# Denumerable models of complete theories *

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**Introduction.** The following theorem, which characterizes a certain type of complete theories, was established by Ryll-Nardzewski.

0.1. *A necessary and sufficient condition for a complete theory $T$, having infinite models, to be $\aleph_0$-categorical* (1) *is that, for each $n$, there are only finitely many formulas, with free variables $v_0, \ldots, v_{n-1}$, which are inequivalent in $T$.*

A simplification of the proof (of necessity) was found by Ehrenfeucht (2).

In this paper we shall apply methods closely related to those used in proving 0.1 to the study of the denumerable models of some other types of complete theories.

Before the work can be described more fully, some notions must be defined. Let $\mathfrak{A}$ be an infinite model of a theory $T$. We say that $\mathfrak{A}$ is *homogenenous* if, whenever $a_0, \ldots, a_n$ and $a'_0, \ldots, a'_n$ satisfy in $\mathfrak{A}$ exactly the same formulas of $T$, there is an automorphism of $\mathfrak{A}$ carrying $a_i$ into $a'_i$ $(i = 0, \ldots, n)$. $\mathfrak{A}$ is $\aleph_0$-*universal* if $\mathfrak{A}$ is denumerable and is an elementary extension (cf. § 1) of an isomorph of each denumerable model of $T$. $\mathfrak{A}$ is *prime* if every model of $T$ is an elementary extension of an isomorph of $\mathfrak{A}$ (3).

* Many of the results in this paper were announced in [22].

(1) A theory is said to be categorical in the power $\aleph_\alpha$ or, simply, $\aleph_\alpha$-categorical if all its models of that power are isomorphic (cf. [9]). The exact meaning we ascribe to various familiar terms such as "theory", will be specified in § 1; but let it be said now that, herein, "complete" implies "consistent".

(2) Cf. [13] and, also, [10], p. 24. Later, independently, 0.1 was established by L. Svenonius, and by E. Engeler [3].

(3) Notions more or less closely related to "$\aleph_0$-universal" and "homogeneous" have been employed by various authors. Cf. e.g., [1], [4], and [8]; also, see footnote 15. A. Robinson [12] defined "prime" as above, but with "elementary" omitted; however, for the "model-complete" theories he was studying, this omission does not change the extension of the notion.

The two types of complete theories we discuss are those having prime models and those having $\aleph_0$-universal models— or, what, turns out to be the same—those having $\aleph_0$-universal, homogeneous models. It is shown that a model of $T$ of the last sort is unique up to isomorphism, and that the same applies to a prime model. A number of necessary and sufficient conditions for a model to be such a model, or for a theory to have such a model are given in 3.4, 3.5, 4.6, and 4.7; these are the principal results of the paper.

According to a theorem of Ehrenfeucht [2], certain of these conditions are satisfied by theories categorical in a non-denumerable power. Consequently, our results may be applied to show that such theories possess prime models and $\aleph_0$-universal, homogeneous models. Some additional conclusions regarding these theories are also derived in § 5.

In § 6, it is shown that a complete theory cannot have exactly two non-isomorphic denumerable models, answering a question of Raphael Robinson.

**§ 1. Preliminaries.** The *theories* we consider are formalized in the first order logic with identity, and are assumed to have at most $\aleph_0$ non-logical constants (4), each of which is either a relation symbol or an individual constant. (When more than $\aleph_0$ non-logical constants occur, we speak of a *generalized theory.*) A theory specifies a non-repeating list $X_0, \ldots, X_\xi, \ldots (\xi < \eta)$ of its non-logical constants. The distinct individual variables of every theory $T$ are $v_0, v_1, \ldots, v_n, \ldots$ *The set of all formulas of $T$ whose free variables are among* $v_0, \ldots, v_{n-1}$ *is called $F_n(T)$* (5). Let $\varphi, \varphi' \in F_m(T)$. We write $\vdash_T \varphi$ to mean that $\varphi$ is valid in $T$ (i.e., the sentence $\bigwedge v_0 \ldots \bigwedge v_{m-1}\varphi$ is valid in $T$). $\varphi$ and $\varphi'$ are *equivalent in $T$* if $\vdash_T \varphi \leftrightarrow \varphi'$; and $\varphi$ is *consistent with $T$* if $\sim\varphi$ is not valid in $T$. It $\tau_0, \ldots, \tau_{p-1}$ are terms, $\Phi(\tau_0, \ldots, \tau_{p-1})$ is the formula obtained by the proper simultaneous substitution of $\tau_i$ for the free occurrences of $v_i$ in $\varphi$ ($i = 0, \ldots, p-1$). ("Proper" means that bound variables should be changed to avoid collisions.)

If a system $\mathfrak{A}$ is a realization of $T$ and $a_0, \ldots, a_{m-1} \in |\mathfrak{A}|$ (6), we write $\models_{\mathfrak{A}} \Phi[a_0, \ldots, a_{m-1}]$ to mean that $\varphi$ is satisfied in $\mathfrak{A}$ by

(4) Terminology which is explained only partly or not at all is that of [17] and [18].

(5) Letters "i", ..., "r" denote natural numbers 0, 1, ..., i.e., members of $\omega$. "0" also denotes the empty set. "$\xi$", "$\eta$", "$\zeta$" denote ordinals.

(6) $|\mathfrak{A}|$ is the *universe* of $\mathfrak{A} = \langle A, \ldots\rangle$, i. e., the set $A$.

the assignment of $a_0$ to $v_0, \ldots, a_{m-1}$ to $v_{m-1}$. Thus, when $m = 0$, $\models_{\mathfrak{A}} \varphi$ means that $\varphi$ is true in $\mathfrak{A}$, or $\mathfrak{A}$ is a model of $\varphi$. Realizations of the same theory are called *similar*. The similar systems $\mathfrak{A}$ and $\mathfrak{B}$ are *elementarily equivalent* (in symbols, $\mathfrak{A} \equiv \mathfrak{B}$) if they have the same true sentences, or in other words, if they are models of the same complete theory. $\mathfrak{A}$ is said to be an *elementary extension* of $\mathfrak{B}$ and $\mathfrak{B}$ an *elementary subsystem* of $\mathfrak{A}$ (in symbols, $\mathfrak{A} \succ \mathfrak{B}$) if (in addition) $\mathfrak{B}$ is a subsystem of $\mathfrak{A}$ and, in general, $\models_{\mathfrak{B}} \Phi[b_0, \ldots, b_{n-1}]$ implies $\models_{\mathfrak{A}} \varphi[b_0, \ldots, b_{n-1}]$.

Let $\mathfrak{A}_0, \ldots, \mathfrak{A}_\xi, \ldots$ $(\xi < \eta)$ be similar systems such that $\mathfrak{A}_{\xi'} \succ \mathfrak{A}_\xi$ whenever $\eta > \xi' > \xi$ (such a sequence of systems is said to be *elementarily increasing*). Then the union $\bigcup \{\mathfrak{A}_\xi | \ \xi < \eta\}$ is the system whose universe is the ordinary union of $\{|\mathfrak{A}_\xi|/ \ \xi < \eta\}$ and whose $\zeta$th relation or distinguished element is, respectively, the union of the $\zeta$th relations of all the $\mathfrak{A}_\xi$, or the (common) $\zeta$th distinguished element of all the $\mathfrak{A}_\xi$. Given a system $\mathfrak{A} = \langle A, X_0, \ldots, X_\xi, \ldots \rangle_{\xi<\eta}$ and a sequence $Y_0, \ldots, Y_\xi, \ldots$ $(\xi < \zeta)$ of further relations or distinguished elements over $A$, the system $\langle A, X_0, \ldots, X_\xi, \ldots, Y_0, \ldots, Y_{\xi'}, \ldots \rangle_{\xi<\eta, \xi'<\zeta}$ will be indicated by the notation (of S. Feferman) $(\mathfrak{A}, Y_0, \ldots, Y_\xi, \ldots)_{\xi<\zeta}$.

