclidean, spherical, hyperbolic, etc. Whether these geometrical hypotheses are "true" is *not* a mathematical question. As the epigram of [HT20] puts it:

> Geometry doesn't contain the truth about how space is. Geometry is how you view space. Take charge of it – it's yours. Understand how you see things and how you imagine things. Geometry can say something about you and your universe. – David W. Henderson

But this new view leaves open the issue of how we are to understand these 'not known to be true' geometric hypotheses. What are the fundamental notions? What is true about them? What do they imply?

{axcrit}

Methodology 1.3 (SLO1: Criteria for Choosing Axioms)**.** Natural criteria include that axioms should be intuitive and parsimonious. By intuitive, we mean the axioms can be easily illustrated for the students involved. An axiom system is independent if no axiom can be deduced from the others. Parsimony can be violated in two ways: including an independent axiom which is not needed for the intended collection of results; failing to be independent. Independence may not be evident; it took two thousand years to show the parallel postulate independent. Mathematicians were convinced that the parallel postulate was not self-evident but should be provable.

A third natural criteria is that the axioms should be, as Hilbert said in 1.3, complete. But completeness turns out to be a rather complex notion that we will explore REMARK XXX. For now we will say 'an axiom system is complete completeness is if it implies all the propositions it was designed to axiomatize. to axiomatize.

2 Interpretations, Models, and Axioms

{model}

Section 2 Nov. 28.

{synana}

Pedagogy 2.1. *[SLO2: Synthetic and Analytic proof]* Narrative SLO2 prescribes 'understanding different type of proof such as synthetic (from axioms), analytic (using coordinates), and proofs using transformations or symmetries.' This distinction between synthetic and analytic illustrates the difference between proofs from axioms in the language of geometry and proof about interpretations. A *synthentic* proof is a sequence of statements that form a deduction. We may or may not translate these statements into symbolic form. An *analytic* proof is an algebraic proof about the coordinatized plane, which almost always uses symbols. As such, it is a proof *about* an interpretation of the axioms.

We now show coordinate geometry illustrates the concept of interpretation and then formally define the concept. Here are some basic mathematical structures that should be known, but perhaps not so precisely. We use both symbols and words to describe the actual structure that interprets certain mathematical terms. A structure (e.g. 'the rationals') is a set with several kinds of basic terms: specified constants, operations, and relations. The ordered field of rational numbers $\langle \mathbb{Q}, +, \times, -, ^{-1}, 0, 1, =, < \rangle$ consists of the fractions with the specified constants, operations, and relations listed. The word field indicates that both addition and multiplication are groups (satisfy associativity, commutativity with identities $0, 1$ and inverses) and that multiplication distributes over multiplication. 'Ordered' prescribes a linear order relation. The real numbers satisfy the same properties but also satisfy the least upper bound principle. One point of these notes is the least upper bound principle is irrelevant to high school geometry; it will not appear again.

We have given a particular interpretation of the vocabulary of fields (addition, multiplication, additive and multiplicative inverse $0, 1$, equal, less than) in symbols $\langle \mathbb{Q}, +, \times, -, ^{-1}, 0, 1, =, < \rangle$ on a particular set, the rational numbers $\mathbb{Q}$. $(\langle \mathbb{Q}, +, \times, -, ^{-1}, 0, 1, =, < \rangle$ is a *model* of the theory of fields. We say this interpretation is a model of the theory of fields since all the field axioms are satisfied[3].

{moddef}

Definition 2.2. *An* interpretation *of a vocabulary (the basic terms) consists of a set (called e.g. world, domain, universe) and a meaning for each basic term on that domain.*

An interpretation is a model *of a set of axioms if each axiom is satisfied in the interpretation.*

The basic terms of an (incidence) geometry are points, lines and a binary relation between points and lines, 'A lies on ℓ'. We introduce a symbolic vocabulary $\langle P, L, I \rangle$. We need an unfamiliar symbol I because unlike fields, where we routinely work in the model, synthetic proof can be done in English with symbols only naming particular points and lines. The interpretation of the statement, 'the point A is on the line ℓ is $I(A, \ell)$.

The 'coordinate plane' over F is an interpretation for the incidence geometry language for any field F. By the coordinate plane $\pi(F)$ over a field F we mean the interpretation $\langle P, L, I \rangle$ with points the ordered pairs in $F \times F$ and the geometry whose lines are the solutions of linear equations over F. That is, $A = (u, v)$

[3]Since addition does not distribute over multiplication, if we had perversely interpreted addition as $\times$ and multiplication as $+$, we would still have an interpretation; but not a model.

is on the line ℓ (determined by) $y = mx + b$ if $v = mu + b$. We say $\pi(F)$ satisfies the statement 'A lies on ℓ' or formally $I(A, \ell)$.

We now give a very different interpretation of the language of incidence geometries.

{alllinesparalle

Exercise 2.3. *For a different interpretation of $\langle P, L, I\rangle$; keep P and L the same but change incidence I to I' and say $I'(A, \ell)$ holds of A and the line ℓ (now determined by a single field element a) if $A = (u, v)$ for any v if $u = a$. Draw a picture of the lines in this plane.*

For some more interesting examples, we consider axioms for projective planes since they are much simpler than those for Euclidean geometry. For obstructions to students understanding proofs and in particular to understanding the next exercises on projective planes see the distinction between the intuitive axiomatic (Greek) and structural conception (Hilbert) in [Har99].

{projgeom}

Definition 2.4 (Projective Plane)**.** *An incidence geometry is a projective planes if it satisfies the axioms: (P1) Any two distinct points lie on a unique line. (P2) Any two distinct lines meet in a unique point. (P3) There exist at least four points of which no three are collinear (on the same line).*

{projpl}

Exercise 2.5.

1. **Fano Plane** *Draw a picture of the projective plane with* 3 *points on each line. (Hint: it has 7 points and 7 lines.)*

2. *Prove that in a projective plane there are four lines such that no three of them intersect in a common point.*

3. *Suppose (P, L, I) is a projective plane and there are n points on a given line ℓ. Prove each line has n points and there are $n^2 - n + 1$ points in the plane*[4]*.*

Items 1) and 2) have very different nature; the first is a *theorem of projective geometry*; it is expressed in the vocabulary of geometry. The second is *theorem about projective geometry*. There are no numbers in projective geometry; the result describes the models of projective geometry using concepts it cannot express.

3 Common Notions and Postulates

We now discuss Euclid's distinction between general and geometric premises and the 19th century quest for an autonomous basis for geometry.

Remark 3.1 (Common notions vs postulates)**.** Euclid's distinction between principles that are true everywhere in mathematics and those that are true only of a particular topic remains important today. But it is answered in a different way. Euclid was interested only in geometry and natural number (positive integers) arithmetic. His common notions essentially describe the properties of equality and order (among classes of 'comparable objects', i.e. magnitudes of various sorts). Length and area are incomparable for Euclid. In modern mathematics (almost) all topics can be studied on a common basis in set theory.

Nineteenth century geometers insisted that applicability of the common notions be explicitly based within geometry [Gio21]. Postulates describe the relation of the fundamental concepts of a particular subject. The best example for over 2000 years were Euclid's axioms for geometry. Thus the geometrical consequences of the common notions must be derived from the postulates; this required some additions.

[4]The first author took a course in projective geometry while in college. His future wife, who had no college mathematics solved this problem.

These are the common notions of Euclid. They apply equally well to geometry or numbers. Following modern usage we call Euclid's postulates either 'axiom' and 'postulate'.

Common notion 1. Things which equal the same thing also equal one another.
Common notion 2. If equals are added to equals, then the wholes are equal.
Common notion 3. If equals are subtracted from equals, then the remainders are equal.
Common notion 4. Things which coincide with one another equal one another.
Common notion 5. The whole is greater than the part.

{cn1}

Methodology 3.2 (SLO5,7: Common Notion 1)**.** Euclid used 'equal' in a number of ways: to describe congruence of segments and figures, to describe that figures have the same measure (length, area, volume). The only *numbers* for Euclid are the positive integers. He did however discuss the comparison of what we now interpret as lengths. Following Hilbert we build an 'algebra of segments' (a semi-field) and explain how to consider the segments as 'numbers' in Section 7.

CN1 asserts that equality is transitive. For various notions (e.g. congruence) we may need to make this property (as well as symmetry) an explicit axiom.

{cn4}

Methodology 3.3 (SLO5,7: Common Notion 4)**.** What Euclid means by coincide and equal is unclear ([Euc56, p 224, 248]). We adopt the view that X coincides with Y means 'one is mapped to the other by a rigid motion'; we follow the usual interpretation is that 'equal' in this context means congruent.

So Euclid CN4 asserts any figure is congruent with itself. That is one ingredient of Hilbert's congruence axioms. The other more contentious properties, symmetry and transitivity of congruence are discussed in Axiom 5.4.1.

{fundnot}

Methodology 3.4 (SLO1,5,8 **Definition**)**.** Euclid begins with a list of *definitions*. Some ('A line is breadthless length') are really just an *indicative definition*; it points to an intuition. These indicative definitions become the basic terms (vocabulary of Definition 2.2. Others (When a straight line standing on a straight line makes the adjacent angles equal to one another, each of the equal angles is **right**) are *stipulative definitions*. They precisely describe a new concept in terms of previous definitions. The geometric definitions in this chapter are stipulative.

Euclid and Hilbert chose point, line, incidence (a point is on a line), plane, and congruence as the most basic concepts. They regard triangles and other polygons as built from points and straight lines and facts about them follow from the axioms.

For Euclid, words in the proof refer to ideal geometric objects. But Hilbert's attitude is different. These basic concepts are named by words in the vocabulary. For him, the meaning of those words is given implicitly by the axioms [Dem94]. Blumenthal reported, 'One must be able to say at all times–instead of points, straight lines, and planes – tables, chairs, and beer mugs'.

Before giving axioms we clarify some of the stipulative definitions in Euclid.

Activity 3.5. SLO5, CC Standard G-C0 1. Know precise definitions[5] of angle, circle, perpendicular line, parallel line, and line segment, based on the undefined notions of point, line, distance along a line, and distance around a circular arc.

Why is distance along a circular arc given as an undefined notion? Can we define the length (congruence) of a circular arc in terms of the length (congruence of line segments)? Remember SSS. Why is the length of the chord a less good measure than the length of the arc?

[5] Activity G-C01: definition.pdf

As noted [Har00, p 114], congruence of arcs can be defined by rigid motions. But in general the length of an arc may not be the length of a straight line segment in a particular interpretation. E.g. in the plane over the real algebraic numbers which does not contain π. The circle of radius 1 about the origin is the solutions of $x^2 + y^2 = 1$. The length of the semicircle is π which is not in the interpretation.

