Strongly minimal Steiner Systems: Model Theory, Universal Algebra, Combinatorics
UIC logic seminar

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Three. Goals

1. Context:
   1. Fraïssé constructions are explored in infinite combinatorics.
   2. Hrushovski refined the construction to solve problems of Zilber and Lachlan.
   3. Baldwin and Paolini modified that construction to find ‘strongly minimal’ Steiner systems.
2. Today: we discuss the combinatorial consequences of construction 3) and variants.
3. Diversity and Fine Structure:
   1. Illustrate many of the variations on the construction.
   2. Gesture at the proof that many (most??) strongly minimal sets given by an *ab initio* Hrushovski construction do not admit elimination of imaginaries and have essentially unary definable closure

Thanks to Joel Berman, Gianluca Paolini, Omer Mermelstein, and Viktor Verbovskiy.

Steiner Systems

A Steiner system with parameters t, k, n written $S(t, k, n)$ is an n-element set S together with a set of k-element subsets of S (called blocks) with the property that each t-element subset of S is contained in exactly one block.

We always take $t = 2$ and allow infinite n.

Some History

For which n's does an $S(2, k, n)$ exist?
for $k = 3$

Necessity:
$n \equiv 1$ *or* $3 \pmod 6$ is necessary.
Rev. T.P. Kirkman (1847)

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Sufficiency:
$n \equiv 1$ *or* $3 \pmod 6$ is sufficient.
(Bose $6n + 3$, 1939); Skolem ($6n + 1$, 1958)

Linear Spaces

Definition: linear space

The vocabulary contains a single ternary predicate R, interpreted as collinearity. A linear space satisfies

1. R is a predicate of sets (hypergraph)
2. Two points determine a line

α is the iso type of $(\{a, b\}, \{c\})$ where $R(a, b, c)$.

Groupoids and quasi-groups

1. A groupoid (magma) is a set A with binary relation $\circ$.
2. A quasigroup is a groupoid satisfying left and right cancelation (Latin Square)
3. A Steiner quasigroup satisfies $x \circ x = x, x \circ y = y \circ x, x \circ (x \circ y) = y$.

The connection between Steiner systems and quasigroups

1. Every Steiner triple system is a quasigroup.
   I.E. R is the graph of $*$.
2. Every p^n-Steiner system *admits* a compatible quasigroup structure. [GW75]
3. The [BP21] strongly minimal p^n-Steiner systems are not quasigroups (unless $p^n = 3$).
4. There are strongly minimal Steiner groups $(A, R, *)$, that induce q-Steiner systems for every prime power q.

Constructing generic models

$\leq$-amalgamation Classes

A $\leq$-amalgamation class $(\boldsymbol{L}_0^*, \leq)$ is a collection of finite structures for a vocabulary σ (which may have function and relation symbols) satisfying:

1. $\leq$ is a partial order refining $\subseteq$.
2. $\leq$ satisfies joint embedding and amalgamation.
3. $A, B, C \in \boldsymbol{L}_0^*$, $A \leq B$, and $C \subseteq B$ then $A \cap C \leq C$.
4. $\boldsymbol{L}_0^*$ is countable

Theorem

For a $\leq$-amalgamation class, there is a countable structure M, the *generic model*, which is a union of members of $\boldsymbol{L}_0^*$, each member of $\boldsymbol{L}_0^*$ embeds in M, and M is $\leq$-homogeneous.

For Fraïssé, the language is finite relational, the class is closed under substructure, and $\leq$ is $\subseteq$.

Existentially closed 3-Steiner Systems

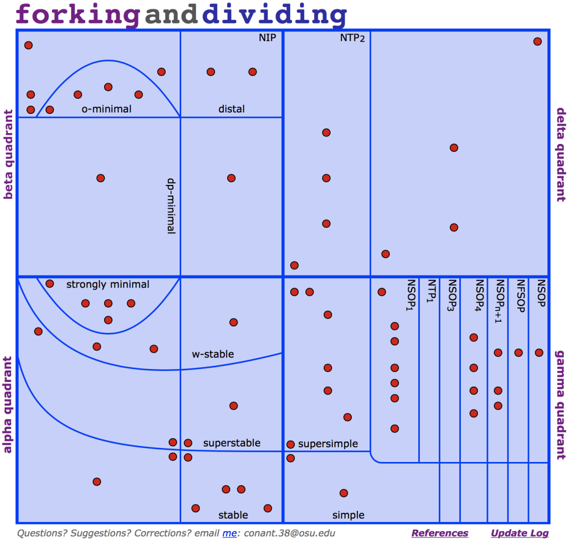
Barbina-Casanovas

[BC19] Consider the class $\tilde{\boldsymbol{K}}$ of finite structures (A, R) which are each the graph of a Steiner quasigroup.

