Images in Mathematics

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Abstract

Mathematical images occur in lectures, books, notes, posters, and on the internet. We extend Kennedy's [Ken17] proposal for classifying these images. In doing so we distinguish three uses of images in mathematics.

1. iconic images
2. incidental images
3. integral images

An *iconic image* is one that so captures the essence of a concept or proof that it serves for a community of mathematicians as a motto or a meme for an area or a result. A system such as Euclid's can combine such apprehensions with other forms of logical inference and an image that is built into a system of exposition is called an *integral image*. An *incidental image* is an image used by a mathematician to reason with a particular concept.

In addition to this thematic characterization, we also explore one concept, infinity, in some depth by comparing representations of the infinite by mathematicians and artists.

In this paper we will investigate various uses of images by mathematicians and contrast the artistic and mathematical representation of the infinite. A quick glance through my library supports an intriguing observation about mathematics. The serious works of history contain at most few pages of photographs; the philosophy books are all text[1]; the mathematics book often contain images of various sorts. In high school and beginning college texts, the images largely serve to illustrate and thereby inculcate concepts. But others play different roles which we will try to explicate here.

We follow Kennedy [Ken17] in beginning a partial classification of images in mathematics by their use. Our categories are not intended as a rigid taxonomy, but rather as beginning a list of types of use. These categories may overlap. Alternatively, one might classify images by their subject; while we don't address this topic systematically, in Section 4, we contrast representations of infinity by artists and mathematicians. This discussion builds on images related to perspective considered in earlier sections We extend Kennedy's distinction [Ken17] among *Euclidean diagrams*, (using her terms), *incidental diagrams* and *architectural drawing* by adding the notion of an *iconic diagram*. We place the Euclidean diagram as a subcategory of our *integral images*, while

[1] The few exceptions are scattered illustrations in some books on the Philosophy of Mathematics.

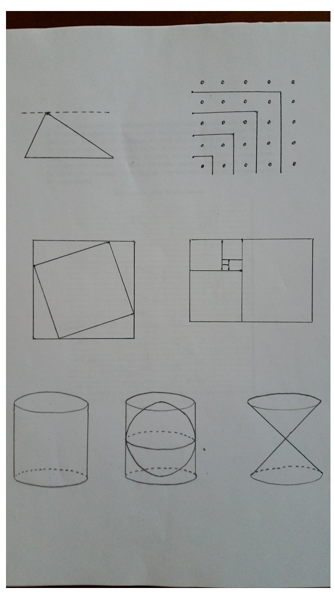
we adopt her usage of 'incidental'; we do not address her notion of architectural drawing which explores the relationship between 'visual' and 'manual' understanding. In Section 1 we explore by example, the notion of iconic image; in Section 2 we clarify the relationship incidental images with other categories. In Section 3 we elaborate on the integral nature of image in Euclidean geometry and in contemporary mathematics.

1 Iconic images

An iconic image so well represents an idea that little context is necessary. It contrasts with the integral images presented in Section 3, precisely because it brings out the idea without supporting context. In this section we display a number of iconic images from various fields of mathematics. Our last example illustrates that the notion of iconic is field-dependent.

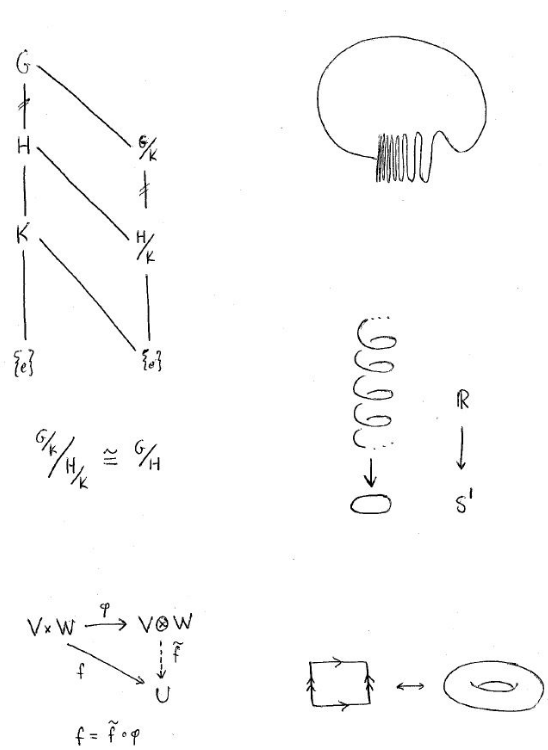
The next set of pictures were contributed by the mathematician John Boller. Each of the sketches represents an iconic result from centuries ago. These images have become iconic as they distill the essence of certain ideas for many people. For example, the upper left hand picture explains why the sum of the angles of a triangle is $180°$ (providing the parallel postulate holds). Similarly, the one labeled 'Eureka' encapsulates the Pythagorean theorem. The three at the bottom summarize Archimedes approach to the calculation of volumes. These diagrams have been emblematic for mathematicians for two millennia.

Iconic traces: John Boller-classical



In contrast, here are some sketches by the same author of some modern icons. The upper left represents the notion of a quotient group. This is a nineteeth century concept. This picture often appears in modern texts in abstract algebra along with the equation $(G/H)(H/K) = G/K$. This equation exploits the notation for a ‘quotient of two structures’ by using it in analogy to the the quotient of two numbers. The diagrams on the right represent ideas that became central in topology during the early twentieth century.

