From Geometry to Algebra: Multiplication is not repeated addition

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Advanced Mathematics from an Elementary Standpoint

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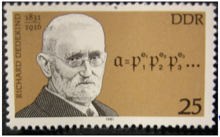
Pedagogical points

1. to teach the idea of proof it is necessary to make hypotheses clear.
2. The Hilbert/Euclid geometric axioms provide this clarity while the Birkhoff system used in U.S. schools for the last 50 years impedes the understanding of proof.

Mathematical points

1. The notions of multiplication as ‘repeated addition’ and ‘scaling’ are distinct mathematical notions.
2. The second provides a geometric foundation for area and proportionality which avoids the use of limits.

Arithmetization of Analysis and Geometry:

Dedekind 

Weierstrass 
Birkhoff 

(late 19th century) Arithmetization of Analysis

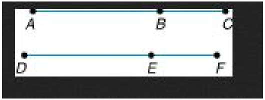
Dedekind, Weierstrass, Kronecker etc. built analysis up from the natural numbers to solve problems about limits, continuity, differentiability etc.

(G.D. Birkhoff) Arithmetization of Geometry

G.D. Birkoff (A set of postulates for plane geometry, Annals of Mathematics, 1932)
proposed a set of axioms for Euclidean Geometry assuming the properties (including completeness) of the real numbers.
Geometric properties were transferred to the reals by the ‘ruler’ and ‘protractor’ axioms in high school texts.

Johnny's first proof (from Glencoe geometry)

Geometry: Proving segment relations: For the proof shown, provide statement 2.



Given: $\overline{AC} \cong \overline{DF}$
$\overline{BC} \cong \overline{EF}$

Prove: $\overline{AB} \cong \overline{DE}$

Proof:

Statements	Reasons
1. $\overline{AC} \cong \overline{DF}$, $\overline{BC} \cong \overline{EF}$	**a.** Given.
2. ___?___	**b.** Definition of congruent segments
3. $AC - BC = DF - EF$	**c.** ___?___
4. $AC - BC = AB$, $DF - EF = DE$	**d.** Segment Addition Postulate
5. ___?___	**e.** Substitution Property (=)
6. $\overline{AB} \cong \overline{DE}$	**f.** ___?___

A. $AC = DF, BC = EF$

B. $AB = BC, DE = EF$

C. $AC = DE, AB = DF$

D. $AB = DE$

The ”right” proof

Common notion 3. If equals are subtracted from equals, then the remainders are equal.
http://aleph0.clarku.edu/~djoyce/java/elements/bookI/bookI.html#cns

‘equals’ may be natural numbers, segments (up to congruence), areas, volumes etc.

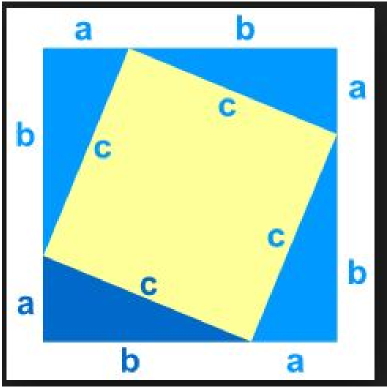
CCSM (Common Core State Standards) on proof

Mathematical Practice 3

Construct viable arguments and critique the reasoning of others. Mathematically proficient students understand and use stated assumptions, definitions, and previously established results in constructing arguments.

CCSM grade 8 proof

”Understand and apply the Pythagorean Theorem. 6. Explain a proof of the Pythagorean Theorem and its converse.”

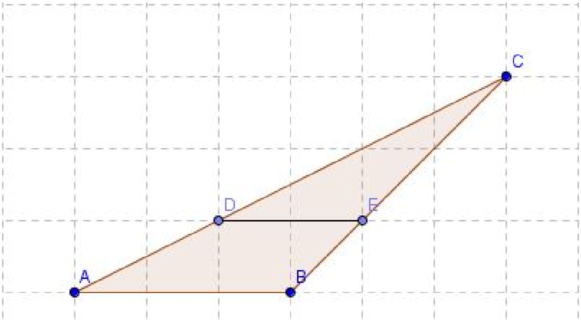


What are the assumptions?