To simplify the description of the next notion, we deal with the case where $T$ has only one non-logical constant, a ternary relation symbol R; from this illustration the general situation will be clear. By a *possible relative interpretation* of $T$ in another theory $T_1$ we understand a system $I = \langle \theta, \gamma \rangle$, where $\theta \in F_1(T_1)$ and $\gamma \in F_3(T_1)$. For any formula $\varphi$ of $T$, $\Phi^I$ is the formula (of $T_1$) obtained from $\varphi$ by replacing each atomic formula $\mathrm{R}v_{k_0}v_{k_1}v_{k_2}$ by $\gamma(v_{k_0}, v_{k_1}, v_{k_2})$, and then replacing subformulas of the form $\bigvee v_j \psi$ or $\bigwedge v_j \psi$ by $\bigvee v_j(\theta(v_j) \wedge \psi)$ or $\bigwedge v_j(\theta(v_j) \rightarrow \psi)$, respectively. $I$ is a *relative interpretation* of $T$ in $T_1$ if $\sigma^I$ is valid in $T_1$ whenever $\sigma$ is a sentence valid in $T$. If $\mathfrak{B}$ is a realization of $T_1$, then the *denotation of $I$ in* $\mathfrak{B}$ is the system $\langle A, R \rangle$, where $A = \{x/\models_{\mathfrak{B}} \theta[x]\}$ and $R = \{\langle x, y, z \rangle / \ x, y, z \in A$ and $\models_{\mathfrak{B}} \gamma[x, y, z]\}$.

For later reference we state here the following, easily proved facts:

LEMMA 1.1. (.1) *If* $\mathfrak{A}_\xi \succ \mathfrak{A}_{\xi'}$ *whenever* $\eta > \xi > \xi'$, *then, for each* $\xi < \eta$, $\bigcup \{\mathfrak{A}_\xi / \xi < \eta\} \succ \mathfrak{A}_\xi$.

(.2) *If* $\mathfrak{B}$ *is a model of* $T_1$, $\mathfrak{B} \succ \mathfrak{B}'$, *and* $\mathfrak{A}$ *and* $\mathfrak{A}'$ *are the respective denotations in* $\mathfrak{B}$ *and* $\mathfrak{B}'$ *of a relative interpretation* $I$ *of* $T$ *in* $T_1$, *then* $\mathfrak{A} \succ \mathfrak{A}'$.

(.3) *Suppose that* $\mathfrak{A}$ *and* $\mathfrak{B}$ *are similar systems*, $|\mathfrak{A}| = \{a_n / n \in \omega\}$, *and, for each* $n$, $(\mathfrak{A}, a_0, \ldots, a_{n-1}) \equiv (\mathfrak{B}, b_0, \ldots, b_{n-1})$. *Then* $\{\langle a_n, b_n\rangle / n \in \omega\}$ *is a function mapping* $\mathfrak{A}$ *isomorphically onto an elementary subsystem of* $\mathfrak{B}$ (7).

We turn now to some less familiar notions. *Henceforth it is assumed that* $T$ *is a complete theory having infinite models* (8). (One or both of these assumptions is often dispensable, but usually with little serious gain in generality.) For each $n$, the set $F_n(T)$, together with the operations $\wedge$ (which, when applied to $\varphi$ and $\psi$, yields $\varphi \wedge \psi$), $\vee$, and $\sim$, and the relation of equivalence in $T$, constitutes a Boolean algebra (9)—also denoted by $F_n(T)$. Consequently, the ordinary terminology for Boolean algebras may be employed:

1.2.1. A member $\alpha$ of $F_n(T)$ is an *atom* of $F_n(T)$ provided that $\alpha$ is consistent with $T$ and, for any $\varphi \in F_n(T)$, if $\alpha \wedge \varphi$ is consistent with $T$ then $\vdash_T \alpha \to \varphi$.

1.2.2. $\varphi$ is an *atomless* element of $F_n(T)$ if $\varphi$ is consistent with $T$ and $\vdash_T \alpha \to \varphi$ holds for no atom $\alpha$ of $F_n(T)$.

1.2.3. $F_n(T)$ is *atomistic* if it has no atomless element.

1.2.4. A *prime ideal* of $F_n(T)$ is a non-empty, proper subset $P$ of $F_n(T)$ such that, for any $\varphi, \psi \in F_n(T)$: $\varphi \wedge \psi \in P$ if $\varphi, \psi \in P$; $\psi \in P$, if $\varphi \in P$ and $\vdash_T \varphi \to \psi$; and either $\varphi \in P$ or $\sim\varphi \in P$. (This is what is usually called a "dual prime ideal".)

1.2.5. A prime ideal $P$ is *principal* if, for some $\theta \in F_n(T)$, $P = \{\varphi / \varphi \in F_n(T)$ and $\vdash_T \theta \to \varphi\}$ —or, equivalently, if $P$ contains an atom of $F_n(T)$.

The set of all prime ideals of $F_n(T)$ will be denoted by $\mathcal{P}_n(T)$. If $P \in \mathcal{P}_n(T)$ and $\mathfrak{A}$ is a model of $T$, then we denote by $P(\mathfrak{A})$ the set of all $n$-tuples $\langle a_0, \ldots, a_{n-1}\rangle$ such that, for every $\varphi \in P$, $\vdash_{\mathfrak{A}} \varphi[a_0, \ldots, a_{n-1}]$.

1.3. Clearly there is a natural one-to-one correspondence between $F_n(T)$ and the set $\bar{F}_n(T)$ of all sentences which involve the non-logical constants of $T$ plus the distinct, new, individual

(7) For (.1), cf. [18], Theorem 1.9. For (.3), cf. the proof of 1.12 of [18].

(8) Any reference to the power of $A$ is to be understood as referring to the power of $|\mathfrak{A}|$.

(9) Thus for us a Boolean algebra is a system of the form $\langle A, +, \cdot, -, \approx\rangle$, whose quotient modulo $\approx$ is a Boolean algebra in the more usual sense. Note also that, $T$ being complete, $F_0(T)$ has always only two inequivalent elements; nonetheless, it is included in the discussion, for technical convenience.

constants $c_0, \ldots, c_{n-1}$. A formula $\varphi \in F_n(T)$ goes into the sentence $\bar{\varphi} = \varphi(c_0, \ldots, c_{n-1})$. The induced correspondence (also denoted by $^{-}$) maps the prime ideals of $F_n(T)$ onto the complete theories involving the non-logical constants of $T$ plus $c_0, \ldots, c_{n-1}$. Clearly, $\langle a_0, \ldots, a_{n-1}\rangle \in P(\mathfrak{A})$ if and only if $(\mathfrak{A}, a_0, \ldots, a_{n-1})$ is a model of $\bar{P}$. It is sometimes convenient to think of the theory $\bar{P}$ in place of the prime ideal $P$ of $F_n(T)$.

Lemma 1.4, below, gives an obvious, alternative characterization of the notion of "atom".