1. $\tilde{\boldsymbol{K}}$ has ap and jep and thus a limit theory T^*_{sq}.
2. T^*_{sq} has
   1. quantifier elimination
   2. $2^{\aleph_0}$ 3-types;
   3. the generic model is prime and locally finite;
   4. T^*_{sq} has TP_2 and $NSOP_1$.

[BC19]

Classification of first order theories



Omitting classes of Steiner quasigroups

Horsley- Webb

Consider the class $\tilde{\boldsymbol{K}}$ of finite structures $(A, *)$ which are Steiner quasigroups that are F-free (omit a family $\boldsymbol{F}$ of finite nontrivial STS) and good (there exists an $A \in \boldsymbol{K}$ which neither extends nor embeds in any member of $\boldsymbol{F}$).

1. $\tilde{\boldsymbol{K}}$ has ap and jep and thus
2. $\tilde{\boldsymbol{K}}$ has a countable locally finite generic model.

On locally finite quasigroups their homogeneity is the model theorists ultrahomogeneity. Thus their construction gives $2^{\aleph_0}$ countable ($\aleph_0$ categorical Steiner systems.

Question

Where do they fit on the map?

If $F = \emptyset$, this is T^*_{sq}. The others should be similar.

Strongly minimal Steiner Systems

Definition

A *Steiner* $(2, k, v)$*-system* is a linear system with v points such that each line has k points.

Theorem (Baldwin-Paolini)[BP21]

For each $k \geq 3$, there are an uncountable family T_μ for $\mu \in \mathcal{U}$, of strongly minimal $(2, k, \infty)$ Steiner-systems.

$\mathcal{U}$ will be defined later; it guarantees amalgamation.

The generic is a union of finite relational structures but contains few finite quasigroups.
There is no infinite group definable in any T_μ.

Omitting configurations in Steiner systems

Pasch Configuration

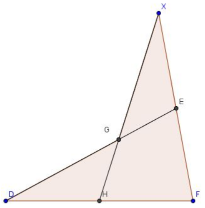


Figure: Pasch configuration: $\mathcal{P}$

In an associative STS any 3 non-collinear points generate a Pasch configuration.

Definition

Let X be finite partial Steiner system. A Steiner system (M, R) is *anti-X* if there no embedding of X into M.

[HW21] ask, Do the finite anti-Pasch triple systems form an amalgamation class?

Model theorists' Pasch

In a strongly minimal structure M, interpret collinearity as algebraic closure. Then the Morley rank of a non-collinear triple as 3, and that of a collinear triple as 2.

Group configuration theorem (roughly)

M has an instance of the Pasch diagram if and only if it defines an infinite group.

Contrasting theorems

1. The standard Hrushovski example and the B-Paolini Steiner systems omit the 'model theoretic' Pasch. So R is not the graph of a quasigroup.
   However, they will have instances of the combinatorial Pasch (e.g. Fano plane).
2. One can modify the amalgamation class so there are strongly minimal anti-Pasch (combinatorial sense) strongly minimal Steiner triple systems. [Bal22, Theorem 3.6].

Mitre and Mia

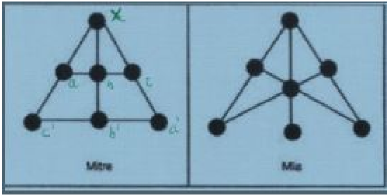


Figure: Mitre and mia configurations

[Fuj06]: The $(5, 7)$-configuration, Mitre, represents the left self-distributive law:

$$x(ab) = (xa)(xb).$$

If the $(5, 7)$ configuration MIA is realized, left multiplication does not preserve lines.

By constructing ∞-sparse configurations below we simultaneously omit the Pasch, mitre, and mia configurations.