Definition 3.6. *A metric on a set X is function d from $X \times X$ into an ordered group (field for us) such that $d(x, x) = 0$, $x \neq y \to d(x, y) > 0$, $d(x, y) = d(y, x)$ and triangle inequality $d(x, z) \leq d(x, y) + d(y, z)$ (a straight line is the shortest distance between two points).*

{congdist}

Methodology 3.7 (SLO5: Congruence vs Distance)**.** A fundamental distinction between Hilbert and Birkhoff is that Hilbert takes the congruence relation as fundamental and proves that one can define a metric (with values in a field). Birkhoff (and SMSG [Ced01, Appendix]) assume the existence of a real valued metric. Indeed, often the term metric space is only used for taking values in the reals. Birkhoff's position is natural for a mathematician who is also studying the analytic properties of the non-Euclidean geometries.

The difficulty for a high school course is that limits, which *Hilbert has shown are irrelevant to geometry*, are used implicitly and basic observations are replaced by long proofs. E.g. Common notion 3 (subtraction of line segments) is sometimes 'reduced' to using the ruler postulate twice and assuming the student knows the laws of algebra well. In this chapter we take congruence of line segments or angles as fundamental, not some measure.

4 SL02, SL08: A guiding problem

{guidepb}{whyprob}

Pedagogy 4.1. *[SL02, SL08: Role of this section]* We began our workshop with this exercise, first used with future middle school teachers, to emphasize the importance of ruler (straight-edge) and compass constructions in basic geometry and with the hope that the questions in the activity will provoke a need for the proof in Sections 5-8. While a solution using analytic geometry is fairly straight forward, the process of creating a purely geometric proof gives a deep insight into '(a) recognize and communicate the distinction between axioms, definitions, and theorems, and describe how mathematical theories arise from them, (b) construct logical arguments within the constraints of an axiomatic system' (SLO 4).

Exercise 4.2. *Each group chooses an odd number n between 2 and 10. After the number is chosen, the group will be asked to fold a string to divide it into as many equal pieces as the number they chose. Other physical models will be used. Activity - Divide a line into n equal pieces.*

{linediv}

Exercise 4.3. SLO8: CCSS G-C0-12 *Here is a procedure to divide a line into n equal segments.*

1. *Given a line segment AC.*
2. *Draw a line through A different from AC and lay off sequentially n equal segments on that line, with end points $A, A_1, A_2, \ldots$. Call the last point D.*
3. *Construct B on the opposite side of AC from D so that $AB \cong CD$ and $CB \cong AD$.*
4. *Starting at B, lay off n equal segments of length AA_1 and call the points so constructed on BC sequentially $B, B_1, B_2, ..., B_n - 1, C$.*
5. *Draw lines A_iB_i.*
6. *The points C_i where C_i is the intersection of A_iB_i with AC are the required points dividing AC into n equal segments.*

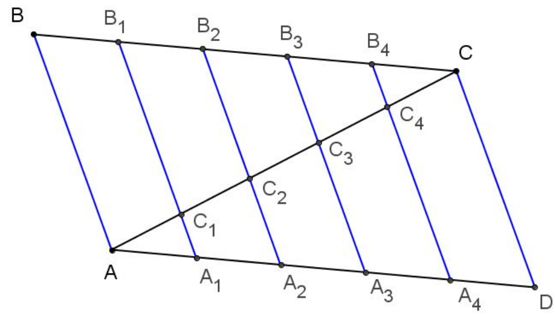


Figure 1: Dividing the line {divdia}

Exercise 4.4.

1. *Use the algorithm described above to divide an arbitrary line segment into 5 equal segments. (could be done in pairs. One person draws the line; the two have to divide it up.)*

2. *Show this construction used only Euclid's first 3 axioms, listed in Axiom 5.1.1 and 5.1.4 below.*

Pedagogy 4.5 (SLO2: Why is this assignment made?)**.** We are really asking, how and why does this construction work? Working in our system we see Euclid's first three postulates suffice to make the construction. See Exercise 5.1.6. But we will need SAS and more to prove it works! We start with this exercise both to give the student a reason to prove and to emphasize this distinction between rule based construction of geometric objects and a deductive verification of geometric propositions.

5 Book I: Propositions 1-34

{bk1}

The construction in the guiding problem Exercise 4.3 is rather straight forward using only Euclid's first three axioms; the proof that the construction works involves much more. To prepare for this argument, we amalgamate the approaches of Euclid and Hilbert, trying to maximize both understanding and rigor. The material adapts some of the results from the first 34 proposition of Book I of Euclid to solving our guiding problem.

Axioms arise from attempts to organize a body of results or describe rigorously some mathematical concept such as the plane. Different axiomatizations arise from different issues. We describe three challenges.

Motivation 5.0.1 (SL02, 7: Euclid's Challenge)**.** *Euclid aimed to provide a unified foundation for earlier geometry, specifically the side-splitter theorem of Thales (Theorem 8.5) and the Pythagorean theorem. The obstacle is incommeasurability in each case. However, using a theory of area, based on what is now called* equidecomposablity *(Section 9), Euclid establishes Pythagoras as the culmination of Book I. By appealing to the Axiom of Archimedes, he establishes a theory of proportion that yields side-splitter in book VI.*

{SLO5, 7: diagr

Pedagogy 5.0.2. *[Reading a diagram]* There was a tradition that carried on from pre-Euclidean time until late in the 19th century that the diagram carried certain information that was part of the proof.

What diagrams meant classically.

Inexact properties can be read off from the diagram: slightly moving the elements of the diagram does not alter the property. Intersections, betweenness and side of a line, inclusion of segments are inexact.

What classical diagrams don't mean

Anything about distance, congruence, size of angle (right angle!) may be deceptive. Since incidence is exact, you can't read off that a point is on a line but you can read off that two lines intersect in a point and then name that point and use the fact that it is on each line.

What high school diagrams mean Classical diagrams are enhanced in modern texts. Besides the inferences allowed above, SAT instructions say "All figures in this test are drawn to scale unless otherwise indicated". E.g. "Figure not drawn to scale". Students are taught tick marks for congruent segments, angle marks for congruent angles, right angle marks, parallel marks. Figures on one side of a line are assumed to be in that half-plane. Points that appear on a line(s) can be assumed to be on that (those) line(s).

{diagdetail}

Extension 5.0.3. *[Diagrams and proof]* Late 19th century mathematicians banished the diagram from (informal) mathematics. The SLO4 narrative defines a theorem as 'a statement than can be proved from the axioms without regard to interpretation'. While correct in spirit, it misses an essential point; how is 'without regard to interpretation guaranteed'. The answer is: clear requirements on what statements are and rules for deducing one from others while preserving truth. Without passing to the technicalities of formal systems, one give a useable specification. First fix a vocabulary as in Methodology 3.4. That is, 'atomic formula' such $B(A, B, C)$, written in English as 'B is between A and C or $I(A, \ell)$ A is on the line ℓ. Combinations of such atoms or their negations by using 'and' and 'or' are *basic statements* Δ that express diagrams. The construction postulates below have the form 'Every set of elements and lines satisfying Δ_1 can be extended a set satisfying Δ_2'. Theorem are even easier; they have the form 'Every set of elements and lines satisfy Δ'. Now the allowed rules of inference are a) manipulate *basic statements* by propositional logic [6] (thinking of each atomic statement, e.g. $B(A, B, C)$ as a proposition) b) apply construction postulates by naming an instance[7] c) from a proof of $\Delta(A_1, A_n, \ell_1, \ell_m)$ deduce Δ is true of any n points and m-lines.

Extension 5.0.4 (The fly in the ointment)**.** In more complicated arguments (unlikely to appear in high school), the location of the witness for a construction postulate in the existing diagram force a different proof[8]

Recent research clarifies and formalizes the ways that diagrams played an essential role in mathematical proof for 2000 years. [Man08] lays out the main issues and historical background. [ADM09] and [Mil07] provide formal systems with the diagram explicit. [ADM09] show their diagram-based system is complete for a set of sentences that include the results of Euclid. See [Bal18, §9] for a summary.

Pedagogy 5.0.5. *We don't have space in this chapter to describe the rules of inference of propositional logic. An excellent reference for grasping these connections is [BE02], which has very helpful software (Tarski's world) to explore the connections between syntax and syntax. We discuss the importance of the equivalence of an implication with its contrapositive in Definition 5.5.5 thru Pedagogy 5.5.7. Understanding this equivalence and fact that such an equivalence fails for an implication and its* converse *is very important; spelling out the connection with inverse is a matter for logicians (of the last century).*

Motivation 5.0.6 (Hilbert's Challenge)**.** 19th century mathematicians Cantor, Dedekind, and Frege revolutionized the foundations of mathematics by making the natural numbers rather than Euclidean geometry

[6] The college students may need supplemental work on propositional logic; it is not proposed the understanding in this comment is needed in high school; it is at the high end of Van Hiele 4.

[7] E.G. There is a line through A and B becomes 'Choose ℓ through A and B).

[8] See the 'proof' that all triangles are isosceles [Gre93, p 48-50] and many explanations on the net.

fundamental. Hilbert aimed for an independent development of geometry. In doing this, he had to meet the new higher standards of rigor. He needed to develop notions of distance and proportion from geometric notions of point, line, between and congruence.

Pedagogy 5.0.7 (Our Challenge)**.** *A prime objection to Hilbert's axiom is that their abstract nature is too hard to grasp for high school. Our aim is to amalgamate the axioms of Hilbert and Euclid to provide a more accessible account. For simplicity and succinctness,* we axiomatize only plane geometry.

5.1 Construction Postulates

{conspost}

Our vocabulary contains unary predicates P , **L**, binary I and ternary B , standing for point, line, incidence and between. We will introduce further vocabulary such as predicates for congruence later. We begin with Euclid's first three postulates. Both Axiom I and II are implied by the betweenness axioms we did not list in detail.

{axI-III}

Axiom 5.1.1 (Euclid's first 3 axioms in modern language)**.**

- **Axiom I** *Given any two points there is a line segment connecting them.*
- **Axiom II** *Any line segment*[9] *can be extended indefinitely (in either direction).*

  Here is a translation of Euclid's Postulate II from a rule *for a construction into a Hilbertian* assertion *that for any witness to Euclid's 'given', there are further witnesses for his conclusion.*

  For any point A and B and any C with B between A and C, there is a D such that C is between A and D.
- **Axiom III** *Given a point and any segment there is a circle with that point as center whose radius is the same length as the segment.*

Hilbert's first three axioms assert that two points determine a line and there are three non-collinear points. They follow from Euclid's first three, (Axiom 5.1.1).

{circ}

Remark 5.1.2. Circles Euclid chooses a fundamental notion that does not appear in Hilbert: This leads to an anomaly in any Hilbert or Birkhoff based high school text; there are no postulates about circles. Hence, we include I.3 which replaces [Hil71, Axiom III,1]. In addition to grounding the work students will do with circles, Axiom III is a much more tangible way to transfer distance than Hilbert's. [Har00, p 102-3] describes three Hilbert's of tools which somewhat awkwardly allow one to duplicate the constructions.