LEMMA 1.4. *A necessary (and sufficient) condition for a member $a$ of $F_n(T)$ to be an atom of $F_n(T)$ is that, for any models $\mathfrak{A}$ and $\mathfrak{B}$ of $T$, if $\models_{\mathfrak{A}} a[a_0, \ldots, a_{n-1}]$ and $\models_{\mathfrak{B}} a[b_0, \ldots, b_{n-1}]$, then $(\mathfrak{A}, a_0, \ldots, a_{n-1}) \equiv (\mathfrak{B}, b_0, \ldots, b_{n-1})$.*

**§ 2. Existence of models.** In this section we shall prove the following

THEOREM 2.1. (.1) *There is a denumerable model $\mathfrak{A}$ of $T$ such that*

($*$) *every finite sequence of elements of $|\mathfrak{A}|$, of any length $m+1$, satisfies in $\mathfrak{A}$ either an atom or an atomless member of $F_{m+1}(T)$.*

(.2) *If, for each $j \in \omega$, $P_j$ is a non-principal prime ideal of $F_{p_j+1}(T)$, then there is a denumerable model $\mathfrak{A}$ of $T$ such that*

($**$) *$P_0(\mathfrak{A}), P_1(\mathfrak{A}), \ldots$ are all empty.*

(.3) *Indeed, under the hypothesis of (.2), a denumerable model $\mathfrak{A}$ of $T$ can be found for which both ($*$) and ($**$) hold.*

2.1.2 was proved by Ehrenfeucht. Its special case in which there is only one $P_j$ was used by him to give a simple proof of the necessity in Ryll-Nardzewski's theorem, 0.1 ([2]), and will be used in § 3 and § 6, below. 2.1.1 was established by the author. In § 3 it will be applied in the special case in which each $F_n(T)$ is atomistic. Thus, 2.1 is, in a number of ways, stronger than what is needed in the rest of the paper. The strong form has been stated because it is no more difficult to prove and may, perhaps, be of some intrinsic interest. Note that (.2) allows us to assume that countably many, *arbitrary*, non-principal prime ideals will be empty for $\mathfrak{A}$. On the other hand, (.1) says that *certain* non-principal prime ideals, possibly $2^{\aleph_0}$ in number, can be made empty for $\mathfrak{A}$ ([10]).

([10]) As one easily verifies, these are, in fact, those non-principal prime ideals of each $F_n(T)$ which—as points of the topological space corresponding to $F_n(T)$ by Stone's representation theorem ([15])—are the limit of a sequence of isolated points.

Proof. We proceed by a modification of Henkin's proof of the completeness theorem (Cf. [6] and pp. 42-43 of [5]). Let $c_0, c_1, \ldots$ be distinct, new individual constants. Clearly all entities of the form $\Pi = \langle P, c_{k_0}, \ldots, c_{k_q}\rangle$ —where $q$ is arbitrary, $k_0, \ldots, k_q$ are distinct, and either $P = 0$ or else $P$ is a $P_i$ with $p_i = q$ —may be enumerated in a list $\Pi_0, \Pi_1, \ldots$ Let $T_1$ be the theory whose constants are those of $T$ plus $c_0, c_1, \ldots$, and whose axioms are the valid sentences of $T$. The members of $F_1(T_1)$ may be enumerated in a list $\varphi_0, \varphi_1, \ldots$ For later reference, we note the well-known principle, which holds for any $\psi \in F_r(T)$:

(1) $\psi$, or $\bigvee v_0 \ldots \bigvee v_{r-1}\psi$, is consistent with $T$ if and only if $\psi(c_0, \ldots, c_{r-1})$ is consistent with $T_1$.

We are going to define recursively sentences $\sigma_0, \sigma_1, \ldots$ of $T_1$ in such a way that, for each $n$, $\sigma_0 \wedge \ldots \wedge \sigma_{n-1}$ is consistent with $T_1$. Suppose that $\sigma_0, \ldots, \sigma_{n-1}$ have been defined and $\sigma_0 \wedge \ldots \wedge \sigma_{n-1}$ is consistent with $T_1$. Let $\nu$ be the smallest number such that $c_\nu$ occurs in none of $\varphi_n, \sigma_0, \ldots, \sigma_{n-1}$. Then as is well known, the sentence

$$\gamma\colon\ [\bigvee v_0\varphi_n \rightarrow \varphi_n(c_\nu)] \wedge \sigma_0 \wedge \ldots \wedge \sigma_{n-1}$$

is (yb (1)) consistent with $T_1$. Let $\Pi_n$ be $\langle P, d_0, \ldots, d_q\rangle$, and let $e_0, \ldots, e_{r-1}$ be the distinct $c_i$'s occurring in $\gamma$ and not equal to any of $d_0, \ldots, d_q$. Clearly, there is a formula $\theta \in F_{q+1+r}(T)$ such that $\theta(d_0, \ldots, d_q, e_0, \ldots, e_{r-1})$ is logically equivalent to $\gamma$. Then the formula

$$\beta\colon\ \bigvee v_{q+1} \ldots \bigvee v_{q+r}\theta$$

of $F_{q+1}(T)$ is consistent with $T$. We now distinguish two cases:

Case (i). For some atom $\alpha$ of $F_{q+1}(T)$, $\vdash_T \alpha \rightarrow \beta$. Choosing a definite $\alpha$, we take for $\sigma_n$ the sentence $\alpha(d_0, \ldots, d_q) \wedge \gamma$. Since $\alpha$ (as an atom) is consistent with $T$, one easily sees (applying principle (1)) that $\sigma_0 \wedge \ldots \wedge \sigma_n$ is consistent with $T_1$.

Case (ii). $\beta$ is an atomless element of $F_{q+1}(T)$. If $P$ is the empty set, we take for $\sigma_n$ simply the sentence $\gamma$; then, certainly, $\sigma_0 \wedge \ldots \wedge \sigma_n$ is consistent with $T_1$. Otherwise, $P$ is a non-principal prime ideal of $F_{q+1}(T)$. Then, clearly, there is a formula $\delta \in F_{q+1}(T)$ such that $\sim\delta \in P$, while $\beta \wedge \delta$ is consistent with $T$. For $\sigma_n$ we take $\delta(d_0, \ldots, d_q) \wedge [\bigvee v_0\varphi_n \rightarrow \varphi_n(c_\nu)]$. Again, applying (1), we see that $\sigma_0 \wedge \ldots \wedge \sigma_n$ is consistent with $T_1$.

Thus, $\sigma_0, \sigma_1, \ldots$ are defined, and the theory $T_2$, obtained from $T_1$ by taking $\sigma_0, \sigma_1, \ldots$ as additional axioms, is consistent. Moreover, the construction assured that

(2) for each $n$, there is a $c_i$ such that $\vdash_{T_2} \bigvee v_0\varphi_n \rightarrow \varphi_n(c_i)$.

By a well-known result of Henkin ([6]), it follows from (2) that $T_2$ has a model $(\mathfrak{A}, c_0, \ldots, c_n, \ldots)$ such that $\mathfrak{A}$ is a model of $T$ and $|\mathfrak{A}| = \{c_0, \ldots, c_n, \ldots\}$. We shall see that our construction above has also ensured that $\mathfrak{A}$ fulfills $(*)$ and $(**)$.

Indeed, suppose that $a_0, \ldots, a_q \in |\mathfrak{A}|$. Then there are *distinct* $c_{k_0}, \ldots, c_{k_q}$ such that $a_0 = c_{k_0}, \ldots,$ and $a_m = c_{k_q}$. (This is because any given $c_j$ is of the form $c_i$, for infinitely many $i$. The latter fact is easily seen by noting that there are infinitely many $\varphi_i$'s of the form $\mathrm{v}_0 = \mathrm{c}_j \wedge \sigma$, where $\sigma$ is tautologous [11]). Now, at some point in defining $\sigma_0, \sigma_1, \ldots$, we had $\Pi_n = \langle 0, \mathrm{c}_{k_0}, \ldots, \mathrm{c}_{k_q}\rangle$. If case (i) occurred, then clearly $c_{k_0}, \ldots, c_{k_q}$ satisfy in $\mathfrak{A}$ an atom of $F_q(T)$—namely, the $\alpha$ of case (i). Otherwise, $\beta$ was atomless, and, since the construction insured that $\vdash_{T_2} \beta(\mathrm{c}_{k_0}, \ldots, \mathrm{c}_{k_q})$, we see that $c_{k_0}, \ldots, c_{k_q}$ satisfy in $\mathfrak{A}$ an atomless member of $F_n(T)$. Thus, $(*)$ holds. Now suppose further that $p_j = q$, so that for some $n'$, $\Pi_{n'} = \langle P_j, \mathrm{c}_{k_0}, \ldots, \mathrm{c}_{k_q}\rangle$. If at the $n'$th step case (i) occurred, then $c_{k_0}, \ldots, c_{k_q}$ satisfy in $\mathfrak{A}$ an atom, say $\alpha'$. Since $P_j$ is non-principal, $\sim\alpha' \in P$, and hence $\langle c_{k_0}, \ldots, c_{k_q}\rangle \notin P_j(\mathfrak{A})$. If, instead, case (ii) occurred in the $n'$th step, then we have explicitly ensured in that step that $\langle c_{k_0}, \ldots, c_{k_q}\rangle \notin P_j(\mathfrak{A})$. Thus $(**)$ holds, and the theorem is proved.

