Elimination of Imaginaries in strongly minimal sets with flat geometries
Conference in honor of Viktor Verbovskiy, Almaty

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Strongly Minimal Theories

STRONGLY MINIMAL

Definition

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e.g. acf, vector spaces, successor

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a is in the algebraic closure of B ($a \in \mathrm{acl}(B)$) if for some $\phi(x, \mathbf{b})$:
$\models \phi(a, \mathbf{b})$ with $\mathbf{b} \in B$ and $\phi(x, \mathbf{b})$ has only finitely many solutions.

Theorem

If T is strongly minimal algebraic closure defines matroid/combinatorial geometry.

The trichotomy???

Zilber Conjecture

The acl-geometry of every model of a strongly minimal first order theory is

1. disintegrated (lattice of subspaces distributive)
2. vector space-like (lattice of subspaces modular)
3. (non-locally modular)
   1. very Ample Zariski Geometry iff mutually interpretable with acf
   2. flat $\Rightarrow$ cm-trivial $\Leftrightarrow$ not 2-ample
   3. Anything else?

Zilber: geometries $\leftrightarrow$ canonical structures

Hrushovski gave a method of constructing strongly minimal sets that have flat geometries and admit no associative binary function with infinite domain.

There is no apparent canonical structure - only a (very flexible) method.

Baizhanov's Question

Question (1990's)

Does every strongly minimal set that admits elimination of imaginaries interpret an algebraically closed field?

Partial Answer

1. Infinite language: No! Verbovskiy [Ver06]
2. finite language:
    1. Yes! for constructions of [Hru93, BP21].
    2. A program for other flat geometries

Groups, definable closure, and elimination of imaginaries

This section is about arbitrary strongly minimal theories not just Hrushovski constructions.

T^{eq} and elimination of imaginaries

Definition

1. M^{eq} : Add a sort U_E for each definable over $\emptyset$ equivalence relation E on M^n for each n and a map from M^n to U_E taking a to a/E. The a/E are dubbed 'imaginary'.
2. A theory T admits *elimination of imaginaries* if $M \models T$ implies for every formula $\varphi(\overline{x}, \overline{y})$ and $\overline{a} \in M^n$ there exists $\overline{b} \in M^m$ such that for every automorphism $f \in \mathrm{aut}(\mathrm{M})$, f fixes **b** iff f fixes $\varphi(M, \overline{a})$.
3. A theory T admits weak elimination of imaginaries iff for every formula $\phi(\overline{x}, \overline{a})$ there exists a formula $\psi(\overline{x}, \overline{y})$ such that there are only finitely many parameters $\overline{b}_1, \ldots, \overline{b}_n$ such that each of $\psi(\overline{x}, \overline{b}_1), \ldots, \psi(\overline{x}, \overline{b}_n)$ is equivalent to $\phi(\overline{x}, \overline{a})$.

Fact: Elimination of imaginaries

A theory T admits *elimination of imaginaries* if its models are closed under definable quotients. ACF: yes; locally modular: no

Finite Coding

Definition

A finite set $F = \{\overline{a}_1, \ldots, \overline{a}_k\}$ of tuples from M is said to be coded by $S = \{s_1, \ldots, s_n\} \subset M$ over A if

$$\sigma(F) = F \Leftrightarrow \sigma|S = \mathrm{id}_S \quad \text{for any } \sigma \in \mathrm{aut(M/A)}.$$

We say $T = \mathrm{Th}(M)$ has *the finite set property* if every finite set of tuples F is coded by some set S over $\emptyset$.

(weak) elimination of imaginaries and finite coding

Fact

If T admits weak elimination of imaginaries then T satisfies the finite set property if and only T admits elimination of imaginaries.

Since every strongly minimal theory with $\mathrm{acl}(\emptyset)$ infinite has weak elimination of imaginaries, [Pil99], we have

A strongly minimal T with infinite $\mathrm{acl}(\emptyset)$ admits elimination of imaginaries iff it has finite coding.

Group Action and Definable Closure

Fix I, a finite set of independent points in the model $M \models T$.