Fine historical point. Euclid does not explicitly mention that overlapping pairs of circles and circles overlapping a line actually intersect and Hilbert never mentions circles. The additions below to Axiom 3 make the assumption precise. In thinking about Exercise 5.1.3 consider why Euclid's notion of diagrams might have caused him to think no further Postulate was necessary to prove Proposition I.1.

{EI1}

Exercise 5.1.3. CCSS G-CO.13 *Prove Proposition I.1 of Euclid: To construct an equilateral triangle on a given finite straight line.*

Following [Har00] we label this axiom E for Euclid as it treats circle while Hilbert doesn't.

[9]If Euclid is being used as a supplement, emphasize to students that a line for Euclid is a line segment for us.

Axiom 5.1.4 (Axiom E: Circle Intersections). *If from points A and B, circles with radius AC and BD are drawn such that each circle contains points both in the interior (those points that are connected to the center of the circle by segments that don't cross the circle) and in the exterior of the other, then they intersect in two points, on opposite sides of AB.* {circint}

As Hartshorne notes, one can conclude from E a line circle axiom: If a line contains a point inside a circle it intersects the circle (twice!). In many expositions (e.g. [Gre93, p. 80]), Axiom 5.1.4 is deduced from the continuity axiom and used to prove the circle propositions from Euclid's Book III and IV. But Hartshorne [Har00, p 114, 203] shows that only the theory EG (Notation **??**) is needed for the circle theorems.

Lemma 5.1.5 (Euclid's Proposition 2). *To place a straight line (segment)*[10] *for equal to a given straight line segment with one end at a given point. In modern language: Given any line segment AB and point C, one can construct a line segment of length AB and end point C.* {EP2}

In straight-edge and compass constructions, we transfer segments by measuring with the compass, then copy that length to any other place on the paper (that is when we do the construction, our 'rusty compass' does not change the radius). The Rusty Compass Activity in the supplement lays out the geogebra construction (SLO6) to prove Lemma 5.1.5. See Euclid for the proof of Theorem 5.1.5 from the axioms already given.

The first problem is solved!

Exercise 5.1.6. *Using the axioms and theorem in this subsection show the algorithm in Section 4 can be carried out.* {doit}

The following exercise gives the student the chance to understand satisfaction in a model in a fairly familiar example and to look at independence where the models are straightforward. While the college students have seen the analytic geometry over the reals, here we note that the construction can act on *any* field.

Definition 5.1.7. *The Cartesian plane over a field F has a points P the elements of $F \times F$ and as lines the solution sets in P of any linear equation with coefficients in F.* {carpl}

Exercise 5.1.8. {conind}

1. *Define carefully each of $(P, \mathbf{L}, I)$ to construct the Cartesian plane over the rational numbers. The description in an Algebra I book answers the question if rational numbers is replaced by real numbers.*
2. *Prove the plane defined in item 1) models Axioms I and II from Axiom 5.1.1 but not Axiom 5.1.4 (Axiom E: Euclid assumed circle-circle intersection).*

5.2 Betweenness, Order, and Planarity

{bot}

Hilbert's 2nd group of axioms [Hil71, §I.3] is labeled *Axioms of Order* and his Theorem 6 that roughly describes a linear order. Szmielew [Szm78, §7.1] gives ten axioms for betweenness (think of statements that are true of a symmetric ($\mathrm{B}\,(A, B, C) \leftrightarrow \mathrm{B}\,(C, B, A)$) and then carefully derives the definition below of a relation $\leq$ that linearly orders the line ℓ through ABC.

Definition 5.2.1 (Linear Order). *A set X is linearly ordered by $<$ if $<$ is asymmetric ($x < y$ implies $y \not< x$), irreflexive ($x \not< x$),transitive ($x < y$ and $y < z$ implies $x < z$), and satisfies trichotomy (for any x, y: $x < y$ or $x = y$ or $y < x$; it is dense if between any two points there is another.* {lodef}

[10] 'line' in Euclid means 'line segment'

Definition 5.2.2. *1. Fix $\ell = \overline{ABC}$ and define $\leq$ for $P, Q \in \ell$ by* {lo}

$$P \leq Q \leftrightarrow (\mathbf{B}(P,Q,B) \wedge \mathbf{B}(P,B,C)) \vee (\mathbf{B}(P,B,C) \wedge \mathbf{B}(A,B,Q,)) \vee (\mathbf{B}(A,B,Q) \wedge \mathbf{B}(B,P,Q,)).$$

In fact, this definition can define either a linear order or its converse, By a tricky argument, treating the rays in each direction separately, Szmielew proves:

Theorem 5.2.3. *[Szm78, §7.1] For any distinct A, B, C with $\mathbf{B}(A,B,C)$ the relation $\leq$ in Definition 5.2.2 is a linear order of ℓ. Assuming for all A, C there exists a B, $\mathbf{B}(A,B,C)$ the order is dense.* {ordthm}

Remark 5.2.4. The difficulty of this argument illustrates the difficulty of proving directly from the betweenness relation. Such details are the reason that Hilbert axioms are not used in high school texts. However, *we will just use Theorem 5.2.3 in our development*. So, plausibly, an alternative axiomatization would be to replace Hilbert's order axioms with our Theorem 5.2.3. {whylo}

Hilbert concludes his first group with Pasch's Axiom, which is implicitly used by Euclid but cannot be derived from his axioms. It asserts:

Axiom 5.2.5. *[Pasch's Axiom:]* Let A, B, C be three non-collinear points and let ℓ be any line which does not meet any of the points A, B, C. If ℓ passes through a point of the segment AB, it also passes through a point of segment AC, or through a point of segment BC. {Pasch}

Like Euclid, Hilbert develops geometry of dimension 3 with plane as a fundamental notion and so a ternary predicate P for coplanar is in his formal vocabulary and the axiom holds when $\mathrm{P}(A,B,C)$. We guarantee the universe is plane by requiring Pasch's axiom to hold for *for any* triplet of points; there is no predicate for plane in our system.

5.3 Angles

{angles}

We define the notion of angle, one of the indicative definitions in Euclid. We need one preliminary.

Definition 5.3.1. *Given a line ℓ and points A, B on ℓ and D, E not on ℓ.* {ray}

1. *the ray $\overrightarrow{AB}$ is all points C on ℓ the same side of A as B (i.e. $\mathbf{B}(A,C,B)$ or $\mathbf{B}(A,B,C)$.*
2. *D and E are in the same half-plane determined by ℓ if the line segment between D and E does not intersect ℓ.*

Definition 5.3.2. *An angle $\angle ABC$ is a pair of distinct rays from a point B. The rays BA and BC split the plane into two connected regions. (A region is* connected *if any two points can be connected by a polygonal path (a sequence of segments such that successive segments share one endpoint)). The region such that any two points are connected by a segment entirely in the region is called the* interior *of the angle.* {angledef}

Definition 5.3.3. *Two angles are* adjacent *if they share a ray but no interior points.* {adjang}

Activity 5.3.4. *What are at least three different units for measuring the size of an angle? (Answers include, degree, radian, turn, grad, house (astrology), Furman.)* {anglemeas}

Activity 5.3.5. *Measure, don't calculate the circumference of a convenient cylinder. To clarfty the exercise, measure the radius or the diameter and then calculate the circumference. We have found this a useful exercise for college freshman and it is intended to urge future teachers to clarify this distinction for their students.* {meascald}

Remark 5.3.6. We differ from Euclid here in allowing straight angles. Thus, we avoid the awkward locution of the 'two right angles' for 'straight angle'. But before even defining 'right angle'; we need to consider congruence.

5.4 Congruence Axioms

{congpost}

This section fills what is generally agreed to be a true gap in Euclid. In Proposition I.4, he purports to prove SAS. The argument implicitly relies on the superposition principal, which we discuss in Remark 5.4.13.

{congax}

Axiom 5.4.1 (Congruence Axioms)**.** As in Euclid, we take the notions of segment congruence ($AB \cong A'B'$) and angle congruence ($\angle ABC \cong \angle A'B'C'$) as primitive. But rather than relying on the common notions, we follow Hilbert [Hil62, §6] and assert: *Congruence is an equivalence relation on undirected line segments (or angles) (reflexive, symmetric, transitive and the sum (difference) of congruent (line segments, angles) is congruent.* The symmetry of angle congruence arises because, following Euclid and Hilbert we studying angles not measuring rotation.

Euclid uses equal where we say congruent for segments and angles.

We stated this axiom in English. Formally, for angles we would add a 6-ary predicate (4-ary for segments) and write C (A, B, C, D, E, F) to translate the axiom for two angles ABC and DEF.

Now we define 'right angle'.

Definition 5.4.2 (Right Angle)**. CCSS G-C0-1** *When a straight line standing on a straight line makes the adjacent angles equal to one another, each of the equal angles is* right*, and the straight line standing on the other is called a* perpendicular *to that on which it stands.* {ra}

{depax}

Comment 5.4.3 (Dependent axioms)**.** *After we introduce the congruence Axiom 5.4.8 (SSS), we see in Exercise 5.4.9 that Euclid's 4th postulate,* All right angles are equal*,* **CCSS G-C0-1** *is a theorem. This represents a dependency in his system since SSS is his Proposition I.8.*

Activity 5.4.4. *Fold paper to make a right angle.*

Now we define what it means for two triangles to be congruent.

{tricondef}

Definition 5.4.5 (Triangle congruence)**. CCSS G-C0-7** *Two triangles are congruent if there is a way to make the sides and angles correspond so that:*
Each pair of corresponding angles are congruent.
Each pair of corresponding sides are congruent.

Comment 5.4.6. Some mathematicians and some high school texts treat congruence as a property of labeled triangles (But then under some permutations of the names of the vertices of a scalene triangle the resulting labeled triangles may not be congruent). By looking at the statement of I.4, it is clear this is not Euclid's intent. As Euclid's concept of superposition (Remark 5.4.13) and saying symmetric triangles are not congruent to themselves without some labeling would redefine rigid motions to remove reflection (Remark 5.4.12). Hilbert treats the idea [Hil71, Appendix II]. But as a weakening of SAS to insist on oriented triangles (and so rigid motions must preserve orientation).

{axchoice}

Remark 5.4.7 (axiom choice)**.** Just as we had a choice of which concepts to name in the vocabulary, we have choices to make for axioms. Euclid's Theorem I.4 (SAS) has been known since antiquity to rely on an implicit 'principle of superposition'. In modern language we express this by saying the group of rigid motions acts transitively on congruent angles.

Hilbert chose to do this by simply making SAS an axiom. Euclid proved without any hidden assumptions that SAS implies SSS, ASA and AAS. We chose SSS and gave SSS implies SAS as an exercise. Here are two reasons for choosing SSS. 1) It is very practical: any three sticks that can form a triangle will always form the same triangle (note: in real-life reflection is not an issue). It is minimalistic: SSS only uses segments in its statement, all others use segments and angles, and defining angles is not trivial.