§ 3. **Prime models.** A model $\mathfrak{A}$ of $T$ will be called *atomic* if each finite sequence of elements of $|\mathfrak{A}|$, of any length $n$, satisfies in $\mathfrak{A}$ an atom of $F_n(T)$ [12].

LEMMA 3.1. *Suppose that $\mathfrak{A}$ is a denumerable, atomic model of $T$ and $\mathfrak{B}$ is an arbitrary model of $T$. Then $\mathfrak{A}$ can be mapped isomorphically onto an elementary subsystem of $\mathfrak{B}$. Moreover, if $a_0, \ldots, a_{m-1} \in |\mathfrak{A}|$, $b_0, \ldots, b_{m-1} \in |\mathfrak{B}|$, and $(\mathfrak{A}, a_0, \ldots, a_{m-1}) \equiv (\mathfrak{B}, b_0, \ldots, b_{m-1})$, then the mapping may be so chosen that it carries $a_i$ into $b_i$ for each $i < m$.*

Proof. Let $a_0, \ldots, a_n, \ldots$ be a list, possibly with repetitions, of the elements of $|\mathfrak{A}|$ (commencing with the given $a_0, \ldots, a_{m-1}$). Suppose that $b_m, b_{m+1}, \ldots, b_{n-1}$ $(n > m)$ have been defined in such a way that

$$(1) \qquad (\mathfrak{A}, a_0, \ldots, a_{n-1}) \equiv (\mathfrak{B}, b_0, \ldots, b_{n-1}) .$$

Since $\mathfrak{A}$ is atomic, there exists an atom $\alpha$ of $F_n(T)$ and an atom $\alpha'$ of $F_{n+1}(T)$ such that $\models_{\mathfrak{A}} \alpha[a_0, \ldots, a_{n-1}]$ and

[11] This detail seems less bothersome than those required in the applications of (1) above, had $\mathrm{d}_0, \ldots, \mathrm{d}_q$ not there been assumed distinct.

[12] This terminology is due to L. Svenonius.

$\models_{\mathfrak{A}} \alpha'[a_0, \ldots, a_n]$. It follows that $\models_{\mathfrak{B}} \alpha[b_0, \ldots, b_{n-1}]$ and that formula $\alpha \wedge \bigvee v_n \alpha'$ is consistent. By 1.2.1, the latter implies that $\vdash_T \alpha \rightarrow \bigvee v_n \alpha'$. $\mathfrak{B}$ being a model of $T$, we infer that $\models_{\mathfrak{B}} (\bigvee v_n \alpha')\ [b_0, \ldots, b_{n-1}]$. Thus, we may choose for $b_n$ an element of $|\mathfrak{B}|$ such that $\models_{\mathfrak{B}} \alpha'[b_0, \ldots, b_n]$. Then, by 1.4, $(\mathfrak{A}, a_0, \ldots, a_n) \equiv (\mathfrak{B}, b_0, \ldots, b_n)$.

Thus, $b_m, b_{m+1}, \ldots$ can be defined recursively in such a way that (1) holds for every $n$. Lemma 3.1 now follows immediately from 1.1.3.

THEOREM 3.2. *If $\mathfrak{A}$ and $\mathfrak{B}$ are denumerable, atomic models of $T$, then $\mathfrak{A}$ is isomorphic to $\mathfrak{B}$* (13).

THEOREM 3.3. *If $\mathfrak{A}$ is a denumerable, atomic model of $T$, then $\mathfrak{A}$ is homogenous.*

Proof of 3.2 and 3.3. The proof of 3.1, above, resembles Cantor's argument showing that any denumerable, simply ordered system is a subsystem of a denumerable, densly ordered system without extreme points. The proof of 3.2 is analogously related to Cantor's proof that any two systems of the latter sort are isomorphic. Roughly, to prove 3.2, we let $|\mathfrak{A}| = \{a_m / m \in \omega\}$ and $|\mathfrak{B}| = \{b_n / n \in \omega\}$ and define recursively couples $\langle a_{j_n}, b_{k_n} \rangle$, $n = 0, 1, \ldots$ The passage from $n$ to $n+1$ is like that in the proof of 3.1 ($j_n$ being defined to be the first $i \neq j_0, \ldots, j_{n-1}$) when $n$ is even; when $n$ is odd, the rôles of $\mathfrak{A}$ and $\mathfrak{B}$ are reversed. The proof of 3.3 is analogously related to the argument proving the second conclusion of 3.1.

It may be noted that the second part of 3.1 could have been derived from the first part, and 3.3 from 3.2, by noting the following easily proved fact: If $\mathfrak{A}$ is atomic and $a_0, \ldots, a_n \in |\mathfrak{A}|$, then $(\mathfrak{A}, a_0, \ldots, a_n)$ is atomic.

THEOREM 3.4. *$\mathfrak{A}$ is prime if and only if $\mathfrak{A}$ is denumerable and atomic.*

Proof. If $\mathfrak{A}$ is atomic and denumerable, then, by 3.1, $\mathfrak{A}$ is prime. On the other hand, if $\mathfrak{A}$ is prime, then clearly (by the Löwenheim-Skolem theorem) $\mathfrak{A}$ is denumerable. Suppose now, that $\mathfrak{A}$ is not atomic. Then some $a_0, \ldots, a_m \in |\mathfrak{A}|$ satisfies no atom (of $F_m(T)$). Hence, clearly, $P = \{\varphi / \models_{\mathfrak{A}} \varphi[a_0, \ldots, a_m]\}$ is a non-principal prime ideal of $F_m(T)$. By 2.1.2, $T$ has a model $\mathfrak{B}$ with $P(\mathfrak{B})$ empty. It obviously follows that $\mathfrak{B}$ cannot be an

(13) This result was also (indeed, earlier) established by Svenonius; of course, a closely related result was proved by Ryll-Nardzewski [13].

elementary extension of an isomorph of $\mathfrak{A}$, a contradiction. This proves 3.4.

THEOREM 3.5. *The following are equivalent:*

(.1) *$T$ has a prime model;*

(.2) *$T$ has an atomic model;*

(.3) *Each $F_n(T)$ is atomistic.*

Proof. It follows immediately from 2.1.1 that, if each $F_n(T)$ is atomistic, then $T$ has a denumerable, atomic model; by 3.4, such a model is prime. Thus, (.3) implies (.1). By 3.4, (.1) implies (.2). The remaining implication is nearly obvious. Indeed, suppose that $T$ has an atomic model $\mathfrak{A}$. Let $\varphi$ be any member of $F_n(T)$, consistent with $T$. Since $T$ is complete, $\vdash_T \bigvee v_0 \dots \bigvee v_{n-1}\varphi$; hence, there are $a_0, \dots, a_{n-1} \in |\mathfrak{A}|$ such that $\models_{\mathfrak{A}} \varphi[a_0, \dots, a_{n-1}]$. $\mathfrak{A}$ being atomic, there is an atom $\alpha$ of $F_n(T)$ such that $\models_{\mathfrak{A}} \alpha[a_0, \dots, a_{n-1}]$. Then $\alpha \wedge \varphi$ is consistent with $T_1$ so that, by 1.2.1, $\vdash_T \alpha \rightarrow \varphi$. Thus, $\varphi$ is not atomless. This argument shows that $F_n(T)$ is atomistic, completing the proof.