2 groups

Let $G_{\{I\}}$ be the set of automorphisms of M that fix I setwise and G_I be the set of automorphisms of M that fix I pointwise.

Definition

1. $\mathrm{dcl}^*(I)$ consists of those elements that are fixed by G_I but not by G_X for any $X \subsetneq I$.
2. The *symmetric definable closure* of I, $\mathrm{sdcl}^*(\mathrm{I})$, consists of those elements that are fixed by $G_{\{I\}}$ but not by $G_{\{X\}}$ for any $X \subsetneq I$.

$\mathrm{sdcl}^*(\mathrm{I}) = \emptyset$ implies T does not admit elimination of imaginaries.

$\mathrm{sdcl}^*(\mathrm{I}) \subseteq \mathrm{dcl}^*(\mathrm{I}) \subseteq \mathrm{dcl}(\mathrm{I})$.

‘Non-trivial definable functions’

Definition

Let T be a strongly minimal theory. A function $f(x_0 \ldots x_{n-1})$ is called *essentially unary* if there is an $\emptyset$-definable function $g(u)$ such that for some i, for all but a finite number of $c \in M$, and all but a set of Morley rank $< n$ of tuples $\mathbf{b} \in M^n$, $f(b_0 \ldots b_{i-1}, c, b_i \ldots b_{n-1}) = g(c)$.

Lemma

For a strongly minimal T the following conditions are equivalent:

1. for any $n > 1$ and any independent set $I = \{a_1, a_2, \ldots a_n\}$, $\mathrm{dcl}^*(I) = \emptyset$;
2. every $\emptyset$-definable n-ary function ($n > 0$) is essentially unary;
3. for each $n > 1$ there is no $\emptyset$-definable truly n-ary function in any $M \models T$.

Definable closure, finite coding, elimination of imaginaries

Lemma

Let $I = \{a_0, a_1\}$ be an independent set with $I \leq M$ and M is a generic model of a strongly minimal theory.

1. *If $\mathrm{sdcl}^*(\mathrm{I}) = \emptyset$ then I is not finitely coded.*
2. *If $\mathrm{dcl}^*(I) = \emptyset$ then I is not finitely coded and there is no parameter free definable binary function.*

The Hrushovski Construction

The diversity of flat strongly minimal sets

The ‘Hrushovski construction’ actually has 5 parameters:

Describing Hrushovski constructions

1. σ: vocabulary
2. $\boldsymbol{L}_0$: A univerally axiomatized collection of finite σ-structures. (But generalizing to $\forall\exists$ is useful.)
3. ϵ: A submodular (hence flat) function from $\boldsymbol{L}_0^*$ to $\mathbb{Z}$.
4. $\boldsymbol{L}_0$: $\boldsymbol{L}_0^*$ defined using ϵ.
5. μ: a function bounding the number of 0-primitive extensions of an $A \in \boldsymbol{L}_0$ are in L_μ.

To organize the classification of the theories each choice of a class **U** of μ yields a collection of T_μ with similar properties.

Flatness

Definition

Flat pregeometries

1. Suppose (A, cl) is a pregeometry on a structure M with dimension function d and $F_1, \ldots, F_s$ are a sequence finite-dimensional d-closed subsets of A.
   For $T \subseteq \{1, \ldots s\}$ let $F_T = \bigcup_{i \in T} F_i$ and $F_\emptyset = \bigcup_{1 \leq i \leq s} F_i$.
   Then (A, cl) is *flat* if $d(F_\emptyset)$ is $\leq$ the value computed by the include-exclude principal applied to the F_S.
2. (A, cl) is *strictly flat* if it is flat but not *distintegrated* $(\mathrm{acl}(ab) \neq \mathrm{acl}(a) \cup \mathrm{acl}(b))$.

In Hrushovski construction flatness for the d-geometry and algebraic closure are equivalent.

The main result: Classifying dcl [BV22]

Theorem

Let T_μ be a strongly minimal theory as in Hrushovski's original paper. I.e. $\mu \in \mathcal{U} = \{\mu : \mu(A/B) \geq \delta(B)\}$). Let $I = \{a_1, \ldots, a_v\}$ be a tuple of independent points with $v \geq 2$.