A major weakness of many high school texts is to think the equivalence proofs of the congruence propositions are too hard for high school and make each a separate postulate. This destroys one of the main features of axiomatics: the search for a small number of (ideally independent) assumptions from which the theory can be deduced.

Axiom 5.4.8. *[4.5 The triangle congruence postulate: SSS]* **CCSS G-C0-8** *Let ABC and $A'B'C'$ be triangles with $AB \cong A'B'$ and $AC \cong A'C'$ and $BC \cong B'C'$ then $\triangle ABC \cong \triangle A'B'C'$* {sss}

While congruence is a property of triangles not of labeled triangles it is a useful convention to require that $\triangle ABC \cong A'B'C'$ implies that the primes indicate the correspondence. Often, in describing polygon ABCDE... any consecutive letters in the name are consecutive (connected by a side) vertices in the polygon.

This might be given as exercisel; the proof is next.

Exercise 5.4.9 (Challenge)**.** *Prove the 4th postulate from SSS.* {4thpostdep}

Theorem 5.4.10 (SAS)**.** **CCSS G-C0-8, G-C0-10** *Let ABC and $A'B'C'$ be triangles with $AB \cong A'B'$ and $AC \cong A'C'$ and $\angle CAB \cong \angle C'A'B'$ then $\triangle ABC \cong \triangle A'B'C'$* {sas}

Proof. We must show $\triangle ABC \cong \triangle A'B'C'$. Draw arcs with radius AC from A' and with radius BC from B' using Axiom 3. Let them intersect at a point D on the same side of $A'B'$ as C'. Note that triangle $A'DB' \cong ACB$ by SSS. ($AB \cong A'B'$, $BC \cong B'D$ and $AC \cong A'D$). So $\angle CAB \cong \angle DA'B'$. But then by transitivity of equality, $\angle C'A'B' \cong \angle DA'B'$. But then D lies on $A'C'$ and in fact D must be C'. So we have proved the theorem. $\square_{5.4.10}$

Definition 5.4.11. A rigid motion *is a bijection from points to points that preserves betweenness, collinearity (so it induces a bijection on lines), and congruence of segments and angles.* {rmdef}

Note that preserving the first three implies preserving congruence of angles by use of SSS. The existence of enough[11] rigid motions (ERM) is proved in any Hilbert plane in [Har00, §17]. In fact, $HP - SAS \vdash ERM \Leftrightarrow SAS$.

Theorem 5.4.12. *Every rigid motion is a composition of reflections, translations and rotations.* {rmlist}

Proof. A rigid motion ϕ falls into one of four disjoint classes according to the number of points they fix.

1. ϕ fixes all points;$\phi = \psi^2$ where ψ is a reflection.
2. ϕ fixes at least two points A, B but not all. In that case ϕ fixes the line ℓ through AB setwise. So under ϕ each X on ℓ remains the same distance from A and B; thus ℓ is pointwise fixed.

   Suppose $C \notin \ell$ and $\phi(C) = C''$ with $C'' \neq C$ is on the same side of ℓ as C. As ϕ takes the segment AC to AC''. But one is congruent to a proper subset of the other. So $C \notin \ell$ implies $\phi(C) = C'$ is

[11] 'enough' means the group of rigid motions ERM is sufficiently transitive: transitive on points and two further conditions.

on the opposite side of ℓ from C. Then for any $X \in \ell$, $XC \cong \phi(X)C'$ and $\phi(X) \in \ell$. In particular $AC \cong A\phi(C)$ and $BC \cong BC'$

Let ℓ' be the line extending CC'. It is distinct from ℓ, so intersects ℓ only in one point D. But since ϕ fixes all lines setwise $\phi(D)$ is on $\ell \cap \ell'$, i.e, $\phi(D) = D$. So $DA \cong DB$ and $DC \cong DC'$. Thus $\triangle DBC \cong \triangle DBC'$ and $\triangle DAC \cong \triangle DAC'$. So $\angle CDB$ is a right angle and $\ell \perp \ell'$. Now we can see that ϕ is a reflection in ℓ.

Let ℓ'' denote the image of ℓ under ϕ.

3. ϕ fixes a single point A. Then since ϕ preserves lines, it must be a rotation around A (not equal to a full turn).

4. ϕ fixes no point. Since ϕ sends lines to lines and no points are fixed; if for any ℓ, $\ell \parallel \phi(\ell)$; ϕ is a translation, if not it is a glide reflection [CK17, p 82].

□

{superposition}

Remark 5.4.13. Euclid proves SAS by an implicit use of the principle of superposition: If a rigid motion takes one figure to another then they are congruent. This principle is an immediate consequence of the definition of rigid motion. [Har00, §17] shows the existence of rigid motions (ERM) in any Hilbert plane with SAS and conversely that from the axioms for a Hilbert plane without SAS, ERM implies SAS. This is essentially Euclid's proof of Proposition I.4. The most immediate formalization of rigid motions is to add second order quantifiers over arbitrary permutations of the set of points. But one can add a new sort $\mathbf{M}$ for motions and a ternary relation $\mathbf{R}$ on P $\times$ P $\times$ $\mathbf{M}$ that for each f in $\mathbf{M}$ the pairs $\langle a, b\rangle$ such that $\mathbf{R}(\mathbf{a}, \mathbf{b}, \mathbf{f})$ is the graph of a rigid motion.

Pedagogy 5.4.14 (SLO1: Van Hiele level of Transformational geometry)**.** Taking into account the necessity for a deep understanding of the notion of abstract function[12], one might posit a further 'Van Hiele' level (thought not geometric): Ability to work with abstract functions. This may not be an issue for college students but additional work on functions might be helpful (and appear in the supplement).

The HS teacher testifies against this, 'At the HS level we successfully work with transformations without using functions. Working in the coordinate system, given two possibly congruent shapes, visually draw a series of transformations of that shape to find out if the two coincide after the transformations.'

The method of proving the following important exercise is embedded in the proof of Theorem 5.4.10.

{moveangle}

Exercise 5.4.15 (Move Angle)**.** *Let ABC be an angle. For any segment DE, choose a point F so that $\angle ABC \cong \angle DEF$.*

{cp1}

Construction 5.4.16 (Constructing perpendiculars)**. CCSS G-C0-12** *Given a line AD there is a line perpendicular to the line through AD at D.*

Proof. Extend AD and let B be the intersection of that line with the circle of radius AD centered at D. Now construct an equilateral triangle with base AB by using Axiom 5.1.1 twice to construct the vertex C. Draw CD. SSS implies $\triangle ACD \cong \triangle BCD$; so $\angle CDA \cong \angle CDB$ and therefore $CD \perp AB$. $\square_{5.4.16}$

Extension 5.4.17 (Independence of Congruence Axioms)**.** In the proof we constructed an equilateral triangle using only the first three postulates. We seem to need SSS to finish. [Hil71, p 39] shows by varying the distance formula in the real plane, that the congruence axioms are independent from first two groups.

[12] See [Har14] for an argument against the use of transformation based systems in high school; the unfamilarity of sophomores with functions is a key point.

Definition 5.4.18 (Straight Angle)**.** *An angle $\angle ABC$ is called a* straight *angle if A, B, C lie on a straight line and B is between A and C.*

Since Euclid does not introduce a measure for angles, he has names for the most important, straight and right, and rough indications of size such as acute and obtuse.

Note a perpendicular creates two right angles side of a line. Constructing a perpendicular at the vertex of a straight angle and apply Euclid's fourth postulate yields:

{stangle}

Theorem 5.4.19. CCSS G-C0-9 *All straight angles are equal (congruent).*

Proof. Let $\angle ABC$ and $\angle A'B'C'$ be straight angles. Construct lines BD and $B'D'$ perpendicular to AC and $A'C'$, respectively. Now $\angle ABD + \angle DBC = \angle ABC$ and $\angle A'B'D' + \angle D'B'C' = \angle A'BC'$. By Axiom 5.4.1, $\angle ABD = \angle A'B'D'$ and $\angle DBC = \angle D'B'C'$. $\square_{5.4.19}$

Theorem 5.4.19 is statement about the uniformity of the plane. In terms of transformations it says any point and a line through it can be moved by a rigid motion to any other point and any line through it.

{vertang}

Definition 5.4.20. *If two lines cross, non-adjacent (Definition 5.3.3) angles that have only the vertex in common are called vertical angles.*

Deduce from Theorem 5.4.19:

Exercise 5.4.21 (CCSS G-C0-9)**.** *Prove that vertical angles are equal.*

Definition 5.4.22 (Isosceles)**.** *A triangle is* isosceles *if at least two sides have the same length. The angles opposite the equal sides are called the* base angles*. (Note some textbooks require exactly to sides have the same length).*

Activity 5.4.23 (SLO8, 10: G-CO 11,12)**.** *Make a) an isosceles and b) an equilateral triangle in Geogebra using translations*

Activity 5.4.24. G-CO 10 *Activity: Prove the isosceles triangle and exterior angle theorems (Euclid Translation Activity.pdf). Compare 'paragraph' and 'two column' proof.*

{isosbase}

Theorem 5.4.25. CCSS G-C0-10 *The base angles of an isosceles triangle are equal (congruent).*

Proof. Let ABC be an isosceles triangle with $AC \cong BC$. We will prove $\angle CAB \cong \angle CBA$. The trick is to prove $\triangle ABC \cong \triangle BAC$. ($\triangle BAC$ is obtained from $\triangle ABC$ by flipping the triangle over its altitude.) We have two ways to prove the congruence. We know $BC \cong AC$ and $AC \cong BC$. We can also note $AB \cong BA$ and use SSS or $\angle ACB \cong \angle BCA$ and use SAS. In any case, since the triangles are congruent $\angle CAB \cong \angle CBA$. $\square_{5.4.25}$

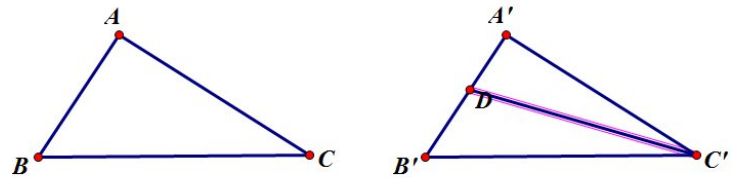
Activity 5.4.26. *Prove the angles of an equilateral triangle are equal. (Note that there are two proofs, using either SSS or SAS, and they are distinguished by which correspondences are made in defining the congruence. Explain this by considering the theorem in terms of rotational or reflective symmetry.)*

We include the proof of the following result to show a typical use of *proof by contradiction*.