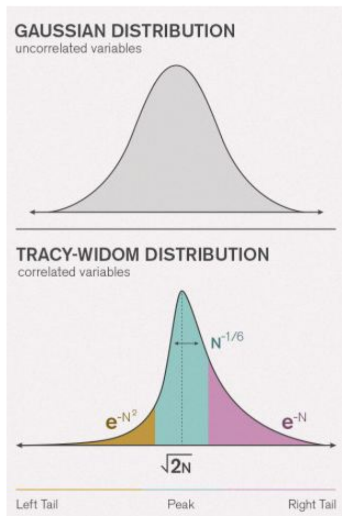
Iconic traces: John Boller–modern



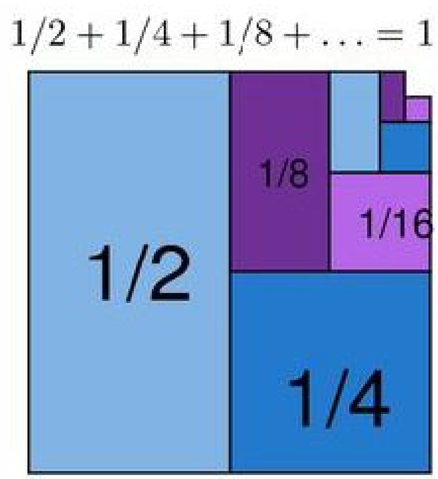
The lower left diagram is an example of the kind of category-theoretic diagram we discuss in the next section, the definition of a *tensor product*.

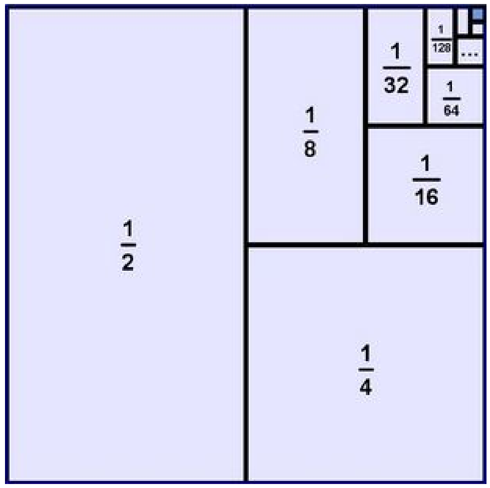
Iconic traces: Post-Modern

The upper picture is the iconic picture of 20th century statistics, a normal distribution for independently distributed events. The lower one represents the result when there are correlations among the variables. They were published in Quanta magazine 10/15/2014.



One type of iconic image is the so-called 'proof by picture', an image so clear that it 'immediately' convinces the reader of the proof of the proposition. Many useful examples occur in [Nel93]. But some care is needed; compare the following two diagrams that purport to demonstrate that $\Sigma_{1\leq n<\infty}\frac{1}{2^n} = 1$.



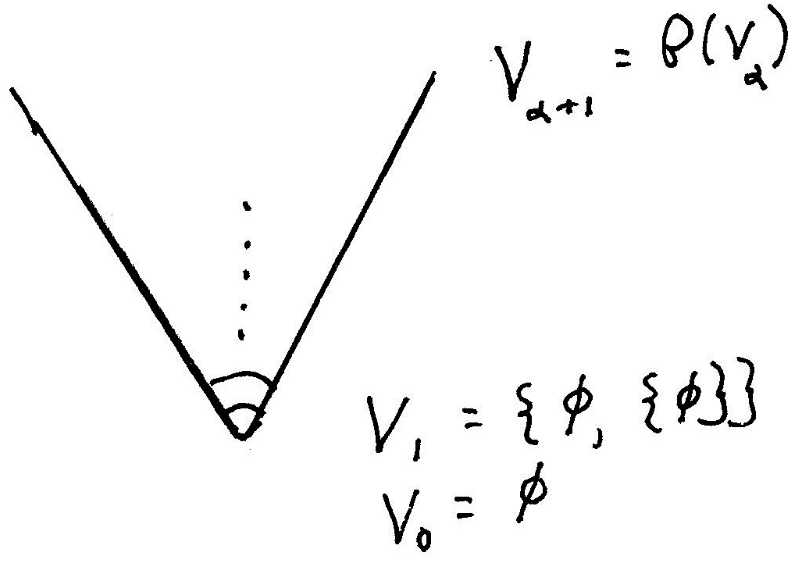


The three dots near the upper right hand corner of the second diagram are essential to indicate that this procedure does not stop. And in fact, after Bolzano (see below), we may think more care is needed to make this proof complete.

The next anecdote illustrates that 'iconic' should really be 'iconic' for a specified group of (perhaps all) mathematicians.

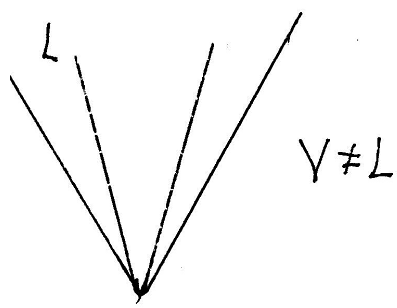
Iconic traces: The universe of sets: V

The modern conception of the cumulative hierarchy envisions all sets being generated by a sequence V_α where V_0 is the empty set and $V_{\alpha+1}$ is the power set of V_α. The iconic picture is vee-shaped.

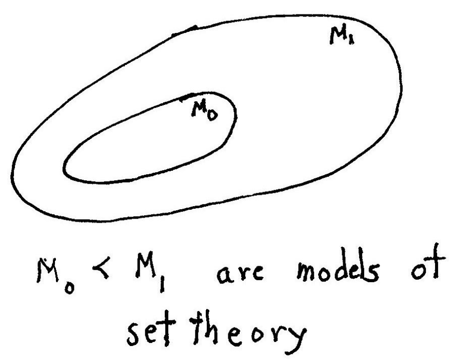


A standard way to show that a proposition ϕ (e.g. the continuum hypothesis) is consistent with ZFC is to construct a substructure of V which satisfies ϕ, an inner model. For example, the following picture, where L denotes the class of 'constructible sets' which satisfies the continuum hypothesis, illustrates the consistency of the continuum hypothesis.