G_I If T_μ triples

$$\mathcal{U} \supseteq \mathcal{T} = \{\mu : \mu(A/B) \geq 3\}$$

then $\mathrm{dcl}^*(I) = \emptyset$,
$\mathrm{dcl}(I) = \bigcup_{a \in I} \mathrm{dcl}(a)$,
and every definable function is essentially unary (Definition 10).

$G_{\{I\}}$ In any case $\mathrm{sdcl}^*(\mathrm{I}) = \emptyset$
$\mathrm{sdcl}(\mathrm{I}) = \bigcup_{\mathrm{a} \in \mathrm{I}} \mathrm{sdcl}(\mathrm{a})$
and there are no $\emptyset$-definable symmetric (value does not depend on order of the arguments) truly v-ary function.

In both cases T_μ does not admit elimination of imaginaries and the algebraic closure geometry is not disintegrated.

Amalgamation and Generic model

We study classes $\boldsymbol{K}_0$ of finite structures A
with $\delta(A') \geq 0$, for every $A' \subset A$.
basic example: one ternary relations $\delta(A) = |A| - \#$(realizations of R.
$d_M(A/B) = \min\{\delta(A'/B) : A \subseteq A' \subset M\}$.

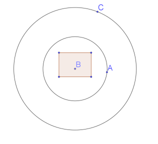
$A \leq M$ if $\delta(A) = d(A)$.
When $(\boldsymbol{K}_0, \leq)$ has joint embedding and amalgamation there is unique countable generic.

Primitive Extensions and Good Pairs

Definition

Let $A, B, C \in \mathbf{K}_0$.

1. C is a 0-*primitive extension* of A if C is minimal with $\delta(C/A) = 0$.



2. C is good over $B \subseteq A$ if B is minimal contained in A such that C is a 0-*primitive extension* of B. We call such a B a *base*.

α is the isomorphism type of $(\{a, b\}, \{c\})$,

Overview of construction

Realization of good pairs

1. A good pair C/B *well-placed* by A in a model M, if $B \subseteq A \leq M$ and C is 0-primitive over X.
2. For any good pair (C/B), $\chi_M(B, C)$ is the maximal number of disjoint copies of C over B appearing in M.
3. For $\mu \in \mathcal{U}$, $\boldsymbol{K}_\mu$ is the collection of $M \in \boldsymbol{K}_0$ such that $\chi_M(A, B) \leq \mu(A, B)$ for every good pair (A, B).

Adequacy Condition

For every good pair A/B, $\mu(A/B) \geq \delta(B)$.

Guarantees amalgamation (and more!)

If C/B is well-placed by $A \leq M$, $\chi_M(B, C) = \mu(B/C)$

The structure of $\mathrm{acl}(X)$

G-decomposable sets

Definition

$\mathcal{A} \subseteq M$ is G-decomposable if

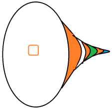
1. $\mathcal{A} \leq M$
2. $\mathcal{A}$ is G-invariant
3. $\mathcal{A} \subset_{<\omega} \mathrm{acl}(I)$.

Fact

There are G-decomposable sets.
Namely for any finite U with $d(U/I) = 0$,

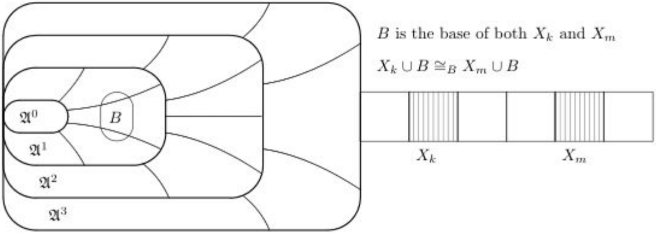
$$\mathcal{A} = \mathrm{icl}(\mathrm{I} \cup \mathrm{G}(\mathrm{U}))$$

Linear Decomposition



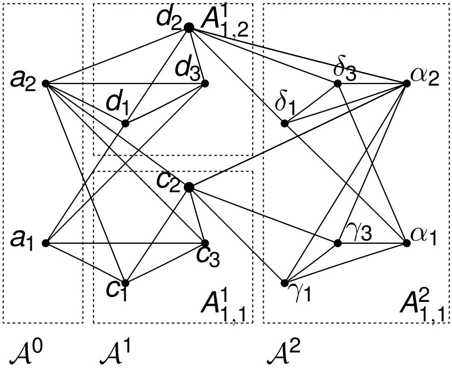
Constructing a G-tree-decomposition I

$\mathfrak{A}_0 = \mathrm{icl}(\mathrm{I})$ so has dimension 2.