Hrushovki's basic construction vs Steiner

Example

1. σ has a single ternary relation R;
2. $\boldsymbol{L}_0$: All finite σ-structures
   finite linear spaces
3. $\epsilon(A)$ is $|A| - r(A)$, where $r(A)$ is the number of tuples realizing R.
   $\delta(A) = |A| - \sum_{\ell \in L(A)}(|\ell| - 2)$.
4. $A \in \boldsymbol{L}_0^*$ if $\epsilon(B) \geq 0$ for all $B \subseteq A$.
   Replace ϵ by δ.
5. **U** is those μ with $\mu(A/B) \geq \epsilon(B)$.
   $\mu(\boldsymbol{\alpha}) = q - 2$ gives line length q.

Definition

A Steiner triple system (M, R) is ∞-sparse if there is no $A \subseteq M$ with $|A| \geq 6$ and $\delta(A) = 2$.

Blocking ∞-sparse configurations

[CGGW10, page 116] construct by induction continuum many countable ∞-sparse configurations.

Definition

Let $\boldsymbol{L}_0^{sp}$ be the subclass of $\boldsymbol{L}_0$ (linear spaces) such that for every $B \subseteq A$:

$$(\#)\ |B| > 1 \ \rightarrow\ \delta(B) > 1\ \&\ |B| > 3 \rightarrow \delta(B) > 2.$$

Theorem

The system $(\boldsymbol{K}_0^{sp}, \leq)$ has $\leq$-amalgamation. And so for any $\mu \in \mathcal{U}$, $\boldsymbol{K}_\mu^{sp}$ has $\leq$-amalgamation. So there are $2^{\aleph_0}$ strongly minimal sparse 3-Steiner systems of every infinite cardinality.

So this also blocks mia and mitre.

Cycle and Path Graphs

Cycle graph in STS

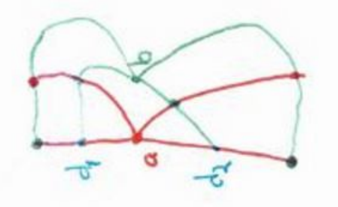


Figure: Cycle graph in STS

Extends to infinite STS ([CW12])

Path in 4-Steiner system

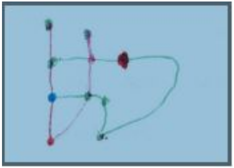


Figure: path graph in 4-Steiner System

Paths and Fans have dimension 1.

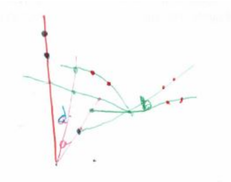


Figure: fan in 4-Steiner System

Strongly Minimal Theories

STRONGLY MINIMAL

Definition

T is strongly minimal if every definable set is finite or cofinite.

e.g. acf, vector spaces, successor

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Definition

a is in the algebraic closure of B ($a \in \mathrm{acl}(B)$) if for some $\phi(x, \mathbf{b})$:
$\models \phi(a, \mathbf{b})$ with $\mathbf{b} \in B$ and $\phi(x, \mathbf{b})$ has only finitely many solutions.

Theorem

If T is strongly minimal algebraic closure defines a matroid/combinatorial geometry.

Combinatorial Geometry: Matroids

The abstract theory of dimension: vector spaces/fields etc.

Definition

A closure system is a set G together with a dependence relation

$$cl : \mathcal{P}(G) \to \mathcal{P}(G)$$

satisfying the following axioms.

A1. $cl(X) = \bigcup\{cl(X') : X' \subseteq_{fin} X\}$
A2. $X \subseteq cl(X)$
A3. $cl(cl(X)) = cl(X)$

(G, cl) is pregeometry if in addition:
A4. If $a \in cl(Xb)$ and $a \notin cl(X)$, then $b \in cl(Xa)$.

If $\mathrm{cl}(x) = x$ the structure is called a geometry.

Usually this acl pre-geometry is not definable.

Towers

A prime model of a theory T is the unique model that can be elementarily embedded in each model.

If T is strongly minimal there is a tower (elementary chain: $M_n \prec M_{n+1}$) $(\langle M_j \colon 0 \leq j < \omega + 1 \rangle)$ of countable models of T, with M_0 the prime model; then M_ω is isomorphic to the generic structure $\mathcal{G}_{\mu,V}$ [BP21, Lemma 5.29].
One might think each M_n is prime with an acl-basis of cardinality n. This is true when $\mathrm{acl}(\emptyset)$ is infinite; but not in general.