As was to be expected the results in § 2 and § 3 have as a consequence 0.1, the theorem of Ryll-Nardzewski. Indeed, the proposed condition clearly implies that any model of $T$ is atomic; the $\aleph_0$-categoricity of $T$ then follows, by 3.2. As already remarked (after 2.1) the reverse implication is easily derived by using 2.1.2 and the completeness theorem—as was noted by Ehrenfeucht. (It may also rather easily be derived from 2.1.1.) The argument depends on the well-known fact:

3.6. *A Boolean algebra contains infinitely many inequivalent elements if and only if it has a non-principal prime ideal.*

In Ryll-Nardzewski's theorem a semantical condition is shown to be equivalent to a purely syntactical statement. The equivalence, proved above, between 3.5.1 and 3.5.3 has the same character.

**§ 4. Saturated models.** A model $\mathfrak{A}$ of $T$ will be called *weakly saturated* if, for any $P \in \mathcal{P}_n(T)$ ($n$ arbitrary), $P(\mathfrak{A})$ is not empty. $\mathfrak{A}$ is said to be *saturated* if (in addition) the conditions $P \in \mathcal{P}_n(T)$, $P \subseteq Q \in \mathcal{P}_{n+1}(T)$, and $\langle a_0, \dots, a_{n-1}\rangle \in P(\mathfrak{A})$ imply that there exists an element $x$ such that $\langle a_0, \dots, a_{n-1}, x\rangle \in Q(\mathfrak{A})$.

LEMMA 4.1. *Suppose that $I$ is a relative interpretation of $T$ in a theory $T_1$, $\mathfrak{B}$ is a denumerable model of $T_1$, and $\mathfrak{A}$ is the denotation of $I$ in $\mathfrak{B}$. Let $n \in \omega$, $P \in \mathcal{P}_n(T)$, $P \subseteq Q \in \mathcal{P}_{n+1}(T)$, and*

$\langle b_0, \dots, b_{n-1}\rangle \in P(\mathfrak{A})$. *Then there exists a denumerable, elementary extension* $\mathfrak{B}^*$ *of* $\mathfrak{B}$, *in which the denotation of* $I$ *is a system* $\mathfrak{A}^*$ *having an element* $d$ *such that* $\langle b_0, \dots, b_{n-1}, d\rangle \in Q(\mathfrak{A}^*)$.

Proof. We employ a type of argument which has been used by A. Robinson [14]. Let $b_0, \dots, b_{n-1}, b_n, \dots$ be all the elements of $|\mathfrak{B}|$, and let $\mathrm{d}, \mathrm{b}_0, \dots, \mathrm{b}_n, \dots$ be distinct, new individual constants. Let the axioms of the theory $T_1'$ be all sentences of the form $\psi(\mathrm{b}_{j_0}, \dots, \mathrm{b}_{j_{k-1}})$, such that $\psi \in F_k(T_1)$ and $\models_{\mathfrak{B}}\psi[b_{j_0}, \dots, b_{j_{k-1}}]$, plus all sentences $\Phi^I(\mathrm{b}_0, \dots, \mathrm{b}_{n-1}, \mathrm{d})$ for which $\Phi \in Q$. If $\Phi_0, \dots, \Phi_p \in Q$ then, clearly, the formula $\bigvee v_n(\Phi_0 \wedge \dots \wedge \Phi_p) \in P$; therefore, there is a $y$ such that, for each $i \leqslant p$, $\models_{\mathfrak{A}}\Phi_i[b_0, \dots, b_{n-1}, y]$, i.e. $\models_{\mathfrak{B}}\Phi_i^I[b_0, \dots, b_{n-1}, y]$. It easily follows that any finite set of axioms of $T_1'$ has a model, so that $T_1'$ is consistent. By the completeness theorem, $T_1'$ has a model $B_1'$. $B_1'$ is of the form $(\mathfrak{B}_1, u_0, \dots, u_n, \dots)$, where $\mathfrak{B}_1$ is a model of $T_1$. Clearly, $\mathfrak{B}_1$ is isomorphic to a system $\mathfrak{B}^*$ having the desired properties.

THEOREM 4.2. *Suppose that each* $\mathcal{P}_n(T)$ *is countable. Let* $T_1$ *be a consistent theory, and let* $I_0, \dots, I_n, \dots$ *be relative interpretations of* $T$ *in* $T_1$. *Then* $T_1$ *has a denumerable model in which the denotation of each* $I_n$ *is a saturated model of* $T$.

Proof. We note first that given any denumerable model $\mathfrak{B}$ of $T_1$, a system $\mathfrak{B}^*$ can be found, with the following properties:

(1) $\mathfrak{B}^*$ is a denumerable elementary extension of $\mathfrak{B}$; (2) if $k, m \in \omega$, if $\mathfrak{A}$ and $\mathfrak{A}^*$ are the (respective) denotations of $I_k$ in $\mathfrak{B}$ and $\mathfrak{B}^*$, and if $P \in \mathcal{P}_m(T)$, $\langle a_0, \dots, a_{m-1}\rangle \in P(\mathfrak{A})$, and $P \subseteq Q \in \mathcal{P}_{m+1}(T)$, then there exists an $x$ such that $\langle a_0, \dots, a_{m-1}, x\rangle \in Q(\mathfrak{A}^*)$. Indeed, since each $P_n(T)$ is countable, we may enumerate all tuples $\langle I_k, P, Q, a_0, \dots, a_{m-1}\rangle$ for which the hypothesis of (2) holds. Let $\mathfrak{C}_0 = \mathfrak{B}$ and (recursively) let $\mathfrak{C}_{n+1}$ be a system of the type whose existence is asserted in Lemma 4.1—as applied to $\mathfrak{C}_n$ and to the entities $I_k, P, Q, a_0, \dots, a_{n-1}$ constituting the $n$th-tuple in our enumeration. One easily sees, using 1.1.1 and 1.1.2, that the system $\mathfrak{B}^* = \bigcup\{\mathfrak{C}_k/k \in \omega\}$ has the properties (1) and (2).

Now let $\mathfrak{B}_0$ be a denumerable model of $T_1$, and (recursively) let $\mathfrak{B}_{m+1}$ be to $\mathfrak{B}_m$ as $\mathfrak{B}^*$ is to $\mathfrak{B}$ in (1) and (2). Then, clearly, the system $\bigcup\{\mathfrak{B}_k/k \in \omega\}$ is as demanded in 4.2.

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(14) Cf. theorems 2.1, 2.2, and 2.5 of [11]. One could, indeed, derive 4.1 from these.

A special case of 4.2 is the following: a sufficient condition for $T$ to have a denumerable, saturated model is that each $\mathcal{P}_n(T)$ be countable. Later, in 4.7 and 4.4, we shall see that this condition is also necessary, and that there is at most one denumerable, saturated model of $T$ up to isomorphism. Consequently, the full Theorem 4.2 states, roughly, that denumerable, saturated models have further "saturation properties"—in addition to those in the definition of "saturated"; in particular, a kind of second order saturation.