Iconic traces: Inner Models



Iconic traces: Elementary extensions



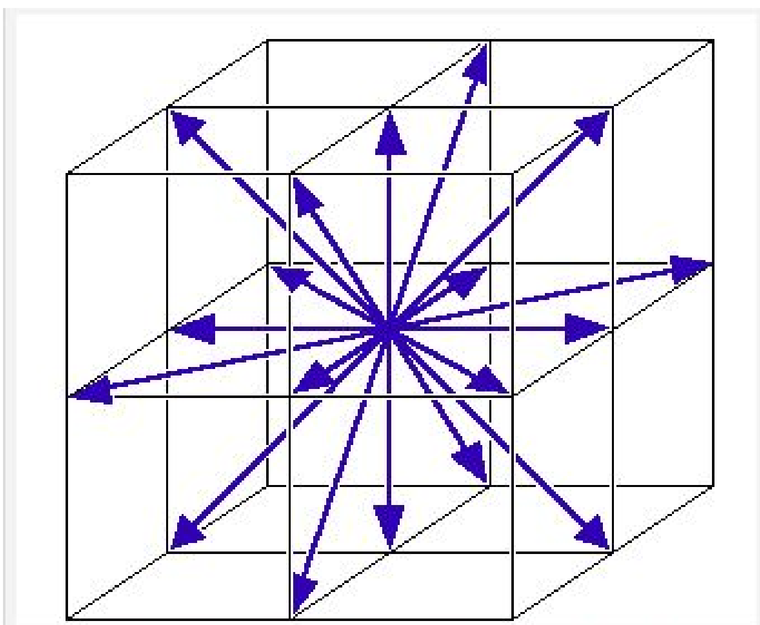
Model theorists have a different picture for a similar notion. The diagram above represents that M is an elementary submodel of N — they satisfy the same sentences. As a model theorist, I drew it on the blackboard for a group of set theorists. M_0 and M_1 were supposed to be models of set theory; the room exploded in laughter. As the referee pointed out, the set theoretic picture has a more refined meaning; it emphasizes that V is wider than L while they have the same height. But this was not an important feature of the model extension I was drawing. The fields have established certain cultural norms, dare I say meme, for representing the concept of model extension; these images reflect more specific aspects. But the particular image is expected even when that aspect is not salient. Even a single model of set theory is drawn (by set theorists) as a V.

Sasha Borovik reports a complementary example; the ability of the same diagram to illustrate different ideas. He wrote,

> Once upon a time, three of my postgraduate students worked on three completely different problems:
>
> 1. groups of finite Morley rank,
> 2. symplectic matroids and
> 3. probabilistic recognition of finite simple groups.
>
> I used to talk to each of them referring to the same picture that was sitting for a month or two on the blackboard in my office. It represented the root system of type B_3 – an eternal object. Could the same be done without a blackboard (or at least a whiteboard)? I doubt it.

Here is the diagram.

The root system B_3



Root system B_3. Picture by Maria Borovik

Our last example has a more social aspect. Pictured below is the T-shirt of the 'Forking Festival' held in Chicago for three years in the early 80's when I was writing the first full textbook in stability theory. This was deliberate attempt to both educate a group of young logician in a new technology and an attempt to popularize the subject. Thus, various images which had largely been incidental in the previous decade (and unknown before), are displayed to show the world the new field. While the individual images have iconic significance for the participant, the entire shirt proclaims a motto. The word 'forking' names in stability theory (an area of model theory, which itself is a branch of mathematical logic) an important generalization of the notion of algebraic independence which is a bulwark of modern algebra. In reaching the final formulation of this notion, Shelah successively defined: splitting, strong splitting, dividing. All of these words describe a 'bad behavior' of an extension of a type: more information has been introduced. That is, it is 'good' when the type p is a non-splitting extension of the type q. When he finally came on the right notion he asked Chang for another English[2] word in this family. Chang suggested forking and that stuck. However, when Lascar and Poizat approached the concepts, they looked first at the positive side and thinking of an extension as a 'fils' (son), the non-forking extension became the 'fils aîneé'. And Harnik translated 'fils aîneé' as heir. I mention this string of names to emphasize that finding a name for a concept is a crucial part of making the definition.

The T shirt (designed by Gisela Ahlbrandt who was a graduate student at the time) contains a number of themes. The top line is read, 'Forking dominates Chicago' as $p \triangleright q$ represents the technical relation of p dominates q. The tree of forking extensions plays a central role in the theory; here, the various leaves are names of important concepts. The crossed forks represent the visits each evening to one of the ethnic restaurants of Chicago. Here, the icons are being created for propaganda purposes; in advertising the field the workers in the (then very small) field of stability theory are telling the mathematical world some of the key ideas by diagrams which are incidental (as described below) to the workers but might become iconic (and some have).

Forking Dominates Chicago

[2]He had one in Hebrew.

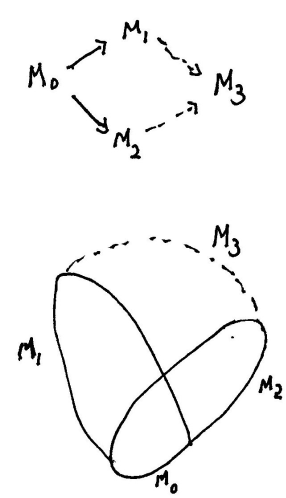
2 Incidental images

In this section we show some images that may stay on the mathematicians blackboard for months (perhaps altered slightly from day to day) and yet may or may not appear in the published paper. Crucially, even if published, the image is illustrative but not integrated. It is an aid to understanding rather than a step in the proof.

Kennedy wrote in two separated comments in her paper

> And there are the informal, illustrative drawings, as one might call them, the incidental drawings mathematicians create while working alone or in conversation with other mathematicians, or while giving lectures.
>
> These drawings are not proofs, though that claim has been made for them on occasion; and they are almost always discarded—in fact it seems wrong to preserve them, as if to circumscribe an epiphany, an intimate, rational moment, with a souvenir. [Ken17]

Our notion follows the first paragraph above, but not the second. Especially as it becomes easier to publish illustrations, incidental diagrams may appear in print. They may double as iconic images for a particular concept. But perhaps this hits on the crucial distinction. A particularly apt incidental diagram can be become an icon for a concept. But the proof (modulo the Manders-style revision discussed in Section 3) is a sequence of assertions. It is the ability to tinker with the diagram that allows one to reach this precision.