No perfect strongly minimal Steiner systems

An STS is perfect if each cycle graph G(a, b) has a single cycle

Perfect infinite STS exist. [CW12]

Let $R\text{-}\mathrm{cl}(X)$ denote the subquasigroup generated by X.

None of these strongly minimal Steiner systems are perfect

In these strongly minimal examples for finite X, $\mathrm{acl}(X) - R\text{-}\mathrm{cl}(X)$ is infinite.
QED

Finite and infinite (pseudo-cycles)

Results

$\mathrm{acl}_M(\emptyset) \neq \emptyset$

1. If $\mathrm{acl}_M(\emptyset) \neq \emptyset$ there are infinitely many disjoint (over $\mathrm{icl_M(a, b)}$) finite pseudocycles in $G_M(a, b) = \mathrm{acl}_M(a, b) - \mathrm{icl(a, b)}$.
2. If $\mathrm{acl}(a, b) \neq M$, all paths in $M - \mathrm{acl}(a, b)$ are infinite.
3. If $M \not\leq N$ and $\dim(N/M) \geq 1$, M is covered by a union of 'fans' that each intersect at most one other fan.

Uniform Path graphs

[

Uniform model] A model $(M, *, R)$ of $T^q_{\mu', V}$ is *uniform*, if for any (a, b), (a', b'), $G_M(a, b) \simeq G_M(a', b')$.

Lemma

1. If $(M, *, R)$ is a model of a theory T generated by a Hrushovski class of linear spaces such that every two element set A satisfies $A \leq M$, the automorphism group of $(M, *, R)$ acts 2-transitively on (M, R).
2. Clearly, if the automorphism group of $(M, *, R)$ acts 2-transitively on $(M, *, R)$, $(M, *, R)$ is uniform.

Key point If every two element set A *in the prime model* satisfies $A \leq M$, then it holds in all models.

Constructing Strongly minimal Steiner systems

The trichotomy

Zilber Conjecture

The acl-geometry of every model of a strongly minimal first order theory is

1. disintegrated (lattice of subspaces distributive)
2. vector space-like (lattice of subspaces modular)
3. 'bi-interpretable' with an algebraically closed field (non-locally modular)

Hrushovski gave a method of constructing strongly minimal sets that have flat geometries and admit no associative binary function.

Zariski Geometries aim at canonical structures with more restrictions.

The flexibility of the Hrushovski construction

The ‘Hrushovski construction’ actually has 5 parameters:

Describing Hrushovski constructions

1. σ: vocabulary $\boldsymbol{L}_0^*$ is the collection all finite σ-structures. $\boldsymbol{L}^*$ is the collection all σ-structures.
2. $\boldsymbol{L}_0$: A $\forall\exists$ axiomatized subclass of $\boldsymbol{L}_0^*$
3. δ: A function from $\boldsymbol{L}_0^*$ to $\mathbb{Z}$ that induces a dimension on the definable subsets of the generic.
4. $\boldsymbol{L}_0 \subseteq \boldsymbol{L}_0^*$ defined using δ.
5. L_μ: the $A \in \boldsymbol{L}_0$ satisfying that the number of 0-primitive (B/C) are bounded by $\mu(B/C)$.

To organize the classification of the theories choosing nice classes **U** of μ yields a collection of T_μ with similar properties.

For Hrushovski, the ‘standard’ **U** is $\mathcal{U} = \{\mu : \mu(C/B) \geq \delta(B)\}$.

Obtaining strong minimality

Primitive Extensions and Good Pairs

Let $A, B, C \in \mathbf{K}_0$.

1. C is a 0-*primitive extension* of A if C is minimal with $\delta(C/A) = 0$.
2. C is good over $B \subseteq A$ if B is minimal contained in A such that C is a 0-*primitive extension* of B. We call such a B a *base*.

Bounding realization of good pairs

1. For any good pair (C/B), $\chi_M(B, C)$ is the maximal number of disjoint copies of C over B appearing in M.
2. For $\mu \in \mathcal{U}$, $\boldsymbol{K}_\mu$ is the collection of $M \in \boldsymbol{K}_0$ such that $\chi_M(A, B) \leq \mu(A, B)$ for every good pair (A, B).

This guarantees strong minimality.