LEMMA 4.3. *Suppose that $\mathfrak{B}$ is a denumerable, saturated model of $T$ and $\mathfrak{A}$ is an arbitrary, denumerable model of $T$. Then $\mathfrak{A}$ can be mapped isomorphically onto an elementary subsystem of $\mathfrak{B}$. Moreover, if $a_0, \ldots, a_{m-1} \in |\mathfrak{A}|$, $b_0, \ldots, b_{m-1} \in |\mathfrak{B}|$, and $(\mathfrak{A}, a_0, \ldots, a_{m-1}) \equiv (\mathfrak{B}, b_0, \ldots, b_{m-1})$, then the mapping can be chosen so that it carries $a_i$ into $b_i$ for each $i < m$.*

Proof. Let $a_0, \ldots, a_{m-1}, a_m, \ldots$ be all the elements of $|\mathfrak{A}|$. We can define by recursion a sequence $b_m, b_{m+1}, \ldots$ of elements of $|\mathfrak{B}|$ such that, for any $n$,

$$(\mathfrak{A}, a_0, \ldots, a_{n-1}) \equiv (\mathfrak{B}, b_0, \ldots, b_{n-1}) . \tag{3}$$

Indeed, (3) holds for $n < m$, by hypothesis. Suppose that (3) holds for a given $n \geqslant m$. Then, clearly (cf. 1.3), for some $P \in \mathcal{P}_n(T)$, $\langle a_0, \ldots, a_{n-1}\rangle \in P(\mathfrak{A})$ and $\langle b_0, \ldots, b_{n-1}\rangle \in P(\mathfrak{B})$. Now, for some $Q \in \mathcal{P}_{n+1}(T)$, $\langle a_0, \ldots, a_n\rangle \in Q(\mathfrak{A})$. But then $P \subseteq Q$, so that, since $\mathfrak{B}$ is saturated, we may choose $b_n \in |\mathfrak{B}|$ such that $\langle b_0, \ldots, b_n\rangle \in Q(\mathfrak{B})$. Then (3) holds with "$n+1$" for "$n$", by 1.3.

4.3 now follows from 1.1.3.

THEOREM 4.4. *Any two denumerable, saturated models of $T$ are isomorphic.*

LEMMA 4.5. *Any denumerable, saturated model of $T$ is homogeneous.*

Proof. The proofs of 4.4 and 4.5 are obtained by modifying that of 4.3 in a manner completely analogous to the one in which the proofs of 3.2 and 3.3 were obtained from that of 3.1.

THEOREM 4.6. *For a denumerable model $\mathfrak{A}$ of $T$, the following conditions are equivalent:*

(.1) *$\mathfrak{A}$ is saturated;*

(.2) *$\mathfrak{A}$ is $\aleph_0$-universal and homogeneous;*

(.3) *$\mathfrak{A}$ is weakly saturated and homogeneous.*

Proof. By 4.3 and 4.5, (.1) implies (.2). From the Gödel-Löwenheim-Skolem theorem, one sees at once that an $\aleph_0$-universal model is weakly saturated, so that (.2) implies (.3). Suppose (.3) holds, $a_0, \ldots, a_{n-1} \in P(\mathfrak{A})$, $P \in \mathcal{P}_n(T)$, and $P \subseteq Q \in \mathcal{P}_{n+1}(T)$. Since $\mathfrak{A}$ is weakly saturated, some $\langle a'_0, \ldots, a'_n\rangle \in Q(\mathfrak{A})$. Then $\langle a'_j, \ldots, a'_{n-1}\rangle \in P(\mathfrak{A})$, and hence, $\mathfrak{A}$ being homogeneous, there is an automorphism $f$ of $\mathfrak{A}$ taking $a'_i$ into $a_i$ for $i < n$. Consequently, $\langle a_0, \ldots, a_{n-1}, f(a'_n)\rangle \in Q(\mathfrak{A})$. Thus, (.3) implies (.1).

THEOREM 4.7. *The following conditions are equivalent:*

(.1) *Each $\mathcal{P}_n(T)$ is countable;*

(.2) *$T$ has a denumerable, saturated model;*

(.3) *$T$ has an $\aleph_0$-universal model;*

(.4) *$T$ has a weakly saturated, denumerable model* [15].

Proof. We have already remarked that "(.1) implies (.2)" is a special case of 4.2. By 4.3 or 4.6, (.2) implies (.3). As noted above, (.3) implies (.4). Finally, that (.4) implies (.1) is obvious.

COROLLARY 4.8. *If $T$ has an $\aleph_0$-universal model, then $T$ has a prime model.*

Proof. As is well-known, Boolean algebras with countably many prime ideals are atomistic. Hence 4.8 follows from 4.7, and 3.5. (While this proof of 4.8 depends on 2.1.1, it may be noted that another proof could be constructed depending, rather, on 2.1.2—since, under the hypothesis of 4.8, $\bigcup\{\mathcal{P}_n(T)/n \in \omega\}$ is countable.)

One is tempted to say, by analogy with the discussion in the last paragraph of § 3, that condition 4.7.1 is purely syntactical. Indeed, in 4.7.1, no reference to any semantical concept, such as "model", is made. However, a little thought convinces one that a notion of "purely syntactical condition" wide enough to include (.1) would be so broad as to be pointless.

In § 5 and § 6, we will see that the results of § 2-§ 4 can be applied to establish some general properties of models of certain kinds of theories. On the other hand, the chances that these results can be usefully applied in the study of a particular

(15) In view of 4.6, it follows that 4.7.1 or 4.7.3 is, also, a necessary and sufficient condition for $T$ to have an $\aleph_0$-universal, homogeneous model. In [20] and [21], the author announced some results concerning the existence in powers $\aleph_\alpha > \aleph_0$ of "$\aleph_\alpha$-universal models" for arbitrary theories and of $\aleph_\alpha$-universal, "homogeneous" models for complete theories. (For the meanings of "$\aleph_\alpha$-universal" and of "homogeneous" intended here, cf. [20] and [21].) The author takes this opportunity to state that he has learned that results very closely related to those in [20] and [21] were obtained several years earlier by Mr. Michael Morley. Morley's work is not yet published.

relational system or complete theory seem not too good. This is due at least in part to the fact that the notion "elementary subsystem" rather than "elementarily equivalent subsystem" is involved in such notions as "prime" or "$\aleph_0$-universal". Thus, for example, to establish that a theory $T$ fulfills any one of the conditions in 3.5 or 4.8 one would need to have already a good deal of metamathematical (and not just algebraic) information concerning $T$.

It may, however, be worthwhile, for the sake of illustration, to give some examples of theories which fulfill the condition of 3.5 or 4.7. (But it should be noted that the results of § 3 and § 4 yield no new information about these examples.) The theory $T_1$ of infinite, discretely ordered systems with, say, a first but no last element, is one in which each $\mathcal{P}_n(T_1)$ is countable. That this is so is easily verified, because the known decision procedure for $T_1$ provides a description of all possible definable sets and relations in models of $T_1$ (16). The $\aleph_0$-universal, homogeneous model of $T_1$ is the system of order type $\omega + (\omega^* + \omega)\cdot \eta$. It may be remarked that $T_1$ has $2^{\aleph_0}$ non-isomorphic denumerable models.

For the theory $T_2$ of real closed fields, the set $\mathcal{P}_1(T_2)$ obviously has $2^{\aleph_0}$ members. However, as is known (cf. [16], [12]) the field of real, algebraic numbers is a prime model of $T_2$ and each $F_n(T_2)$ is atomistic.

§ 5. **$\aleph_{1+a}$-categorical theories** (17). Ehrenfeucht has proved that

5.1. *If, for some $a$, $T$ has less than $2^{\aleph_0}$ non-isomorphic models of power $\aleph_a$, then each $\mathcal{P}_n(T)$ is countable* (18).

Consequently, the results of § 3 and § 4 may be applied to such theories $T$.

An immediate consequence of 3.5, 4.8, and 5.1 is

(16) For a brief discussion of $T_1$ and references, cf. [18], pp. 90-91.

(17) Examples of $\aleph_a$-categorical theories are given in [9] and [19]. It may be noted that, as pointed out in [9] and [19], any such theory, which has no finite models, is necessarily complete.

(18) For 5.1, cf. [2] (where only the case $n = 1$ is stated). Earlier, in [1], Ehrenfeucht had shown that a $2^{\aleph_a}$-categorical theory $T$ has an "$\aleph_0$-universal" model.

5.1 generalizes its own case where $a = 0$, which is much more easily proved. This case was established in [10], p. 25-26, for some special theories $T$; the method, however, is adequate for any $T$. (One should note the well-known fact that a denumerable Boolean algebra has either countably many or $2^{\aleph_0}$ prime ideals.)