{asa}

Theorem 5.4.27. CCSS G-C0-8, G-C0-10 *If two triangles have two angles and the included side congruent, then the triangles are congruent.*

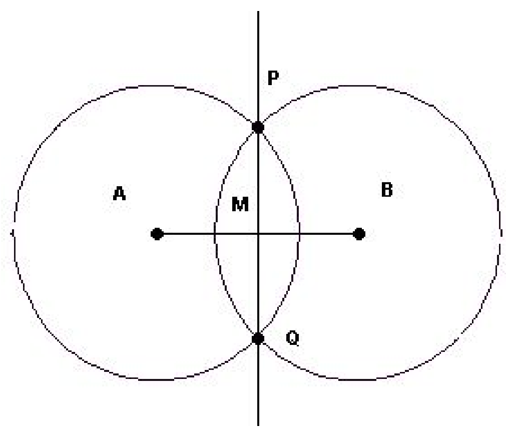
Proof. Suppose ABC and $A'B'C'$ satisfy $\angle ABC = \angle A'B'C'$, $\angle ACB = \angle A'C'B'$ and $BC = B'C'$. We will show the triangles are congruent.



Choose D on $A'B'$ so that $AB \cong DB'$ (We'll assume D is between A' and B' for contradiction. If A' is between B' and D, there is a similar proof.) Now, $AB \cong DB'$, $BC \cong B'C'$ and $\angle ABC \cong \angle A'B'C'$ so by SAS, $\triangle ABC \cong \triangle DB'C'$. Since the angles correspond, $\angle DC'B' \cong \angle ACB$ and so by Common Notion 1, $\angle DC'B' \cong \angle A'C'B'$. But this is absurd since $\angle DC'B'$ is a proper subangle of $\angle A'C'B'$. $\square_{5.4.27}$

{cp2}

Theorem 5.4.28 (Constructing Perpendicular Bisectors)**. CCSS G-C0-12** *For any line segment AB there is a line PM perpendicular to AB such that M is the midpoint of AB.*



Proof. Set a compass at any length at least that of AB and draw two equal circles centered at A and B respectively. Let the two circles intersect at P and Q on opposite sides of AB and let M be the intersection of AB and PQ.

To show PQ perpendicular to AB, note first that $\triangle APQ \cong \triangle BPQ$ by SSS. So $\angle APM \cong \angle BPM$. Then by SAS, $\triangle APM \cong \triangle BPM$. Thus $\angle AMP \cong \angle BMP$. And therefore these are each right angles by Definition 5.4.2. But $\triangle APM \cong \triangle BPM$ also implies $AM \cong BM$ so M bisects AB. $\square_{5.4.28}$

Remark 5.4.29. *Note we could be more prescriptive and just as correct by requiring in the proof of Theorem 5.4.28 that the circle have radius AB. But this is an unnecessary additional requirement.*

5.5 The Parallel Postulate

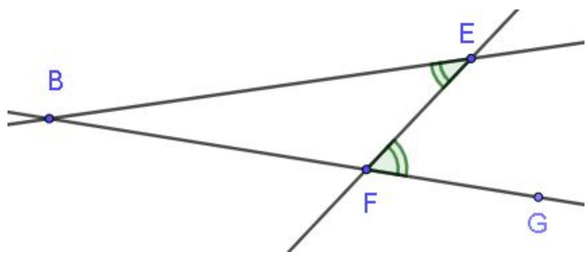
Of course, the change in viewpoint of what axioms mean stems from the proof of the independence of the parallel postulate. We do not rehearse here the well-known history but do discuss a subtle shift in meaning of the phrase 'the parallel postulate'.

Definition 5.5.1. *Two lines are parallel if they do not intersect.*

The difference between several statements which are close to the parallel postulate provides interesting historical and pedagogical background. The most succinct statement is: For a line ℓ and point A not on ℓ, there is at most one line parallel to ℓ through A. Observe that Euclid proved the existence of parallel lines. So spherical geometry, which was studied by the Greeks, could not have been seen as example to show the independence. Playfair and Hilbert rephrased the postulate as the existence of unique parallel lines which was confused even by prominent mathematicians [HT05].

The definitions of corresponding, interior, and exterior angles can be found in any geometry text.

Theorem 5.5.2. *[Euclid I.27] If two lines are crossed by a third and alternate interior angles are equal, the lines are parallel.* {I.27}



Proof. The hypothesis says the exterior angle EFG to triangle BFE is equal to the interior angle FEB. That contradicts the exterior angle theorem **??**.

Remark 5.5.3 (Parallels Exist)**.** *Since the hypothesis of Theorem 5.5.2 is easily constructed, Euclid has proved the existence of parallel lines.* {parex}

Now consider the converse of Theorem 5.5.2. We will show it is equivalent Euclid's 5th postulate.

Axiom 5.5.4 (Euclid's 5th postulate)**.** *If two parallel lines are cut by a transversal then each pair of alternate interior angles contains two equal angles.* {E5}

The rest of this subsection is illustrate the use of contraposition by important result: in HP the two versions of the 5th postulate, Axiom 5.5.4 and Axiom 5.5.8, are equivalent.

Definition 5.5.5 (Contraposition)**.** *Let A and B be mathematical statements. The* contrapositive *of 'A implies B' is '$\neg B$ implies $\neg A$'* {contrap}

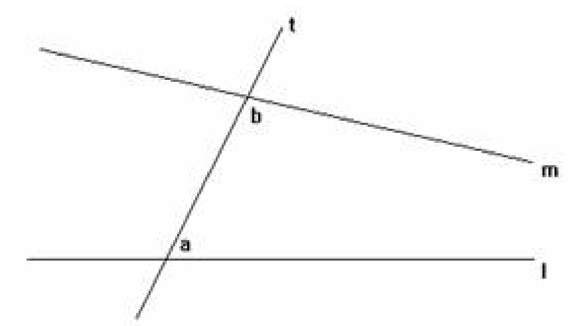
Fact 5.5.6 (Logical fact)**.** *Any implication is* equivalent *to its contrapositive.*

Pedagogy 5.5.7. SLO1 This is easily checked by truth tables. High school geometry texts sometimes ask students to memorize the names of the four variants on a conditional (if-then) statement. One is the inverse that I know only from such books. This is counter-productive; only the conditional, converse and contrapositive are used frequently. A frequent difficulty is to understand why 'A implies B' is declared true when B is false. The first author found it useful in undergraduate logic courses to emphasize that we are formalizing English. The ambiguity between inclusive and exclusive (but not both) or is easy to illustrate. Logicians decided use $\vee$ to mean inclusive or. A similar decision was made for implication $\rightarrow$. Of course if the instructor finds explanations that convince students that's even better. {tt}

{HE5}

Axiom 5.5.8. *Heath's statement of Euclid's 5th postulate:*

If a straight line crosses two straight lines in such a way that the interior angles of the same side are less than two right angles, then, if the two straight lines are extended, they will meet on the side on which the interior angles are less than two right angles.



Using the earlier axioms this statement is seen to be equivalent to my phrasing of Euclid's 5th postulate Axiom 5.5.4. The contrapositive (and so equivalent) to Heath's version of Euclid's 5th postulate reads: If the two lines crossed by a straight line do not meet on one side of that straight line, then the interior angles on that side are not less than two right angles. Applying the contrapositive to the angles on each side of the transversal, the two interior angles on each side of the transversal are not less than two right angles. But the sum of the four interior angles is two straight angles (considering the top and bottom pairs.) So each pair on a given side are supplementary. Now for each pair of alternate interior angles, both of the angles are supplemental to the same angle; so the alternate interior angles are equal.

5.6 Degrees in a triangle and classifying quadrilaterals

{degclass}

A key equivalent to the parallel postulate is that the measures of the angles in a triangle sum to 90^o. In fact, the simplest definition of a degree is $\frac{1}{90}$ of a right angle. Non-Euclidean geometries can be classified by whether that sum is more (semi-elliptic) or less (semi-hyperbolic[13]) than a straight angle [Har00, p 311].

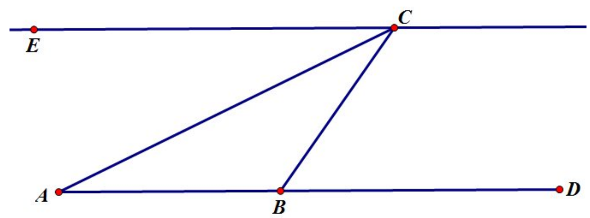
We discuss here Euclid's treatment of the sum of the angles of a triangle,

using degree notation; we point to the situation in non-Euclidean geometries in Section 11

{sumdeg}

Theorem 5.6.1. *[HP5]* **CCSS G-C0-10** *The sum of the angles of a triangle is* 180^o.

Proof. That is, we must show the sum of the angles of a triangle is a straight angle.



1. Draw EC so that $\angle BCE \cong \angle DBC$. (Exercise 5.4.15)
2. Then $EC \parallel AD$. (Theorem **??**)
3. So $\angle BAC \cong \angle ACE$ (Axiom 5.5.4)
4. So $\angle BAC + \angle ACB = \angle DBC$
5. But $\angle ABC + \angle DBC$ is a straight angle.
6. So $\angle ABC + \angle BAC + \angle ACB$ is a straight angle.

$\square_{5.6.1}$

{class}

Remark 5.6.2 (Classification)**.** The classification of quadrilaterals is a major topic in high school geometry. It is essential to first clarify the notion of 'classify'; it does not help to say 'a square is a rectangle just as a parallelogram is a quadrilateral' (heard from a high school teacher). The analogy the student needs is 'dogs and cats are animals'.

Classifications may be 'inclusive' or 'exclusive'. Thus Euclid requires an isosceles to triangle to have exactly two equal sides while modern texts include classifications that are inclusive; all squares are rectangles.

{defpar}

Definition 5.6.3. *A parallelogram is a quadrilateral such that the opposite sides are parallel.*

{eqpar}

Theorem 5.6.4. CCSS G-CO.11 *If the opposite sides of a quadrilateral are equal, the quadrilateral is a parallelogram.*

Proof. Suppose $ABCD$ is the parallelogram; draw diagonal AC. Then $\triangle ABC$ and $\triangle ACD$ are congruent by SSS. Therefore $\angle BAC \cong \angle ACD$. Now since alternate interior angles are equal, $AB \parallel DC$. Similarly (which angles?) $BC \parallel AD$. $\square_{6.0.3}$

{oppsideeq}

Theorem 5.6.5 (Euclid I.34)**. CCSS G-CO.11** *In any parallelogram the opposite sides and angles are equal. Moreover each diagonal splits the parallelogram into two congruent triangles.*

Proof. Immediate from our results on parallelogram and the congruence theorems.

{exercise565}

Exercise 5.6.6. CCSS G-CO.11

If one pair of opposite sides of a quadrilateral are equal and parallel, the figure is a parallelogram .