The Amalgamation

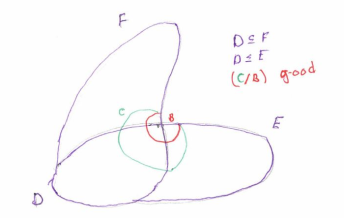


Figure: 0-primitive extensions

The Amalgamation

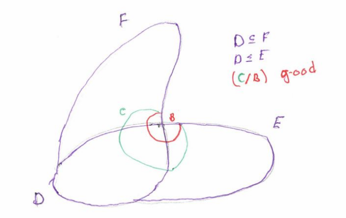
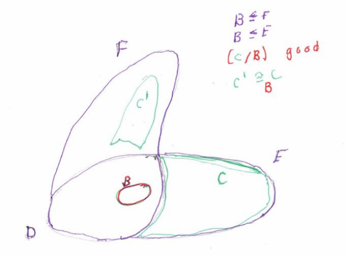


Figure: 0-primitive extensions



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   Replace ϵ by δ.
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   $\mu(\boldsymbol{\alpha}) = q - 2$ gives line length q.

Strongly minimal linear spaces

Fact

Suppose (M, R) is a strongly minimal linear space where all lines have at least 3 points. There can be no infinite lines.

An easy compactness argument establishes

Corollary

If (M, R) is a strongly minimal linear system, for some k, all lines have length at most k.

The construction with $\mu(\alpha) = q - 2$ gives a q-Steiner system.

Coordinatization by varieties of algebras

2 VARIABLE IDENTITIES

Definition

A variety is binary if all its equations are 2 variable identities: [Eva82]

Definition

Given a (near) field $(F, +, \cdot, -, 0, 1)$ of cardinality $q = p^n$ and an element $a \in F$, define a multiplication $*$ on F by

$$x * y = y + (x - y)a.$$

An algebra $(A, *)$ satisfying the 2-variable identities of $(F, *)$ is a block algebra over $(F, *)$

This block algebra is a Steiner quasigroup with cardinality q.

Coordinatizing Steiner Systems

Weakly coordinatized

A collection of algebras V '(weakly) coordinatizes' a class $\mathcal{S}$ of $(2, k)$-Steiner systems if

1. Each algebra in V definably expands to a member of $\mathcal{S}$
2. The universe of each member of $\mathcal{S}$ is the underlying system of some (perhaps many) algebras in V.

Coordinatized

A collection of algebras V definably coordinatizes a class $\mathcal{S}$ of k-Steiner systems if
in addition the algebra operation is definable in the Steiner system.

Coordinatizing Steiner Systems

Key fact: weak coordinatization [Ste64, Eva76]

If V is a variety of binary, idempotent algebras and each block of a Steiner system $\mathcal{S}$ admits an algebra from V then so does $\mathcal{S}$.

Definition [Pad72]

An (r, k) variety is one in which every r-generated algebra has cardinality k and is freely generated by every n-elements.

Definition: Mikado Variety

A variety V of binary, idempotent algebras, $(2, k)$ algebras is called Mikado.
Thus, each $A \in V$ determines a Steiner k-system(The 2-generated subalgebras).
And each Steiner k-system admits a weak coordinatization.

Can this coordinatization be definable in the strongly minimal (M, R)?
NO; the BP examples cannot.

Constructing a strongly minimal quasigroup

Definition: $\boldsymbol{K}^q$

1. Fix a prime power q and a Mikado variety V of quasigroups such that F_2, the free algebra in V on 2 generators has q elements.
2. Let $\boldsymbol{K}^q_V$ be the collection of finite (H, R)-structures A such that
   1. (A, R) is a linear space;
   2. $(\forall a_1, a_2, a_3)H(a_1, a_2, a_3) \to R(a_1, a_2, a_3)$;
   3. Each line (maximal R-clique) has q points.
   4. If $A{\restriction}R$ is a maximal clique (line) ℓ with respect to R, then on ℓ, $A{\restriction}H$ is the graph of the free algebra $F_2 \in V$.
   5. Any collinear triple extends to a q-element clique. (A $\forall\exists$ sentence.)

Since V is axiomatized by 2-variable equations, if $A' \in \boldsymbol{K}^q_V$, $A'{\restriction}H$ is the graph of an algebra in V. In the generic model *each pair* is included in a q-element line; but not in the finite structures.