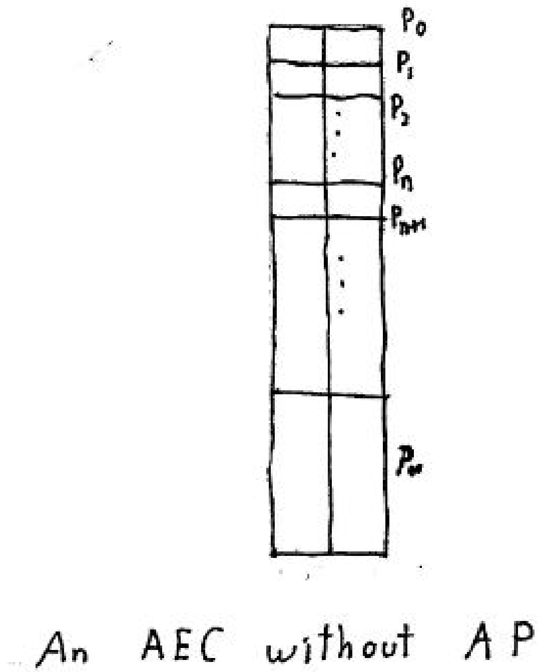
The Amalgamation Property

The amalgamation property, illustrated above, is a fundamental notion in model theory. A class of models $\boldsymbol{K}$ has the amalgamation property if for each embedding

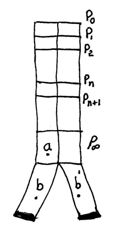
f_1, f_2 of a model $M_0 \in \boldsymbol{K}$ into models $M_1, M_2 \in \boldsymbol{K}$ there are a model M_3 and maps f_1', f_2' of M_1, M_2 into M_3 such that $f_1' \circ f_1$ and $f_2' \circ f_2$ agree on M_0. This property is true for first order theories but is an important contingent property in infinitary logic. These two representations are iconic; but the second will be embelished on the board with indications of obstructions to amalgamations that don't show up in the paper but help the author to find the proof.

I was trying to explain by email the example below to a colleague and could not make myself understood. Since this conference, where I intended to speak on the role of images, was approaching, I carefully drew and emailed the diagram below as I would have drawn them on the blackboard if he were present.

A mathematical conversation I



The diagram above shows the basic structure: an equivalence relation E with two classes (the two columns) and a descending chain of unary predicates P_n such that each annulus $P_n - P_{n+1}$ is finite and each equivalence class intersects the annulus in the same number of points. The 'type' p_∞ is satisfied by an element that is in all of the U_n. An infinitary axiom asserts that all realizations are E-equivalent. This picture was not convincing.

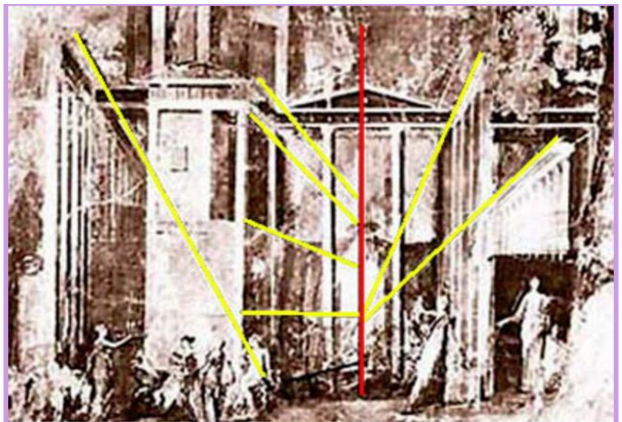


Showing the failure

The second diagram shows that for an element a in the ground model M_0 where no element satisfies p_∞, there are two extensions, one with a b such that $p_\infty(b)$ and $E(a, b)$ and another where $\neg E(a, b)$ holds. These two models cannot be amalgamated. This picture satisfied my colleague. Note this diagram is a variation on the iconic amalgamation diagram. I continued the straight lines of Conversation I, but by splitting the two possible extensions I have turned over the amalgamation diagram. The 'Y' is upside-down.

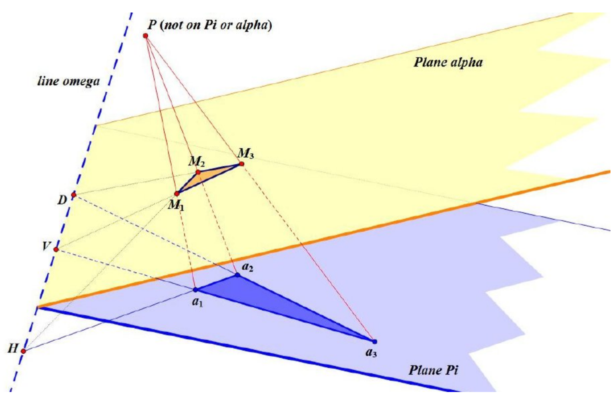
One of the prime interactions between mathematics and art is the notion of *perspective*. The use of *axial perspective* dates back to Etruscan times. A pair of lines that are intended to appear parallel are drawn to intersect and all such pairs intersect on the same line. A painting found at Pompeii[3] (below) illustrates this idea. Another clear example is Giotto's marriage at Cana which is easily available on-line. Compare various 'horizontal lines' on the two walls coming towards you and the grill work at the top.

[3]Image taken from a photo by Esther M. Zimmer Lederberg which is on her memorial web site with the illustrative yellow lines: `http://www.estherlederberg.com/EImages/Extracurricular/Renaissance+Baroque/Perspective+Art/Axial\%20Perspective.html`.



Vanishing "point" replaced by a "vanishing axis" (in red)
Yellow parallel lines forming a "fish bone" shape

In the renaissance Axial perspective was replaced by 'linear perspectivity' and the use of a vanishing point. The remarkable theorem of Desargues asserts that these are the same idea.



The 'vanishing point' of the artist becomes a 'center of a central perspectivity' in projective geometry. In the diagram[4] above, viewed as lying in the plane of the page, the triangles $\langle M_1, M_2, M_3\rangle$ and $\langle a_1, a_2, a_3\rangle$ are centrally perspective by point P and axially perspective by line ω.