6 Proof that the division of a line into n equal parts succeeds

{proofsuc}

We began this excursion into axiomatic geometry by trying to prove that we could divide a line into n equal segments. The construction (Figure 1) used only Euclid's first 3 axioms. *We need to show the segments cut off by the C_i are actually equal.* We use the methods of Section 5 to *almost* prove the procedure in Exercise 4.3 works. We will discover that entirely different methods are needed for the last step in the proof – the side-splitter theorem 8.5.

Looking at the diagram from our guiding problem, since a quadrilateral whose opposite sides are equal is a parallelogram (Theorem 6.0.3), $ABCD$ is a parallelogram. We DO NOT know that A_4B_4CD is a parallelogram. In order to establish that it is, we need some more information about parallelograms.

{class}

Pedagogy 6.0.1. The classification of quadrilaterals is a major topic in high school geometry. It is essential to first clarify the notion of 'classify'; it does not help to say 'a square is a rectangle just as a parallelogram is a quadrilateral' (heard from a high school teacher). The analogy the student needs is 'dogs and cats are animals'.

Classifications may be 'inclusive' or 'exclusive'. Euclid requires an isosceles triangle to have exactly two equal sides while modern texts include classifications that are inclusive; all squares are rectangles.

{defpar}

Definition 6.0.2. *A parallelogram is a quadrilateral such that the opposite sides are parallel.*

{eqpar}

Theorem 6.0.3. CCSS G-CO.11 *If the opposite sides of a quadrilateral are equal, the quadrilateral is a parallelogram.*

Proof. Suppose $ABCD$ is the parallelogram; draw diagonal AC. Then $\triangle ABC$ and $\triangle ACD$ are congruent by SSS. Therefore $\angle BAC \cong \angle ACD$. Now since alternate interior angles are equal, $AB \parallel DC$. Similarly $BC \parallel AD$. $\square_{6.0.3}$

The argument also shows:

{oppsideeq}

Theorem 6.0.4 (Euclid I.34)**. CCSS G-CO.11** *In any parallelogram the opposite sides and angles are equal. Moreover each diagonal splits the parallelogram into two congruent triangles.*

{ex565}

Lemma 6.0.5. CCSS G-CO.11 *If one pair of opposite sides of a quadrilateral $ABCD$, labeled as in Figure 1, are equal and parallel, the figure is a parallelogram.*

Proof. Draw the diagonal AC. By alternate interior angles $\angle BCA \cong DAC$. The triangles ACB and ACD are congruent by SAS, using the hypothesis and that they share a side. So $\angle BAC \cong \angle ACD$. Now viewing AC as a transversal of BA and CD, they are parallel and we finish.

{getpar}

Lemma 6.0.6. *If $ABCD$ is a parallelogram, labeled as in Figure 1, and two points X, Y are chosen on the opposite sides BC and AD so that $XC \cong YD$ then $XDCY$ is a parallelogram.*

Proof. Apply Lemma 6.0.5 taking X for A and Y for C.

{sspreview}

Motivation 6.0.7. *SLO1,SLO2 We are giving the proof in reverse to show how the abstract side-splitter theorem is needed to solve a concrete problem. The proof of it requires a new central idea - proportionality. The next two sections are devoted to providing a firm foundation for proportion. By using Hilbert's proof rather than Euclid's we avoid reliance on the Archimedean axiom.*

To finish the proof we need a very strong result:

Theorem 6.0.8. *Euclid VI.2: Side-splitter* **CCSS G-SRT.4** *If a line is drawn parallel to the base of a triangle the corresponding sides of the two resulting triangles are proportional and conversely.* {sidespl}

Proof of the guiding problem assuming sidesplitter: By repeating the argument for Lemma 6.0.6, we show all the lines $A_iC_iB_i$ are parallel. In particular the line C_4B_4 cuts the triangle B_3C_3C and is parallel to the base B_3C_3. Applying Theorem 8.5, we complete our proof as follows:

$$\frac{CB_4}{CB_3} = \frac{CC_4}{CC_3}.$$

But we constructed $B_4C \cong B_3B_4$, so $C_4C \cong C_3C_4$, which is what we are trying to prove. Now move along AC, successively applying this argument to each triangle. $\square$**??**

{sspreview}

Motivation 6.0.9. *SLO1,SLO2 We giving the proof in reverse to show how the abstract side-splitter theorem is needed to solve a concrete problem. The proof of it requires a new central idea - proportionality. The next two notions are devoted to providing a firm foundation for proportion. By using Hilbert's proof rather than Euclid's we avoid reliance on the Archimedean axiom.*

To finish the proof we need a very strong result:

Theorem 6.0.10. *Euclid VI.2: Side-splitter* **CCSS G-SRT.4** *If a line is drawn parallel to the base of a triangle the corresponding sides of the two resulting triangles are proportional and conversely.* {sidespl}

Proof of the guiding problem assuming sidesplitter: By repeating the argument for Lemma 6.0.6, we show all the lines $A_iC_iB_i$ are parallel. In particular the line C_4B_4 cuts the triangle B_3C_3C and is parallel to the base B_3C_3. Applying Theorem 8.5, we complete our proof as follows:

$$\frac{CB_4}{CB_3} = \frac{CC_4}{CC_3}.$$

But we constructed $B_4C \cong B_3B_4$, so $C_4C \cong C_3C_4$, which is what we are trying to prove. Now move along AC, successively applying this argument to each triangle. $\square$**??**

7 Finding the underlying field

{num}

We reduced our cutting the line problem to the side-splitter theorem VI.2; that is, to the fundamental result about the similarity of triangles. Hilbert defines a (semi)-field of segments (addition and multiplication on the positive elements of an ordered field). He thus has the modern algebraic theory of proportion and VI.2 follows easily (Section 8). Then (Section 9) he defines a measure of area function which recovers Euclid's theory of area and connects it with numerical measures of area.

In this section we outline Hilbert's proof. A fuller proof, with activities for understanding, and more context in on the web at XXXXXXXXXXXX.o

Motivation 7.1 (SLO8 **Irrationality: the Pythagorean scandal).** The geometry course is an excellent place to organize historically and conceptually the students understanding of irrational and transcendental numbers (Section 10). Two or more magnitudes are *commensurable* if they share a common measure. Two feet and three feet are commensurable, each being a multiple of a foot; but the diagonal and side of a square are incommensurable. Thus, the irrationality of $\sqrt{2}$ is usually attributed to 5th century BCE Pythagoreans.

A solution to comparing irrationals was developed by Eudoxus in the 4th century BCE and expounded in Euclid Book V on proportion, perhaps a century later. Crucially, this was a study of 'magnitudes' of various dimensions. The notion of ascribing a number to a measure of area was only adopted in geometry during the 19th century AD and put on a firm footing by Stolz and Pasch as expounded in [Hil62]. A beauty of Hilbert's approach is that he shows that (a suitable translation) of the (first order) axioms of Euclidean geometry allow the measure of area in any Euclidean plane (Notation **??**) by interpreting a field into the plane. In Section 10, we will note how the real numbers provide the most commonly used example. For further background on Greek study of irrational numbers see [Smo08].

The proof of the side-splitter theorem (Theorem 8.5.) is difficult because the meaning of ratio between two sides is obscure at best. To solve this problem, Hilbert defines *geometrically* a multiplication of line segments. Identify the collection of all congruent line segments and choose a representative segment OA for this class. There are then three distinct historical steps. (For SLO7, see [GG09] and Heath's notes to Euclid VI.12 (`http://aleph0.clarku.edu/~djoyce/java/elements/bookVI/propVI12.html`.) In Greek mathematics numbers (i.e. 1, 2, 3 ...) and magnitudes (what we would call length of line segments) were distinct kinds of entities and areas were still another kind. Numbers simply count the number of some unit; the unit varies from situation to situation. For them the notion of a assigning a number as the length of the diagonal of a unit square is incomprehensible.

We first introduce an addition and multiplication on line segments and then prove the geometric theorems to show that these operations satisfy the field axioms except for the existence of an additive inverse.

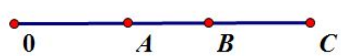
{segeq}

Notation 7.2. *Note that congruence forms an equivalence relation on line segments. We fix* a ray $\overrightarrow{\ell}$ *with end point* 0*. For each equivalence class of segments, we consider the unique segment* $0A$ *on* $\overrightarrow{\ell}$ *in that class as the representative of that class. We will often denote the class (i.e. the segment* $0A$*) by* a*. We say a segment (on any line)* CD *has length* a *if* $CD \cong 0A$*.*

{segadddef}

Definition 7.3 (Segment Addition)**.** *Consider two segment classes* a *and* b*. Fix representatives of* a *and* b *as* OA *and* OB *in this manner: Extend* OB *to a straight line, and choose* C *on* OB *extended (on the other side of B from A) so that so that* $BC \cong OA$*.* OC *is the sum of* OA *and* OB*.*

Diagram for adding segments



Activity 7.4. *Prove that this addition is associative and commutative.*

Of course there is no additive inverse if our 'numbers' are the lengths of segments which must be positive. We discuss finding an additive inverse after Definition **??**. Following Hartshorne [Har00], here is our official definition of segment multiplication.

{segmultdef}

Definition 7.5. *[Multiplication] Fix a unit segment class* 1*. Consider two segment classes* a *and* b*. To define their product, define a right triangle*[14] *with legs of length* 1 *and* a*. Denote the angle between the hypoteneuse and the side of length* 1 *by* α*.*

Now construct another right triangle with base of length b *with the angle between the hypotenuse and the side of length* b *congruent to* α*. The length of the leg opposite* α *is* ab*.*

[14]The right triangle is just for simplicity; we really just need to make the two triangles similar.

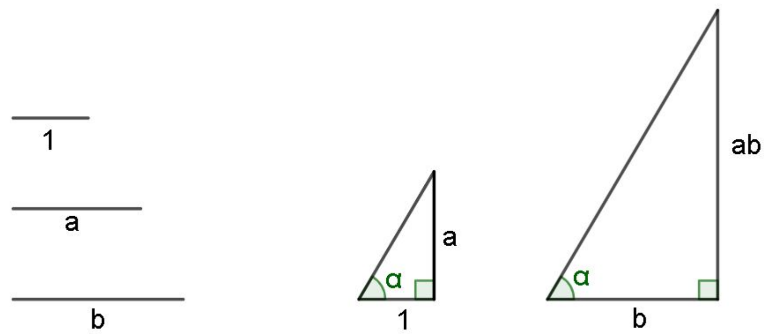


Figure 2: Multiplication {figure1two}

Before we can prove the field laws hold for these operations we introduce a few more geometric facts. The crux of the argument is to prove the multiplication is associative and commutative. Hilbert and many successors give this argument as arising from the Desargues and Pappus theorems which hold in HP5 (neutral geometry plus the parallel postulate). Because the techniques of its proof are more similar to standard high school material, we rely on the cyclic quadrilateral theorem.