Defining δ and μ

1. Define δ, primitive and good extensions on finite (A, R, H) by ignoring H. Let α_q denote the isomorphism type of $(\{c_1, c_2, \ldots c_{q-2}\}/ab)$, where all the c_i satisfy $R(a, b, c_i)$.
2. A μ' mapping $\boldsymbol{K}^q_{0,V}$ into Z is in $\mathcal{U}_{\tau'}$ if it satisfies i) $\mu'(A'/B') \geq \delta_{\tau'}(B)$ and ii) $\mu'(\boldsymbol{\alpha}_q) \geq 1$.
3. Let $D' \in (\boldsymbol{K}^q_{\mu',V}, \leq')$ if and only if $\chi_{D'}(A'/B') \leq \mu'(A'/B')$. To define a q-Steiner system, we set $\mu'(\boldsymbol{\alpha}_q) = 1$.

Finding the generic quasigroup

Theorem

For each $q = p^n$, each $\mu' \in \mathcal{U}_{\tau'}$, and each Mikado-variety of quasigroups V with $|F_2(V)| = q$, there is a strongly minimal theory of quasigroups, dubbed $T^q_{\mu',V}$, that interprets a strongly minimal q-Steiner system.

The amalgamation is an easy modification of the proof in [BP21]; the rest is standard.

Diversity and Classification

No elimination of imaginaries [BV22]

$\mathrm{dcl}^*(X) = \mathrm{dcl}(X) - \bigcup_{Y \subsetneq X} \mathrm{dcl}(Y)$.

Theorem

Let T_μ be a strongly minimal theory as in Hrushovski's original paper. I.e. $\mu \in \mathcal{U} = \{\mu : \mu(A/B) \geq \delta(B)\})$. Let $I = \{a_1, \ldots, a_v\}$ be a tuple of independent points with $v \geq 2$.

G_I If T_μ triples, i.e.

$$\mu \in \{\mu : \mu(A/B) \geq 3\}$$

then $\mathrm{dcl}^*(I) = \emptyset$, $\mathrm{dcl}(I) = \bigcup_{a \in I} \mathrm{dcl}(a)$, and every definable function is essentially unary.

$G_{\{I\}}$ In any case $\mathrm{sdcl}^*(\mathrm{I}) = \emptyset$, $\mathrm{sdcl}(\mathrm{I}) = \bigcup_{\mathrm{a} \in \mathrm{I}} \mathrm{sdcl}(\mathrm{a})$ and there are no $\emptyset$-definable symmetric (value does not depend on order of the arguments) truly v-ary function.

Thus for any $\mu \in \mathcal{U}$, T_μ does not admit elimination of imaginaries and the algebraic closure geometry is not disintegrated.

Examples

A geometry is flat if dimension is computed by inclusion-exclusion on closed subsets.

Strongly minimal theories with non-locally modular algebraic closure

I. the Hrushovski (Steiner) examples $2^{\aleph_0}$ theories of strongly minimal Steiner systems (M, R) with
   1. no $\emptyset$-definable binary function. (i.e. triplable)
   2. Some definable functions (examples in [BV22])
II. $2^{\aleph_0}$ theories of strongly minimal quasigroups $(M, R, *)$ + a 3-Steiner example of Hrushovski
III. strongly minimal Steiner systems with combinatorial interesting properties
IV. Non-Desarguesian projective planes definably coordinatized by ternary fields [Bal95]
V. 2-ample but not 3-ample sm sets (not flat) [MT19]
VI. strongly minimal eliminates imaginaries (flat) INFINITE vocabulary)

Classifying ‘flat’ strongly minimal sets

1. discrete (trivial)
2. non-trivial but no binary function
3. non-trivial but no commutative binary function
4. Non-Desarguesian projective planes definably coordinatized by ternary fields [Bal95]

Key Points

Variations of the Hrushovski construction

1. *k*-steiner for arbitrary *k*.
2. not locally finite
3. Build families of examples for infinite combinatorics: such notions as
    1. families: towers of models of distinct theories.
    2. anti-Pasch, sparseness;
    3. generalize cycle graphs (3-Steiner) to path graphs (*q*-Steiner);
    4. construct quasigroups which induce *q*-Steiner systems for arbitrary prime powers;
    5. 2-transitive;
4. strongly minimal – model theoretically well behaved

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