Side-splitter Theorem

Theorem: Euclid VI.2

If a line is drawn parallel to the base of triangle the corresponding sides of the two resulting triangles are proportional and conversely.



$$CD : CA :: CE : CB$$

What does proportional mean?

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commensurability

AB and CD are commensurable if there exists EF, m, n such that AB is n copies of EF and CD is m copies of EF

Euclid's proof of sidesplitter

1. Uses Area
2. Use Eudoxus to deal with incommensurable side lengths.

What does proportional mean?

Hilbert's proof

1. We define a notion of multiplication $\times$ on segments.
2. so that

$$CD : CA :: CE : CB$$

   means

$$CD \times CB = CE \times CA.$$

Thus the side-splitter theorem will be verified for any figure whose lengths are in the model
– with no recourse to approximation or area.

Acknowledgements: Foundations and High School

1. Euclid, Hilbert et al
2. CTTI workshop on Geometry 2012 http://homepages.math.uic.edu/~jbaldwin/CTTIgeometry/ctti
3. Andreas Mueller
4. Hartshorne and Greenberg
5. Harel: Common Core State Standards for Geometry: An Alternative Approach
   http://www.ams.org/notices/201401/rnoti-p24.pdf

Outline

Euclid-Hilbert Axioms

Synthetic vrs analytic geometry

Synthetic Geometry

Develop geometry systematically from

1. a short list of primitive notions
2. postulates about these notions
3. introduce more sophisticated notions by definition

Analytic Geometry

Analytic geometry is the algebra of R and R^2 and R^3. It's hypotheses are thus whatever one assumes about the reals-complete archimedean ordered field. It is not really a matter of *proving* theorems, but of *calculating* results. (Craig Smorynski)

But is analytic geometry a good vehicle for the initial teaching of proof?

Why analytic geometry is not a good vehicle for introducing proof

1. Proof is not the same as calculation.
2. While mathematicians often prove with the hypothesis implicit, they know that there are explicit hypotheses that they could excavate.

Once students have learned to prove with explicit hypotheses, they can proceed to the more refined skills of

1. unearthing the hypotheses of a proposed argument and
2. understand that calculation is a tool for some kinds of proofs.

Thesis

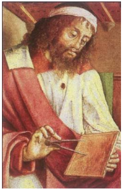
elementary geometry: A pun

1. Greenberg: the geometry of lines and circles – straight-edge and compass construction
2. geometry axiomatized in first order logic

To meet Detlefsen's demand for descriptive completeness, we must show the consequences of these axioms are the 'commonly accepted sentences' pertaining to this subject area.

1. first order: (Hilbert)
   1. the side-splitter theorem
   2. The area of every triangle is measured by a segment. That is, justify area formulas.
2. first order: (new)
   1. Formulas for area and circumference of circle
   2. In some models all angles have measure.

Euclid-Hilbert formalization 1900:

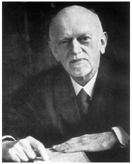
Euclid  Hilbert 

The Euclid-Hilbert (the Hilbert of the Grundlagen) framework has the notions of axioms, definitions, proofs and, with Hilbert, models.
But the arguments and statements take place in natural language.

Euclid uses diagrams essentially; Hilbert uses them only heuristically.

For Euclid-Hilbert logic is a means of proof.

Hilbert-Gödel-Tarski-Vaught formalization 1918-1956:

Hilbert  Gödel  Tarski  Vaught 

In the Hilbert (the founder of proof theory)-Gödel-Tarski framework, logic is a mathematical subject.

There are explicit rules for defining a formal language and proof. Semantics is defined set-theoretically.

First order logic is complete. The theory of the real numbers is complete and easily axiomatized. The first order Peano axioms are not complete.