{cquad}

Corollary 7.6. *[**CCSS G-C.3: Cyclic Quadrilateral Theorem**] Let $ACED$ be a quadrilateral. The vertices of $ACED$ lie on a circle (the ordering of the name of the quadrilateral implies A and E are on the opposite sides of CD) if and only if $\angle EAC \cong \angle CDE$.*

The proof of this theorem and the next are in the supplement.

{mult2works}

Theorem 7.7. *The multiplication defined in Definition 7.5 satisfies:*

1. *For any a,*
$$a \cdot 1 = a$$
2. *For any a, b*
$$ab = ba.$$
3. *For any a, b, c*
$$(ab)c = a(bc).$$
4. *For any a there is a b with $ab = 1$.*
5. *$a(b + c) = ab + ac$.*

We have a *semi-field* because the addition does not form a group. This is important for Hilbert because he is giving an entirely geometric proof. The above semi-field can be modified to become a field by taking points on a line rather than a ray, and then have both positive and negative numbers, therefore getting additive inverses. Details are included in the supplement. With this geometrically based field we give in the next section an algebraic basis for the theory of proportion which allows us to prove side-splitter.

8 Similarity, Proportion, and Side-splitter

{prop}

In this section, we define proportion using Section 7 and then prove the side-splitter theorem. We need a couple of definitions.

Recall that in Section 7 we defined a field whose elements were line segments on a fixed line $\overline{01}$. So we make the following definitions using a, b etc. to range over such segments. Most texts will have identified these segments with real numbers. We emphasize that the results are much more general than that.

Definition 8.1. *Let a, b, a', b' be segments on a fixed line $\overleftrightarrow{01}$. Then we say the ratios $a : b$ and $a' : b'$* satisfy the proportion $a\colon b = a' \colon b'$ *(also written $a\colon b :: a' \colon b'$ or $\frac{a}{b} = \frac{a'}{b'}$) if $ab' = ba'$.*

Definition 8.2. *Two triangles $\triangle ABC$ and $\triangle A'B'C'$ are similar if under some correspondence of angles, corresponding angles are congruent; e.g. $\angle A' \cong \angle A$, $\angle B' \cong \angle B$, $\angle C' \cong \angle C$.*

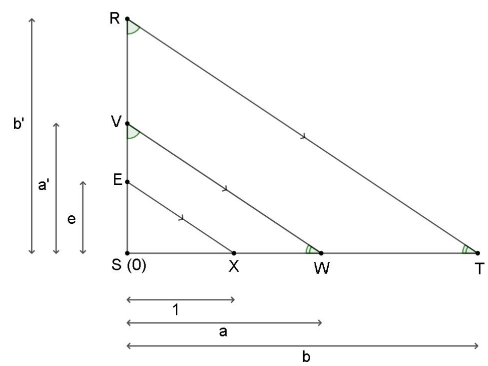
Activity 8.3. *Various texts define 'similar' as we did, or as corresponding sides are proportional or require both. Discuss the advantages of the different definitions. Why are all permissible?*

{simprop}

Theorem 8.4. *Similar triangles have proportional sides.*

Proof. Suppose SVW and SRT are similar triangles as displayed in the diagram below we show

$$\frac{SV}{SR} = \frac{SW}{ST}.$$



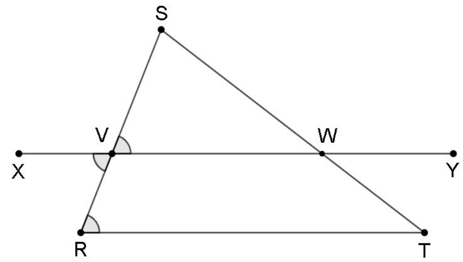
Consider the special case that $\angle RST$ is a right angle. Label SW as a, ST as b, SV as a', SR as b', Then think of S as 0 and pick a point X of ST with $SX \cong 01$. Now using segment multiplication the diagram shows $ab' = ba'$. So by definition $a : b = a' : b'$ or $\frac{SW}{ST} = \frac{SV}{SR}$. [Hil71, p. 56] gives the half page argument that the restriction to a right angle is unnecessary. $\square_{8.5}$

Theorem 8.5. *Euclid VI.2: Side-splitter* **CCSS G-SRT.4** *If a line is drawn parallel to the base of a triangle the corresponding sides of the two resulting triangles are proportional and conversely.* {sidespl}

{simpar}

Lemma 8.6. *Suppose a line VW is drawn connecting points on two sides a triangle SRT, SR and ST. VW is parallel to RT if and only if SVW is similar to SRT.*

Proof. As in the following diagrams, extend VW to a line and pick points X and Y on VW on opposite sides of the triangle.

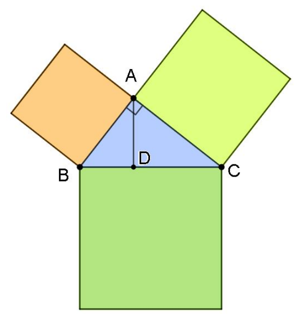


Now $\angle XVR$ and $\angle VRT$ are alternate interior angles for the transversal RS crossing the two lines XY and RT. So $\angle XVR \cong \angle VRT$ if and only if $VW \parallel RT$. But $\angle XVR \cong \angle SVW$ since they are vertical angles. So $\angle SVW \cong \angle VRT$ if and only if $VW \parallel RT$. Similarly $\angle SWV \cong \angle STR$ if and only if $VW \parallel RT$. $\quad \square_{8.6}$

Now from Lemmas 8.4 and 8.6, we have deduced Theorem 8.5 immediately finishing the solution of the guiding problem.

As we will sketch in Section 9, Euclid developed the notion of area (he says equal figure.) in I.35-I.48, *Commentators agree that this was specifically to avoid the use of proportion in the proof of Pythagoras.* In particular, Euclid needed the Archimedean axiom for his theory of proportion and so to prove the side-splitter. Hilbert grounds the theory of proportion *purely geometrically* without assuming Archimedes' axiom.

Exercise 8.7 (Euclid VI.31 CCSS G-SRT4)**.** *Prove the Pythagorean theorem using similarity.*



The supplement suggests several other proofs of Pythagoras including one due to President Garfield.

9 Area of Polygons

{whyarea}

Pedagogy 9.1. SL01, 02,05,07: Experience with students in precalculus and calculus who react to min-max problems by saying 'I know the formula is lw or $2l + 2w$ but I don't know which' motivates this section. The connection between (equi)-decomposition and area needs to be made in high (if not middle school). While the argument in argument in the supplement is too technical for high school, it provides future high school teachers with a necessary perspective.

This section has both methodological and pedagogic content. We reserve the details of the methodological concerns to the supplement.

Methodology 9.2 (SLO4: What is area?)**.** This section expounds the differences among three methods of computing area that are frequently conflated in high school texts. Euclid begins by (implicitly) defining what it means for two figures to have same area (Euclid-equal). By this means, he is able to prove the Pythagorean theorem without invoking the notion of proportion -showing it is a fully *geometric* result. In contrast, modern notions of measuring area rely fundamentally on approximating figures by infinitely many smaller figures and taking limits.

Using the field defined in Section7, Hilbert defines 'equal area' by a slightly different notion (Hilbert-equal) and introduces a finite procedure of assign a numeric value as the area of a polygon. In fact, these three notions of equality are the same. However, they cannot be proved the same as equi-decomposable (scissors-congruent) without the use of the Archimedean axiom.

We established a linear order on (congruence classes) of segments by $[AB] < [CD]$ if $AB \cong A'B'$ for some proper subsegment $A'B'$ of CD. This is not so easy in two dimensions; a long skinny rectangle might or might not 'be bigger' than a short fat one. What sorts of objects we can assign area to and when two 'figures' have the same area?

{rectilineal}

Definition 9.3. *1. A* (rectilineal) figure *is a subset of the plane that can be expressed as a finite union of disjoint triangles (Sides may overlap; interiors can't.).*

2. A polygon *is a closed figure whose sides are line segments that intersect only at their endpoints and each endpoint is shared by exactly two segments. Closed means you can trace the outer edges and come back to where you started without any repetition.*

The term figure is introduce to allow for various types of decomposition.
However, there are at least two ways to implement Method 2.

{assmeas}

Definition 9.4. *[Two ways to measure]*

'equal' area Define an equivalence relation[15] $E(P, P')$ on figures and define $[P_1] < [P_2]$ if some representative of $[P_1]$ is congruent to a proper subset of a representative of $[P_2]$.

equal numerical measure **Analytic measure** Fix a unit; say, a square; tile the plane with congruent squares. Then to measure a figure, continually refines the measure by cutting the squares in quarters and counting only those (possibly fractional) squares which are contained in the figure.

geometric measure (Hilbert) Decompose the figure into finitely many disjoint triangles, which are each assigned area $\frac{bh}{2}$, and add those areas.

We call the last geometric area because the multiplication is the geometric multiplication of Section 7. We consider three further ways to implement Method 1. Before giving the formal definition we see how these methods are abstracted from the proof of Euclid I.35.

Theorem 9.5. *Proposition I.35 Parallelograms which are on the same base and in the same parallels equal one another.*

Proof. The terms 'Euclid equal' and 'Hilbert equal' are defined below (generalized from this argument. Euclid says triangle 1 + 4 is congruent to triangle 3 + 4 . Subtract 4 from the first to get trapezoid 1 and from the second to get trapezoid 3. So 1 and 3 have the same area. Add 2 to each to see the two parallelograms, 1 + 2 and 3 + 2, have the same area.

[15] In the supplement we define two such equivalence relations, Euclid-equal and Hilbert-equal, and prove they each agree with geometric measure.

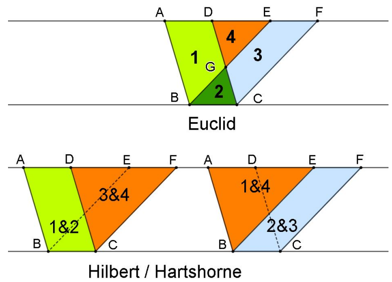


Figure 3: Euclid I.35 {fig:0prim}

Hilbert says adding triangle 3&4 to parallelogram 1&2 gives the same as adding triangle 1&4 to parallelogram 2&3, and 1&4 and 3&4 are equidecomposable (in this case congruent) so we can conclude the two parallelograms have equal area. The distinction is that the weaker condition 'equidecomposablity' on the triangles 1&4 and 3&4 allows him to build scissors decomposition into his notion.

{Eeq}

Definition 9.6. *[Euclid-equal polygons] For figures A and B:*

1. *A has* 1*-equal area with B there is a figure C such that $A + C$ is congruent to $B + C$ or there is a C such that $A - C$ congruent to $B - C$.*
2. Euclid-equal *is the transitive closure of the symmetric and reflexive relation* 1*-Equal content.*

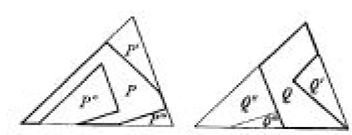
{scon}

Definition 9.7 (Scissor Congruence)**.** *Two polygons are* scissor-congruent *or* equidecomposable *if one can be cut up into a finite number of triangles which can be rearranged to make the second.*

SLO7: It is a sign of Euclid's genius that he realized that a type of refinement of scissors-congruent, dubbed *equicomplementability* by Hilbert around 1900, allowed the proof of proportionality of area to base and height without Archimedes.