THEOREM 5.2. *If the hypothesis of 5.1 holds, then $T$ has an $\aleph_0$-universal, homogeneous model and a prime model* (18).

The hypothesis of 5.1 is satisfied, in particular, by $\aleph_\alpha$-categorical theories $T$. Of course, 5.2 is of no interest when $T$ is $\aleph_0$-categorical. A typical example of a theory which is categorical in non-denumerable powers, but not in $\aleph_0$, is the theory of algebraically closed fields. Here, as is known (cf. [12] and, also, [18], p. 101), the field of complex, algebraic numbers is a prime model, while the algebraically closed fields of transcendence degree $\aleph_0$ are $\aleph_0$-universal, homogeneous models.

THEOREM 5.3. *Under the assumption that $T$ is $\aleph_1$-categorical, but not $\aleph_0$-categorical, we may say further that a prime model $\mathfrak{A}$ of $T$ is minimal—i.e., that $\mathfrak{A}$ has no proper elementary subsystems.*

Proof. Suppose, on the contrary, that a prime model $\mathfrak{A}$ of $T$ has a proper elementary subsystem $\mathfrak{A}'$. It is clear that $\mathfrak{A}'$ is also prime. Consequently, we can define recursively a transfinite sequence $\mathfrak{A} = \mathfrak{A}_0, \mathfrak{A}_1, \ldots, \mathfrak{A}_\xi, \ldots$ $(\xi < \omega_1)$ of prime models of $T$ such that, for any $\xi < \omega_1$, $\mathfrak{A}_\xi$ is a proper elementary subsystem of $\mathfrak{A}_{\xi+1}$ and $\mathfrak{A}_\xi = \bigcup\{\mathfrak{A}_\eta/\eta < \xi\}$ if $\xi$ is a limit number. Indeed, since all prime models of $T$ are isomorphic, our assumption guarantees that such an $\mathfrak{A}_{\xi+1}$ can be found, given $\mathfrak{A}_\xi$; and, when $\xi$ is a limit number less than $\omega_1$, then $\mathfrak{A}_\xi$ is prime, by 3.4 and 1.1.1. Again by 3.4 and 1.1.1, the model $\mathfrak{B} = \bigcup\{\mathfrak{A}_\xi/\xi < \omega_1\}$, of the power $\aleph_1$, is atomic. On the other hand, since $T$ is not $\aleph_0$-categorical, some $F_n(T)$ has a non-principal prime ideal $P$, by 1.1 and 3.6. By the completeness theorem and the generalized Löwenheim-Skolem theorem (19) (and 1.3), $T$ has a model $\mathfrak{C}$, of power $\aleph_1$, in which $P(\mathfrak{C})$ is not empty. Since $P(\mathfrak{B})$ is empty, $\mathfrak{B}$ and $\mathfrak{C}$ are not isomorphic, contrary to our hypothesis. Thus, 5.3 is established.

A conjecture of Łoś [9] is that a theory $T$ which is categorical in some non-denumerable power is categorical in all such powers. From this it would follow that, in 5.2, "$\aleph_1$", could be replaced by "$\aleph_{1+\alpha}$". We have been unable to prove this stronger version of 5.3.

In the following theorem we establish a result which would easily follow from Łoś' conjecture (and which, it would seem, might possibly be useful in establishing it).

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(19) Cf. [18], p. 92, line 5 and, for references, footnote 4 on the same page.

THEOREM 5.4. *Suppose that $T$ is $\aleph_1$-categorical and $\theta \in F_1(T)$. Then, in any model $\mathfrak{A}$ of $T$, the set $\{x/\models_{\mathfrak{A}} \theta[x]\}$ is either finite or of the same power as $\mathfrak{A}$.*

Proof. To simplify the notation we assume that $T$ has only one non-logical constant, the ternary relation symbol R; the extension to an arbitrary $T$ is obvious. Suppose that the conclusion is false, so that $T$ has a model $\mathfrak{A}^1$ in which the set $U = \{x/\models_{\mathfrak{A}^1} \theta[x]\}$ has an infinite power, smaller than that of $\mathfrak{A}^1$. By the generalized Löwenheim-Skolem Theorem [20], $\mathfrak{A}^1$ has an elementary subsystem $\mathfrak{A}^2$, having the power of $U$, such that $U \subseteq |\mathfrak{A}^2|$. Clearly, the system $(\mathfrak{A}^1, |\mathfrak{A}^2|)$ is a model of the theory $T'$, whose symbols are those of $T$ plus a new singulary predicate V, and whose axioms are the valid sentences of $T$, the sentences

$$(1) \qquad \bigvee v_0 V v_0, \qquad \bigvee v_1 \sim V v_1, \qquad \text{and} \qquad \bigwedge v_0[\theta(v_0) \to V v_0],$$

and all sentences of the form

$$(2) \qquad \bigwedge v_0 \ldots \bigwedge v_{n-1}[V v_0 \wedge \ldots \wedge V v_{n-1} \to (\varphi \leftrightarrow \varphi^V)]$$

—where $\varphi \in F_n(T)$ and $\varphi^V$ is obtained from $\varphi$ by "relativizing the quantifiers to V" [21].

In the theory $T'$ there are two relative interpretations of $T$—namely, $\langle v_0 = v_0, R v_0 v_1 v_2 \rangle$ and $\langle V v_0, R v_0 v_1 v_2 \rangle$. Since $T'$ is consistent, it follows from 4.2 and Ehrenfeucht's theorem, 5.1, that $T'$ has a denumerable model $\mathfrak{B} = \langle B, R, V \rangle$, such that the system $\mathfrak{A}^* = \langle B, R \rangle$ and its subsystem $\mathfrak{A}$ with universe $V$ are both saturated. Since all sentences of (1) and (2) are true in $\mathfrak{B}$, $\mathfrak{A}^*$ is obviously a proper, elementary extension of $\mathfrak{A}$, and

$$(3) \qquad \{x/\models_{\mathfrak{A}^*} \theta[x]\} = \{x/\models_{\mathfrak{A}} \theta[x]\}.$$

All denumerable, saturated models being isomorphic (by 4.4), we conclude that an arbitrary such system $\mathfrak{A}$ is a proper elementary subsystem of some other such system $\mathfrak{A}^*$ in such a way that (3) holds.

Now take for $\mathfrak{A}_0$ an arbitrary denumerable, saturated model of $T$ (by 5.1), for $\mathfrak{A}_{\xi+1}$ ($\xi < \omega_1$) a system related to $\mathfrak{A}_\xi$ as $\mathfrak{A}^*$ is to $\mathfrak{A}$ above, and for $\mathfrak{A}_\eta$ —when $\eta$ is a limit number $\leqslant \omega_1$ —the system $\bigcup\{\mathfrak{A}_\xi/\xi < \eta\}$. It is clear from 1.1.1 that

[20] —in the form given in [18], p. 92, Theorem 2.1.
[21] —in the sense of [17], p. 24-25.

the union of an elementarily increasing sequence of saturated systems is saturated, so that our construction is justified. By 1.1.1 and induction on $\xi$, we see that for any $\xi \leqslant \omega_1$ and for any $x$, $\models_{\mathfrak{A}_0} \theta[x]$ if and only if $\models_{\mathfrak{A}_\xi} \theta[x]$. Thus $\mathfrak{A}_{\omega_1}$ is a model of $T$, of power $\aleph_1$, in which $\{x/\models_{\mathfrak{A}_{\omega_1}} \theta[x]\}$ has the power $\aleph_0$. On the other hand, by applying the generalized completeness theorem (to the generalized theory $T_1$ obtained by adding to $T$ individual constants $c_0, \ldots, c_\xi, \ldots$ $(\xi < \omega_1)$ and axioms $c_\xi \neq c_\eta$ $(\xi \neq \eta)$ and $\theta(c_\xi)$—for $\xi, \eta < \omega_1$), we see that $T$ has a model $\mathfrak{C}$, of power $\aleph_1$, in which $\{x/\models_{\mathfrak{C}} \theta[x]\}$ has the power $\aleph_1$. Thus $\mathfrak{A}_{\omega_1}$ and $\mathfrak{C}$ are not isomorphic, contrary to the hypothesis that $T$ is $\aleph_1$-categorical. This completes the proof ([22]).