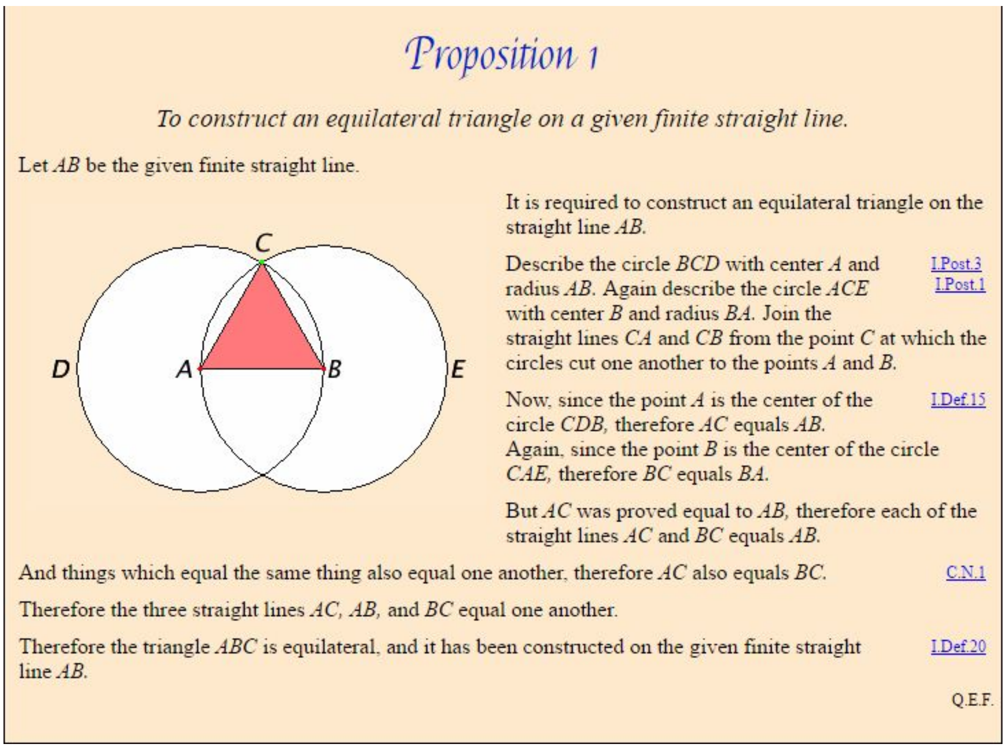
The 16th century Desargues theorem asserts that either perspectivity condition implies the other. Our diagram exhibits this theorem if we change our focus; focusing first on P, we see that central perspectivity; switching our attention to ω we see an axial perspectivity. This diagram illustrating the theorem is certainly iconic. But it and many variants served as incidental diagrams in developing a geometric proof of an interpretation of three-space into a Desarguesian plane Π. (Baldwin-Howard appendix to [Bal13]).

[4]We thank Andreas Mueller for this diagram.

3 Integral Images

Under various interpretations a diagram is taken as a heuristic or as integral part of the mathematical discourse. As Ken Manders [Man08] points out, from the time of Euclid until the mid-nineteeth century, diagram was taken as an essential component of the proof. The very first proposition in Euclid illustrates the issue.

Euclid I.1: Construct an equilateral triangle[5]



Proposition 1

To construct an equilateral triangle on a given finite straight line.

Let *AB* be the given finite straight line.

It is required to construct an equilateral triangle on the straight line *AB*.

Describe the circle *BCD* with center *A* and radius *AB*. Again describe the circle *ACE* with center *B* and radius *BA*. Join the straight lines *CA* and *CB* from the point *C* at which the circles cut one another to the points *A* and *B*. I.Post.3 I.Post.1

Now, since the point *A* is the center of the circle *CDB*, therefore *AC* equals *AB*. Again, since the point *B* is the center of the circle *CAE*, therefore *BC* equals *BA*. I.Def.15

But *AC* was proved equal to *AB*, therefore each of the straight lines *AC* and *BC* equals *AB*.

And things which equal the same thing also equal one another, therefore *AC* also equals *BC*. C.N.1

Therefore the three straight lines *AC*, *AB*, and *BC* equal one another.

Therefore the triangle *ABC* is equilateral, and it has been constructed on the given finite straight line *AB*. I.Def.20

Q.E.F.

Euclid does not have an axiom asserting that two overlapping circles must intersect. Manders (page 66 of [Man08]) asserts: 'Already the simplest observation on what the texts do infer from diagrams and do not suffices to show the intersection of two circles is completely safe.' Manders distinguishes 'exact' and 'inexact' or topological properties. An exact property is one that requires measurement - an assertion that two line segments are congruent. Such statement are always in the text in 2000 years of mathematicians following Euclid's precepts. On the other hand when two lines cross in the diagram (or circles cross) then there is a point of intersection. This convention was discarded by (in particular Pasch and Hilbert) for what seemed to be a good reason. The analysis of the 19th century had discovered 'analytic constructions' that would build

[5]This diagram is taken from the site `http://aleph0.clarku.edu/~DJoyce/java/elements/bookI/propI1.html`.

from rational numbers transcendental numbers that were not on the rational line. Even easier, the construction of $\sqrt{2}$, leaves the rational numbers. In particular, in marked contrast to the view down to Descartes, the 19th century and later view is that there is a straight line segment of length π. But the collection of numbers constructed with a ruler and compass does not contain such a segment. The line of Euclid is found in the plane over the smallest field F such that if $a \in F$ and $a > 0$ then $\sqrt{a} \in F$. So as Manders said, Euclid was perfectly safe.

Thus, the diagrams of Euclid implicitly contain the information that they were constructed by carefully prescribed rules. These rules have been formally reconstructed by [ADM09, Mil07]. See the discussion in [Ken17]. The nineteenth century uncertainty stemmed not from a flaw in Euclid but from a not fully conscious extension of the means of construction.