{ec}

Definition 9.8 (Equal content)**.** *Two figures P, Q have* equal content aka equicomplementable *or* Hilbert equal[16] *if there are figures $P_1' \ldots P_n'$, $Q_1' \ldots Q_n'$ such that none of the figures overlap, each P_i' and Q_i' are scissors congruent and $P \cup P_1' \ldots \cup P_n'$ is scissors congruent with $Q \cup Q_1' \ldots \cup Q_n'$.*



Here is Hilbert's finite scheme for measuring area.

{measwork}

Lemma 9.9. *[Har00, §23] For any n and any triangulation of a figure by n triangles with base b_i and height h_i the sum $\Sigma_n \frac{b_i h_i}{2}$ is the same. That value is the geometric measure of the areas of a polygon. So, the equivalence relation imposed by 'same geometric measure' is well-defined.*

[16] The diagram is taken from [Hil71].

While Euclid-equality is transitive by definition, it is considerably more difficult [Har00, p 199-201] to prove that Hilbert-equality is transitive.

{equivequiv}

Theorem 9.10. *[Har00, §23] In any plane satisfying HP5, figures that have equal under either Hilbert's or Euclid's notion of equal area if and only if they have the same geometric measure.*

However the analytic method is an outlier.

Definition 9.11. *Two figures are* analytically equivalent *if they have the same analytic measure.*

The supplement contains an example showing the Archimidean hypothesis is essential for the next result.

{wbg}

Fact 9.12 (Wallace-Bolyai-Gerwien Theorem)**.** *Two polygons* in an Archimedean plane *are equidecomposable (scissors congruent) if and only if they have the same analytic measure.*

Note that the Archimedean hypothesis is essential. If the line BE in Figure 3 is infinite (Invert the segment $\overline{AB}$ created in Remark **??**.), while all lines in $ABCD$ are finite then the parallelograms $ABCD$ and $EBCF$ are not equidecomposable even though they are Hilbert and Euclid equal. This equivalence often appears in high school text books without making it clear that it requires a vastly stronger hypothesis than any of the other results on polygons.

{DehnS}

Fact 9.13 (Dehn-Sydler Theorem)**.** *Two polyhedra in $\Re^3$ are scissors congruent iff they have the same volume and the same Dehn invariant.*

Dehn [D] proved in 1901 that equality of the Dehn invariant is necessary for scissors congruence. Sydler proved the converse forty years later. We return to one of our original targets.

{htareaprop}

Theorem 9.14 (Euclid VI.I)**.** *If two triangles have the same height, the ratio of their areas equals the ratio of the length of their corresponding bases.*

Proof. Definition 9.4 gave the geometric measure of a triangle to be $\frac{bh}{2}$ and Theorem 9.10 showed geometric measure is equivalent to Euclid equal. So the result is just to realize that $A = \frac{bh}{2}$ can be read as 'the area is jointly proportional to the base and the height. $\square_{9.14}$

In Euclid this result holds for irrationals only by the method of Eudoxus, which is a precursor of the modern theory of limits, but did not envision the existence of arbitrary real numbers. He deduces side-splitter from the proportionality while Hilbert goes in the other direction[17]. In contrast the development here shows that for any triangles which occur in a geometry satisfying the axioms here[18] the areas and their ratios are represented by line segments in the field.

10 Archimedes and Dedekind

{ArchDed}

We quoted in Methodology 1.2, Hilbert's desire 'to choose for geometry a simple and complete set of independent axioms'. In this section we discuss several interpretations of 'complete'. The ability to obtain complete axioms for geometry differs radically depending on the logic in which the axioms are formulated. Thus, to describe the issue we must explore several choices of logic.

[17][Edu09] shows the area of one of two similar figures is r^2 times the area of the other, where r is the constant of proportionality between lengths. They deduce this from side-splitter. It was Al Cuoco of the CME team who alerted the first author to Euclid going in the other direction.

[18]Crucially, neither Archimedean, nor Dedekind complete, is assumed.

Given a collection of statements Φ about possible systems for geometry, there are several ways in which a subset Ψ of Φ can be thought complete for a collection of axioms T. Of course, each $\psi \in \Psi$ must be satisfied in each model of T. And the most natural notion of complete is is *negation complete*; for $\phi \in \Phi$, either ϕ or $\neg\phi$ is satisfied in every model of T. This would clearly be true if T had only one model (T is categorical). A weaker notion discussed in Methodology 1.3, T is *descriptively complete* [Det14] if T implies all the statements in our preexisting list of 'true geometrical statements'.

The main theme of the preceding sections is that Hilbert (1899) showed descriptive completeness of his first four groups of axioms (not only Euclid but establishing Descartes' analytic geometry). He was then writing in natural language. So he had no precise way of expressing negation completeness. He could however add additional axioms in order to establish i) completeness via categoricity ii) analytic geometry as conceived by [Ded63] (1888) in his attempts to provide a firm foundation for calculus. These additional axioms require either infinite disjunctions (Archimedes) or quantification over subsets (Dedekind cuts) (2nd order logic).

Roughly 20 years later, he developed his notion of formal logic. In his general formulation quantification is allowed over individuals, set of individuals, sets of sets of individuals .... He then observed [HA38] that groups I-IV are what we now call first order (for him, the restricted predicate calculus): quantification is only over individuals and only finite conjunctions and disjunctions are allowed in combining statements. Now the key distinction arises from Gödel's completeness theorem: For first order logic, there is a system of inference rules so that θ can be derived from T if and only if θ is true for every model of T. So for first order logic, negation completeness implies the stronger *deductive negation completeness*: for $\phi \in \Phi$, either ϕ or $\neg\phi$ is provable from the axioms of T. But this is impossible in 2nd order logic.

The first order theory T_{rcf} of the Cartesian plane over real numbers is negation complete; one adds to EG the infinitely many axioms that say of the coordinatizing field that every odd degree polynomial has a root [Tar59]. Alternatively, analogously to the Peano axioms for arithmetic, Dedekind cuts are formalized to hold only for first order definable cuts [TG99, p 185].

Hilbert's Group V (continuity axioms) contains two axioms. The Archimedean axiom is usually taken as a property of an ordered group (or field). However for geometry it says for any pair of line segments AB and CD there is a natural number n such that n copies of AB cover CD. Since the n is unbounded, this axiom is not first order but rather in a logic called[19] $L_{\omega_1,\omega}$. Note that the statement of the Archimidean axiom involves some notion of 'addition of lengths'. A standard result shows every Dedekind complete field is Archimedean.

Although expressed in an unusual way, Hilbert's *completeness axiom* is equivalent to Dedekind completeness or the least upper bound axiom in the theory of ordered fields. He proves that there is a unique model of all five of his axiom groups; namely the Cartesian field over $\Re$. This is his only use of these axioms to prove geometric theorems in [Hil62]. The other uses are for proving metamathematical (independence) results.

Euclid uses the Archimedean axiom in Book V on proportion and then to prove VI.2, the side-splitter theorem. As we have seen Hilbert establishes VI.2 on the basis of axiom groups I-IV which are all first-order.

[19]Quantification is allowed only over individuals but infinite conjunctions and disjunctions are allowed. Archimedes asserts: $\bigvee_n \phi_n(A, B, C, D)$ where ϕ_n says n copies of AB cover CD.

Definitely supplement

Theorem 10.1. *[Proof of Existence of non-Archimedean fields] There exists a non-archimedean field.* {nonarch}

Proof. We noted in Remark **??** that Tarski's complete extension of Euclidean geometry, T_{rcf} is the first order theory of the real field. It has models of arbitrary cardinality and most are non-Archimedean. Consider the set Σ of sentences: $\{n \times \overline{AB} < \overline{01}\}$ for $n \in \mathbb{N}$. Clearly every finite subset of Σ is satisfiable. By the compactness Theorem **??**, they are simultaneously satisfiable in some model M of T_{rcf}. Such an $\overline{AB} \in M$ is an infinitessimal. Moreover, no first order extension of EG (Euclidean Geometry; Notation **??**) is finitely axiomatizable [Zie82]. There are uncountably many first order completions of Euclidean geometry.

Hilbert gives a concrete proof of the existence of non-Archimedean fields, taking t to be infinite in an ordering of the rational function field $\Re(t)$. $\square_{10.1}$

supplement: As we noted in Remark **??** Birkhoff's axioms are phrased as in set theory as a complicated description of the field over the reals. With Hilbert's definition of the field, we can make this into a legitimate second order axiomatization.

Theorem 10.2. *Fix two points* $0, 1$ *on a Hilbert plane* M *and the line* ℓ *through them. Let* $<, +, \times$ *be the ordering relation and field operations defined on* ℓ *by Theorem* **??**. *Add the least upper bound axiom:* {formbirk}

$$(\forall X)(\exists y)(\forall x \in X)x < y \rightarrow (\exists z)(\forall w)(\forall x \in X)x < w) \rightarrow y \leq w.$$

The field on ℓ *is a complete ordered field and so is isomorphic to the reals.*

Proof. So clearly the ruler postulate holds on ℓ. But we know by Theorem 5.4.12 that the group of rigid motions acts transitively on lines so the ruler postulate holds on every line and so on M. $\square$

11 Non-Euclidean Geometry

{noneuc}

We showed in Section 7, specifically Methodology **??** that the theories of Euclidean geometry and fields were bi-interpretable. The same is true of Euclidean and hyperbolic geometry. In particular, Poincaire showed that one could interpret hyperbolic geometry in a disc on the Euclidean plane. A geometric argument analogous to that in Section 7 appears in [Har00, §39].

The switch from the old to the new view of geometry that we discussed in Comment **??** stemmed from the proof of independence of the parallel postulate. Most of the modern work on non-Euclidean geometry assumes the existence of a metric (distance function), works with the real field and is *not* done synthetically. However, [Har00] elaborates on some axiomatic non-Euclidean geometry. In neutral geometry, he proves there is a rectangle if and only if the sum of the angles of a triangle is two right angles and introduces an axiomatic trichotomy of semi-Euclidean, semi-hyperbolic, and semi-elliptic geometries depending on the order between the sum of the angles of a triangle and two right angles. Further, he proves that a semi-hyperbolic plane satisfying Hilbert's 'limiting parallel axiom' (*hyberbolic geometry*) defines a field.

UNFULLFILLED PROMISE OF FORMAL DEFINITIION OF INTERPRETING THEORIES

Rather surprisingly, since both hyberbolic geometry and Euclidean geometry are bi-interpretable with the real field, they are themselves bi-interpretable. This emphasizes that interpretation preserves not meaning but consistency.

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