**§ 6. The number of non-isomorphic denumerable models.** Consider the complete theory $T^0$ whose models are all systems $\langle A, R\rangle$ such that $R$ is an equivalence relation over $A$, $R$ has exactly two equivalence classes, and each of these is infinite. Some time ago, Raphael Robinson remarked that $T^0$ has the following property: There are exactly two non-isomorphic models of $T^0$ having the power $\aleph_1$. He raised the question whether there exists a complete theory $T$ having exactly two non-isomorphic models of power $\aleph_0$ ([23]).

As is well-known, the theory $T_1$ of densely ordered systems without extreme points is complete and, by Cantor's theorem, has exactly one denumerable model, up to isomorphism. Ehrenfeucht constructed an example, which he showed to the author, of a complete theory, $T_3$, having exactly three non-isomorphic denumerable models. (He has kindly allowed me to reproduce it here.) $T_3$ has a binary relation symbol $<$ and individual constants $c_0, \ldots, c_n, \ldots$ The axioms of $T_3$ assure that, if $\langle A, <, c_0, \ldots, c_n, \ldots\rangle$ is a model of $T_3$, then $\langle A, <\rangle$ is a model of $T_1$, and $c_i < c_{i+1}$, for $i = 0, 1, \ldots$ That $T_3$ has the stated properties is easily shown; the three isomorphism types of denumerable models $\langle A, c_0, \ldots, c_n, \ldots\rangle$ of $T_3$ are those in which (i) $c_0, c_1, \ldots$ are confinal, (ii) $c_0, c_1, \ldots$ are not confinal and have no limit in $A$, or (iii) $c_0, c_1, \ldots$ have a limit in $A$.

([22]) Again we are unable to establish that "$\aleph_1$" can be replaced by "$\aleph_{1+\alpha}$", in 5.4. However, an argument similar to, but simpler than, the proof just given does show that, in 5.4, "$\aleph_1$-categorical" can be replaced by "$\aleph_\alpha$-categorical and $\aleph_{\alpha+1}$-categorical".

([23]) Robinson formulated this question during a conversation in 1957 with several people including the author.

By a simple modification of $T_3$ we may obtain, for $n = 4, 5, \ldots$, a complete theory $T_n$ having exactly $n$ non-isomorphic models. The non-logical constants of $T_n$ are $<$, $U_0, \ldots, U_{n-3}, c_0, c_1, \ldots$, the $U_i$ being singulary relation symbols. The axioms are those of $T_3$ plus axioms assuring that, in any model

$$(1) \qquad \langle A, <, U_0, \ldots, U_{n-3}, c_0, c_1, \ldots \rangle,$$

the sets $U_0, \ldots, U_{n-3}$ form a partition of $A$, each $U_i$ is dense in $A$, and, for each $n$, $c_n \in U_0$. It is a theorem of Skolem [14] that any two denumerable models of $T_n$ are isomorphic if we ignore their lists of distinguished elements. Using this fact it is easily seen that $T_n$ is complete and that the possible (isomorphism) types of denumerable models (1) of $T_n$ are the two in which (i) or (ii) holds, plus the $n-2$ in which $c_0, c_1, \ldots$ have a limit belonging to $U_i$, $i = 0, \ldots, n-3$.

The theories $T_3, T_4, \ldots$ have infinitely many non-logical constants, but they can easily be converted into complete theories $T'_3, T'_4, \ldots$ having only finitely many—and still having exactly 3, 4, etc., non-isomorphic denumerable models. How to do this, in general, will be clear from the description of $T'_3$. The non-logical constants of $T'_3$ are the binary relation symbols $<$ and R. Its axioms assure that, in any model $\langle A, <, R\rangle$: $R$ is an equivalence relation over $A$ having the substitution property with respect to $<$; the quotient of $\langle A, <\rangle$ modulo $R$ is a densely ordered system without extreme points; for $n = 1, 2, \ldots$, there exists exactly one $R$-equivalence class having exactly $n$ elements; and, for $n = 1, 2, \ldots$, and any $x, y \in A$, *if* the $R$-equivalence classes of $x$ and $y$ have exactly $n$ and $n+1$ elements, respectively, then $x < y$.

Of various simple proofs omitted in the above discussion, perhaps one showing that $T'_3$ is complete should be briefly indicated. Granted that $T'_3$ has only three non-isomorphic models, we may proceed as follows: Let $\mathfrak{A} = \langle A, <, R\rangle$ be a denumerable model of $T'_3$ of the type in which the $R$-equivalence classes having $1, 2, \ldots$ elements are confinal (in $\mathfrak{A}$ modulo $R$). To the theory $T$ whose valid sentences are all true sentences in $\mathfrak{A}$, add an individual constant c and axioms assuring that in any model $\langle A', <', R', c\rangle$, if $xRy$ holds for exactly $n$ elements $y$, then $x <' c$ $(n = 1, 2, \ldots)$. Clearly the resultant theory $T'$ is consistent, so that by the completeness theorem, it has a denumerable model $\langle A'', <'', R'', c''\rangle$;

$\langle A'', <'', R''\rangle$ is then a model of $T_3'$ of one of the other two types. By further applications of the completeness theorem, together with a construction which involves taking the union of an elementarily increasing family of systems, one can show that a denumerable model of $T_3'$ in which the finite equivalence classes do have a limit is elementarily equivalent to one in which they do not have a limit and are not confinal. It then follows easily from the Löwenheim-Skolem theorem that $T_3'$ is complete.

We shall now show that the situation is quite different for $n = 2$, completing the proof of

THEOREM 6.1. *There exists a complete theory having exactly n non-isomorphic denumerable models if and only if $n \neq 2$.*

Remainder of proof. Assume, on the contrary, that $T$ has exactly 2 non-isomorphic denumerable models. Then, by 5.1,

(2) each $\mathcal{P}_n(T)$ is countable.

On the other hand, by 1.1 and 3.6, some $F_m(T)$ has a non-principal prime ideal $P$. By 2.1.2, $T$ has a model $\mathfrak{A}_1$ in which $P(\mathfrak{A}_1) = 0$. If $\mathfrak{A}$ is any system such that

(3) for some $c_0, \ldots, c_{m-1}$, $(\mathfrak{A}, c_0, \ldots, c_{m-1})$ is a denumerable model of the theory $\bar{P}$ (cf. 1.3),

then $P(\mathfrak{A}) \neq 0$, and, hence, $\mathfrak{A}$ is not isomorphic to $\mathfrak{A}_1$. Consequently, all systems $\mathfrak{A}$ for which (3) holds must be isomorphic. Moreover, any such system $\mathfrak{A}$ must be saturated since $\mathfrak{A}_1$ is not saturated and, by 4.7 and (2), some denumerable model of $T$ is. By applying 4.5, we easily conclude that all denumerable models $(\mathfrak{A}, c_0, \ldots, c_m)$ of $\bar{P}$ are isomorphic. Hence, by 1.1, $F_m(\bar{P})$ contains only finitely many formulas inequivalent in $\bar{P}$. This contradicts the fact that the subset $F_m(T)$ of $F_m(\bar{P})$ already contains infinitely many formulas inequivalent in $T$ and, hence, $T$ being complete, in $\bar{P}$.

We may mention here the following unresolved conjecture (which appears to have been made by a number of people):

*A theory which is $\aleph_{1+\alpha}$-categorical but not $\aleph_0$-categorical has exactly $\aleph_0$ non-isomorphic denumerable models.*

A second problem is this:

*Can it be proved, without the use of the continuum hypothesis, that there exists a complete theory having exactly $\aleph_1$ non-isomorphic denumerable models?*

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