Several of the examples of iconic images were of 'picture proofs'; a proposition that is immediately clear from a picture. Kennedy [Ken17] describes this as 'single text view' and gives an example concerning the transitivity of betweenness from [ADM09]. Under the 'single-text' view, the particular image is immediately perceived to verify a conclusion. Following Manders, we argue here that Euclid integrates the text and diagram to form a coherent proof system; we call the diagrams which appear in such a proof integral images. We discuss category theory as another example below.

Kennedy gives a more detailed analysis of the 19th century rejection of the integral diagram emphasizing the important role of Bolzano in stimulating this shift. The intermediate value theorem (IVP) asserts that a continuous function on a closed interval must take on each value in the closed interval $[f(a), f(b)]$. Kennedy summarizes Bolzano's view and its reception.

> To summarize Bolzano's view—one that is now entrenched in the mathematical community: construed as a geometrical truth, the IVP follows immediately from the relevant diagram. Construed as a general mathematical truth, the IVP requires proof.

This assertion of course depends on the meaning of 'geometrical' and 'mathematical'. The revolution in the nineteenth century greatly expanded the notion of a mathematical curve. Perhaps there was a time when mathematical curve and geometrical curve could be considered synonomous[6]. But if Euclidean geometry is to be regarded on its own terms, these two notions of curve must be distinguished. The curves discussed in *The Elements* were (at least, primarily) straight lines or conics. For all of these, the intersection axiom is indeed clear and the picture does not lie. It seems that Bolzano is adopting that meaning of 'geometry'. But if the theorem construed for arbitrary continuous functions a careful proof must be given and at least some modern mathematicians would consider such curves 'geometric'.

This distinction (what does 'geometric' mean?) remains rough in contemporary mathematics. Precise definitions exist but they vary by field. For an algebraic geometer, a curve (more generally, a higher dimensional surface or *algebraic variety*), is defined by a finite conjunction of polynomial equations. For a student of complex manifolds the natural object, an *analytic variety*, is defined by a finite conjunction of zero-sets

[6]This is somewhat dubious; the distinction between geometrical and mechanical arises even in antiquity.

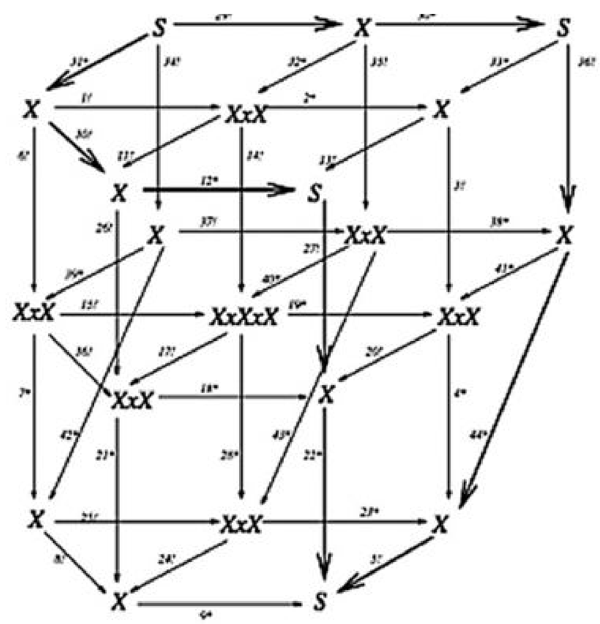
of analytic function. Astonishingly, Chow's theorem asserts that an analytic subspace of complex projective space that is closed (in the ordinary topological sense) is an algebraic subvariety. Which is geometric?

Modern algebraic topology and category theory rely heavily on the use of diagrams. A short exact sequence is written:

$$0 \to A \to C \to B \to 0;$$

It expresses that B is the quotient of C by A. In practice these diagrams become incredibly complicated. For an 'artistic' example see the movie[7] *It's my turn*.

The following diagram is taken from [Voe14].



For the convenience of further reference we numbered all the arrows. The right vertical face of the diagram is the diagram (2) defining the 2-morphism $Id \to \Omega\Sigma$ and the upper horizontal face is the diagram (1) defining the 2-morphism $\Sigma\Omega \to Id$. The whole diagram is the union of the front part which

This three-dimensional diagram is an example of the kind of "formulas" that Voevodsky would have to use to support his arguments about 2-theories.

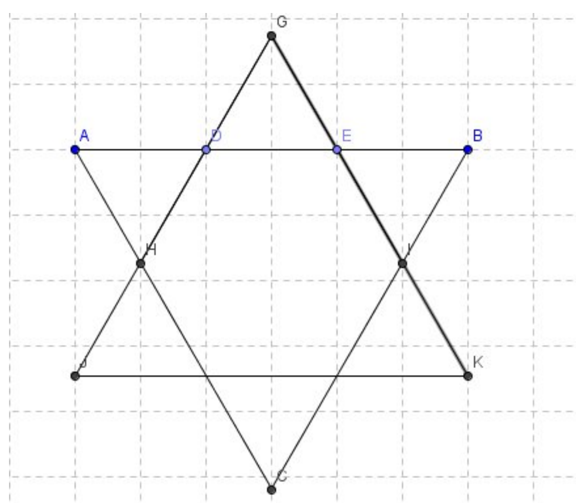
In this paper, Fields medalist Voevodsky explains his mission to establish a firmer basis for mathematics because of errors (listed in the paper) in major articles by senior mathematicians involved in describing calculations involving such diagrams.

[7] Jill Clayburgh proves the *Snake Lemma* at `\{https://www.youtube.com/watch?v=etbcKWEKnvg`.

The fundamental appeal of integral images is that the diagram is surveyable and gives direct evidence of truth. Even though it is possible to draw, at least with a computer, the diagrams described in [Voe14]; they are no longer surveyable. Thus Voevodsky has turned to computer proof.

We pass from the philosophical discussion of the role of images in proof to a question. Raphael paid tribute to the Greeks in his *School of Athens* including a diagram being drawn by Euclid. What theorem is that diagram meant to support?