A non-trivial definable binary function

In the diagrams, we represent a triple satisfying R by a triangle.



Constructing a G- tree-decomposition II

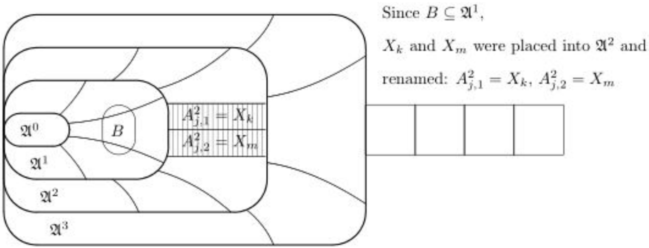


FIGURE 11. From a linear to a tree-decomposition: One Step

Proof idea

Suppose $I \subset \mathfrak{A} \leq M$ and (A/B) is well-placed by $D \subseteq \mathfrak{A}$. Fix a G-normal $\mathcal{A} \leq M \models \hat{T}_\mu$ with height m_0.

1. There are at least two copies of A over $\mathfrak{A}$. Then no element of A is in $\mathrm{dcl}(I)$.
2. **Lemma** Assume that $\hat{T}_\mu$ triples. For $m \geq 1$,
   1. dim_m: $d(E) \geq 2$ for any G_I-invariant set $E \subseteq \mathcal{A}^m$, which is not a subset of $\mathcal{A}^0$.
   2. moves_m: No $A^m_{f,k}$ is G_I-invariant.

   This Lemma is proved by induction on m_0.

Observation

None of these examples are pseudo-finite: $M \models \phi$ implies ϕ has a finite model.
This follows from a theorem of Pillay that any strongly minimal pseudo-finite theory is locally modular.

Conclusion

Strongly minimal theories with non-locally modular algebraic closure

1. Diversity
   1. $2^{\aleph_0}$ theories of strongly minimal Steiner systems (M, R) with no $\emptyset$-definable binary function
   2. $2^{\aleph_0}$ theories of strongly minimal quasigroups $(M, R, *)$ + an example of Hrushovski
   3. Non-Desarguesian projective planes definably coordinatized by strongly minimal ternary fields [Bal95]
   4. 2-ample but not 3-ample sm sets (not flat) [MT19]
   5. strongly minimal eliminates imaginaries (flat) INFINITE vocabulary) (Verbovskiy)
   6. field-like

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   6. field-like
2. Classifying sm sets with flat geometry
   1. discrete
   2. non-trivial but no binary function
   3. non-trivial but no commutative binary function
   4. Non-Desarguesian proj-planes definably coord by ternary fields

Further Problems

Main Conjecture

Take the class $\boldsymbol{L}_0$ to be all finite τ-structures that satisfy the hereditarily positive ϵ dimension discussed above and the adequacy condition on μ.

Conjecture: If there is a natural number N, such that $\mu(A/B) \geq \delta(B)$ for any good pair (A/B) with $\delta(B) \geq N$; then $\mathrm{sdcl}^*(\mathrm{I}) = \emptyset$ for any independent set I with $|I| \geq \max\{N, 5\}$.

It then follows no Hrushovski construction in a finite relational vocabulary τ (that is, where $\boldsymbol{K}_0$ contains all finite τ-structures) has elimination of imaginaries.

More general issues

1. Does any SM set with flat geometry admit elimination of imaginaries?
   Note these include the quasi-groups and ternary fields discussed above.
2. [Eva11] Are Hrushovski's strongly minimal structures in [Hru93] reducts of trivial theories? Evans shows the ω-stable versions are.

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