We work initially in the Euclid-Hilbert mode
but use the insights of model theory to study circles.

Goals

We describe our vocabulary and postulates in a way immediately formalizable as a first order theory T_{Euclid}.

We will show:

1. T_{Euclid} directly accounts for proportionality and area of polygons.
2. We have to extend T_{Euclid} using methods of contemporary model theory to have formulas for arc length and area.

There is no appeal to the axioms of Archimedes or Dedekind.

Vocabulary

The fundamental notions are:

1. two-sorted universe: points (P) and lines (L).
2. Binary relation $I(A, \ell)$:
   Read: a point is incident on a line;
3. Ternary relation $B(A, B, C)$:
   Read: B is between A and C (and A, B, C are collinear).
4. quaternary relation, $C(A, B, C, D)$:
   Read: two segments are congruent, in symbols $\overline{AB} \approx \overline{CD}$.
5. 6-ary relation $C'(A, B, C, A', B', C')$: Read: the two angles $\angle ABC$ and $\angle A'B'C'$ are congruent, in symbols $\angle ABC \approx \angle A'B'C'$.

τ is the vocabulary containing these symbols.

Note that I freely used defined terms: collinear, angle and segment, in giving the reading.

First order *fully* geometric Postulates

1. Incidence postulates
2. the betweenness postulates (after Hilbert) (yield dense linear ordering of any line).
3. One congruence postulate: SSS
4. parallel postulate
   That, if a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles.

(14 first order geometric axioms) Return
Glencoe: 22 geometric/algebraic axioms and field axioms and (hidden) Dedekind completeness

Incidence postulates

Euclid's first 3 postulates in modern language

1. **Postulate 1** Given any two points there is a (unique) line segment connecting them.

$$(\forall p_1, p_2)(\exists \ell)[I(p_1, \ell) \wedge I(p_2, \ell)]$$

$$(\forall p_1, p_2)(\forall \ell_1, \ell_3)[(I(p_1, \ell_1) \wedge I(p_2, \ell_1) \wedge I(p_1, \ell_2) \wedge I(p_2, \ell_2)]) \rightarrow \ell_1 = \ell_2$$

2. **Postulate 2** Any line segment can be extended indefinitely (in either direction).
3. **Postulate 3** Given a point and any segment there is a circle with that point as center whose radius is the same length as the segment.

Section 4: From Geometry to Numbers

From geometry to numbers

We want to define the addition and multiplication of numbers. We make three separate steps.

1. identify the collection of all congruent line segments as having a common ‘length’. Choose a representative segment OA for this class.
2. define the operation on such representatives.
3. Identify the length of the segment with the end point A. Now the multiplication is on points. And we define the addition and multiplication a little differently.

Today we do step 2; the variant of step 3 is a slight extension.

Defining addition I

Adding line segments

The sum of the line segments OA and OB is the segment OC obtained by extending OB to a straight line and then choose C on OB extended (on the other side of B from A) so that $OB \cong AC$.

Properties of segment addition

Is segment addition associative?
Does it have an additive identity?

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No. For this we need 'addition on points'. - a topic for another day.

For the moment think of the algebraic properties of

$$\{a : a \in \Re, a \geq 0\}$$

with ordinary $+$ and $\times$.

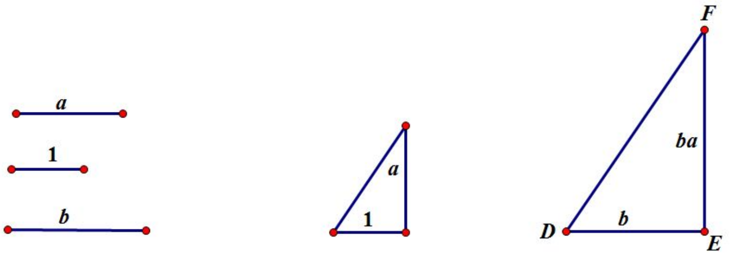
Defining Multiplication

Consider two segment classes *a* and *b*. To define their product, define a right triangle[1] with legs of length *a* and *b*. Denote the angle between the hypoteneuse and the side of length *a* by α.