What is Euclid doing in the lower right-hand corner[8]? Fichtner [Haa12] suggests that he was demonstrating properties of a Star of David:



[8]Photo by Donald Wink

More precisely, Hass proposes (next diagram) that the problem is to prove Fichtner's theorem: in a Star of David, the diagonal $D1$ has equal length to the chord $L2$ (which is not drawn on Euclids slate).

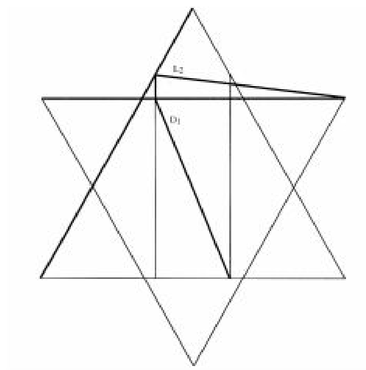
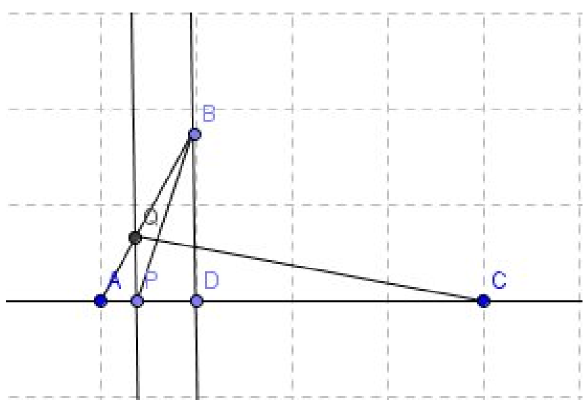


Figure 4: Fichtner's theorem: $D_1 = L_2$.

In a workshop for high school teachers, I, Andreas Mueller, and Donald Wink developed from Fichtner's paper an activity to bring to life the construction in Raphael's painting. Following, Hass [Haa12] we asked the teachers to prove the following theorem



Fichtner's theorem: In the above diagram suppose BD and QP are perpendicular to AC and that the length of AB is twice that of BD and the length of AC is four times that of BD. Show that the length of QC is twice the length of PB.

Hass raised more intriguing questions. He explores how Raphael's knowledge of perspective allows the inference that the diagrams of our discussion of Fichtner's problems are plausible, although the diagrams in the picture are only similar. He also discusses alternative mathematical explanations of the painting.

4 Images of Infinity

In this section we compare some artistic images of infinity with incidental/iconic images of particular infinite sets drawn by mathematicians. The most obvious distinction is that the mathematicians are expressing a notion that was accepted[9] only in the last one hundred and fifty years: there are different sizes of infinity. But, even without distinguishing sizes of infinity, the mathematical motifs underlying different artistic presentations may differ.

Although Andrew Wyeth apparently painted this picture in New England, it evoked Kansas for me and Dorothy staring off after the storm. Rather than the traditional tending to vanishing point to represent immense distance, the big houses are set against the horizon; there is no fading away, just the virtually featureless sky.

`https://www.moma.org/collection/works/78455`

Infinity is empty.

This mirrored room produced by Kusama[10] applies both the perspective effect of suggesting parallelism by lines meeting in the distance and contrasting the symmetry of these lines with the more random placement of the stars.

`http://www.mirrorhistory.com/picture/infinity-mirrored-rooms/`

Infinity is not so empty after all.

In *Citizen Kane*, Welles gave us an infinite sequences of images of Kane — Hearst? But in this still the repetition is precise; there is none of the randomness in Kusama's image. There is perfect symmetry as Kane persists forever.

[9]The scholastic Grossteste had long ago envisioned the concept of orders of infinity [Fre54].

[10]Infinity Mirrored Room The Souls of Millions of Light Years Away by Yayoi Kusama — Photo courtesy of Kusama.Yayoi.

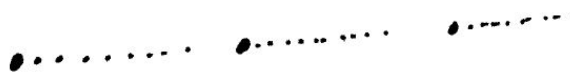
Mathematicians have much more detailed and diverse pictures of infinity. Since, Cantor there are ways to recognize many levels of infinite. In text it is easy to describe this in terms of two operations: taking unions and 'there is always one more'. But forever is hard to visualize so diagrams can only represent the beginning of the sequence. ω is the first infinite ordinal: $1, 2, 3, \ldots$ But it is possible to add infinities. Here are some visualizations I drew.

ω

ω

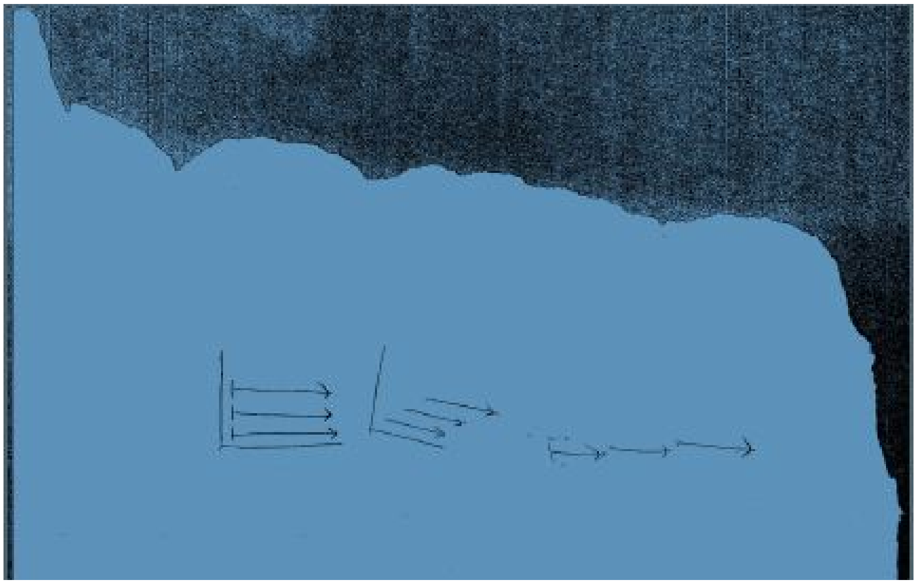
$\omega + \omega$

$\omega + \omega + \omega$



Marynthe Malliaris envisions a relationship. Why is $\omega + \omega + \omega = \omega \times 3$? And explains it on a napkin.