Now construct another right triangle with base of length *b* with the angle between the hypoteneuse and the side of length *b* congruent to α. The length of the vertical leg of the triangle is *ab*.

[1] The right triangle is just for simplicity; we really just need to make the two triangles similar.

Defining segment Multiplication diagram



Note that we must appeal to the parallel postulate to guarantee the existence of the point F.

Parallel Postulate

Is multiplication just repeated addition?

On the one hand, we can think of laying 3 segments of length a end to end.

On the other, we can perform the segment multiplication of a segment of length 3 (i.e. 3 segments of length 1 laid end to end) by the segment of length a.

Only the second has a natural multiplicative inverse *on segments*.

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Only the second has a natural multiplicative inverse *on segments*.

The theory of $(\omega, +, \times)$ is essentially undecidable.

The theory of $(\Re^+, +, \times)$ is decidable and proved consistent in systems of low proof theoretic strength (EFA).

Section 6: Field axioms and Proportionality

Obtaining the field properties

Obtaining the field properties

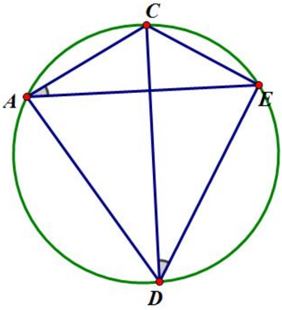
Addition and multiplication are associative and commutative.
There are additive and multiplicative units and inverses.
Multiplication distributes over addition.

(The negative numbers are omitted to avoid complication.)
This particular definition is due to Robin Hartshorne.

Necessary Geometry on Circle

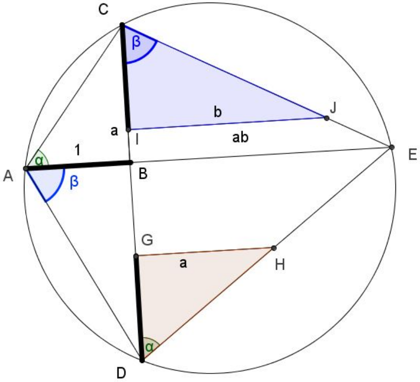
Cyclic Quadrilateral theorem

Let $ACED$ be a quadrilateral. The vertices lie on a circle (the ordering of the name of the quadrilateral implies A and E are on opposite sides of CD) if and only if $\angle EAC \cong \angle CDE$.



Commutativity of Multiplication

Angle α gives right multiplication by a. Angle β gives right muliplication by b.



Thus from the top right quadrant EB has length ab.

But from the bottom right quadrant EB has length ba.

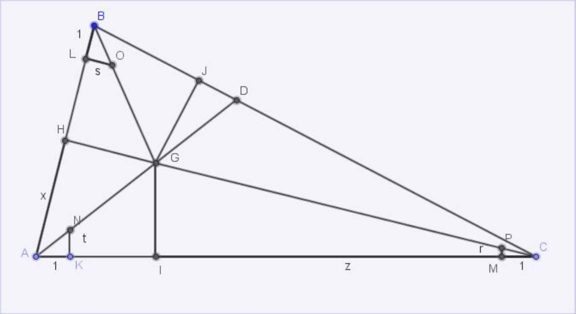
Similar Triangles have proportional sides I.

Theorem

If ABC and $A'B'C'$ are similar triangles then **using the segment multiplication we have defined**

$$\frac{AB}{A'B'} = \frac{AC}{A'C'} = \frac{BC}{B'C'}.$$

Consider the triangle ABC below with incenter G.



Similar Triangles have proportional sides

The point G is the incenter so $HG \cong GI \cong GJ$. Call this segment length a.

Now construct $AK \cong BL \cong MC$ all with standard unit length. Let the lengths of BL be s, NK be t and PM be r.

Let the lengths of $AI \cong AH$ be x, $BH \cong BJ$ be y, and $CI \cong AJ$ be z.

By the definition of multiplication $t \cdot x = r \cdot z = a$. Therefore the length of AC is $\frac{a}{t} + \frac{a}{r} = \frac{a(r+t)}{rt}$.

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The crucial point is that because the angles are congruent r, s, t are the same for both triangles.

And I have defined the ratio of side lengths by expression depending only on r, t. So

$$\frac{A'C'}{AC} = \frac{a'}{a}.$$

The same is true for the other two pairs of sides so the sides of the triangle are proportional.

Consequences

For any model M of the listed postulates: similar triangles have proportional sides.
There is no assumption that the field is Archimedean.

There is no appeal to approximation or limits.

It is easy to check that the multiplication is exactly the usual multiplication on the reals because they agree on the rationals.

The multiplication gives a good theory of area for polygons. (See `http://cmeproject.edc.org/cme-project/geometry-table-contents`)

The key points are Euclid's proof that 'two triangle between the same parallels and on the same base have the area' and showing that the product of the base and height of a triangle does not depend on which base and altitude are chosen.

Area

http://aleph0.clarku.edu/~djoyce/java/elements/bookI/propI35.html

Uses common notions 2 and 3 for *area*

Common Core

G-C01

Know precise definitions of angle, circle, perpendicular line, parallel line, and line segment, based on the undefined notions of point, line, distance along a line, and distance around a circular arc.

Why is the word undefined in this standard? What and how do we know about the undefined notions?

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Why are both 'distance along a line' and 'distance around a circular arc' in the list of undefined concepts instead of just 'distance'?

The field over the real algebraic numbers is a model of these postulates.
But there is no line segment of the same length as the circumference of a unit circle.

Toward adding π

Describing π

Add to the vocabulary τ a new constant symbol π. Let i_n (c_n) be the perimeter of a regular n-gon inscribed (circumscribed) in a circle of radius 1.
Add for each n,

$$i_n < 2\pi < c_n$$

to give a collection of sentences $\Sigma(\pi)$.

A first order theory for a vocabulary including a binary relation $<$ is *o-minimal* if every 1-ary formula is equivalent to a Boolean combination of equalities and inequalities.

Anachronistically, the o-minimality (every definable subset is a finite union of intervals) of the reals is a main conclusion of Tarski.

The theory with π

Metatheorem

The following set T_π of axioms is *first order complete* for the vocabulary τ along with the constant symbols $0, 1, \pi$.

1. the postulates of a Euclidean plane.
2. A family of sentences declaring every odd-degree polynomial has a root.
3. $\Sigma(\pi)$

If the field is Archimedean there is only one choice for the interpretation of π. If not, there may be many but they are all first order equivalent.

Circumference

Definition

The theory $T_{\pi,C}$ is the extension by definitions of the $\tau \cup \{0, 1, \pi\}$-theory T_π obtained by the explicit definition $C(r) = 2\pi r$.

As an extension by explicit definition, $T_{\pi,C}$ is a complete first order theory.

The Circumference formula

Since

1. by similarity, $i_n(r) = ri_n$ and $c_n(r) = rc_n$,
2. by our definition of multiplication, $a < b$ implies $ra < rb$.
3. and by the approximations of π by Archimedes

Metatheorem

In $T_{\pi,C}$, $C(r) = 2\pi r$ is a circumference function. That, for any r, $C(r)$ is bounded below and above by the perimeter of inscribed and circumscribed regular polygons of a circle with radius r.

Summary

1. The first serious introduction to proof should make the hypotheses clear.
2. The hypotheses are more easily understood and the theorem proved more in need of proof from Euclid's axioms than Birkhoffs
3. Choices have to be made as to which lies are being told to students. Rather than never mentioning the notion of limit (but hiding it in 'ruler' and 'protractor axioms') one should develop the geometry that does not need limits.
4. This last includes all of 'elementary geometry'.