$\omega \times 3$

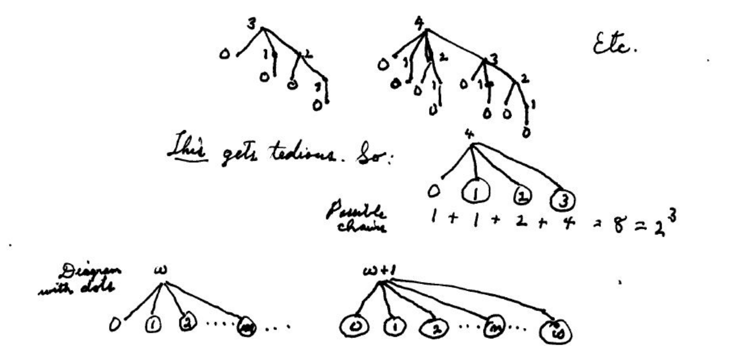


Those were intuitive representations of small infinite ordinal numbers. More formally, an ordinal number is a linear order such that every decreasing sequence is finite. ω is The first infinite ordinal is denoted ω. Bill Howard phrases the problem of showing a linear order as an ordinal graphicall: Envision descending sequences.

He describes this as Gödel's Game. In order that the descending chain principle for the set of ordinals $\leq \alpha$ be *intuitively understandable*, we must be able to

> 'Survey the various structural possibilities that obtain for [the] descending sequences.'

Displaying descending sequences



We thank Bill Howard for telling us of Paul Cohen's intelligence test: for what ordinals can you visualize the termination of descending chains witnessing well-foundedness of the ordinal? Cohen claimed ϵ_0, the least ordinal such that $\epsilon = \omega^\epsilon$. Try it, I don't get nearly that far.

Giaquinto [Gia08] explores this 'cognition of structures' in more detail with a stricter view of what people can grasp. He carefully examines, beginning with finite structures, the process of abstracting from a single diagram to equivalence classes called 'visual images' and then at the next level 'visual categorical specification' (where for example our normal picture of a binary tree of height 2 is equivalent to dividing a line segment in half and repeating on each subsegment). Then he argues that an infinite structure such as $(\omega, <)$ can be visualized by inductively repeating the picture. And he explains the strategy of the 'Cohen game' described above. In contrast to finite and 'short' infinite well-orders, in his view a dense linear order does not admit such a 'visual grasp'. Our notion is more general; we assert that understanding how to repeat a construction can lead to a 'grasp' of an infinite object and in this sense we have similar grasps of $(\omega, <)$ and $(Q, <)$. That is, there is no distinction between a step adding points at the end to form $(\omega, <)$ and a step adding a point between each pair of consecutive elements to form $(Q, <)$. Our grasp of the reals is somewhat more tenuous since it assumes the filling of continuum many cuts, but most mathematicians don't agonize over their grasp of the linear ordering of the reals.

However, all these structures have cardinality at most the continuum. There are few strong intuitions of structures with cardinality greater than the continuum. However, there is a crucial exception to this remark. It is rather easy to visualize a model that consist of copies of single countable or finite object. Consider a vocabulary with a unary function f. Assert that $f(x)$ never equals x but $f^2(x) = x$. Then any model is a collection of 2-cycles. On the one hand we have the notion that there are models of arbitrarily large cardinality but we have no really different image distinguishing among the models of different large cardinalities. This situation generalizes when the number of disjoint copies of the same structure is replaced by the dimension of a vector space or field. Thus we might consider the class of structures A_κ, a direct sum of κ copies of Z_2. The isomorphism type of the model depends solely on the number κ of copies (and not at all on the internal structure of the cardinal κ). That is, the class is κ-categorical.

In this last section we have shown how mathematicians devise specific images to sharpen their grasp of the many facets of infinity.

References

[ADM09] J. Avigad, Edward Dean, and John Mumma. A formal system for Euclid's elements. *Review of Symbolic Logic*, 2:700–768, 2009.

[Bal13] John T. Baldwin. Formalization, primitive concepts, and purity. *Review of Symbolic Logic*, 6:87–128, 2013.

[Fre54] Anne Freemantle. *The Age of Belief*. The Mentor Philosophers. New American Library of World Literature, 1954. Grosseteste quoted page 134.

[Gia08] M. Giaquinto. Cognition of structures. In P. Mancosu, editor, *The Philosophy of Mathematical Practice*, pages 198–256. Oxford University Press, 2008.

[Haa12] Robert Haas. Raphael's school of athens: A theorem in a painting? *Journal of Humanistic Mathematics*, pages 2–26, 2012. DOI: 10.5642/jhummath.201202.03 http://scholarship.claremont.edu/jhm/vol2/iss2/3.

[Ken17] Juliette Kennedy. Three moments in the life of the mathematical diagram: Notes on the syntax/semantics distinction. In L. L. de Freitas, N. Sinclair, and A. Coles, editors, *Mathematical Concepts*. Cambridge University Press, 2017.

[Man08] K. Manders. Diagram-based geometric practice. In P. Mancosu, editor, *The Philosophy of Mathematical Practice*, pages 65–79. Oxford University Press, 2008.

[Mil07] N. Miller. *Euclid and his Twentieth Century Rivals: Diagrams in the Logic of Euclidean Geometry*. CSLI Publications, 2007.

[Nel93] Roger Nelson. *Proofs without Words: Exercises in Visual Thinking*. Mathematical Association of America, Providence, 1993. Classroom Resource Materials.

[Voe14] V. Voevodski. The origins and motivations of univalent foundations; a personal mission to develop computer proof verification to avoid mathematical mistakes. 2014. Website: https://www.ias.edu/ideas/2014/voevodsky-origins.