Math 512: Finite Model Theory RPC and r.e.

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We work in a finite relational language. For simplicity, we assume the language has one ternary relation symbol: R. Following Vaanenen, a finite stucture A is coded as a sequence $c(A)$: $m\#p$. m is binary for the number of elements in A. We assume A is in 'standard position". That is the universe of A is an integer n. p is a sequence of length n^3 of 0's and 1's depending on whether the ith element in a lexiocographic ordering of A^3 (depending on the underlying order of A is or is not in R.

0.1 Definition. A class K of finite L-structures is relativized pseudo-elementary if there is a language L^+ extending L with a new predicate P and an L^+-sentence θ such that for every finite L^+-structure B,

$$B \models \theta \text{ iff } P(B)|L \in K.$$

0.2 Theorem. *Let K be a collection of finite L-structures closed under isomorphism. Then then the set $\{c(A) : A \in K\}$ is recursively enumerable if and only if K is the collection of finite models of a relatively psuedo-elementary class.*

Proof. If K is rpc, we can recursively enumerate K just by examining the finite models of θ.

For the converse, let M be the Turing machine which accepts (codes of) members of K. Fix the language L^+ to have two unary predicates P and Q, a binary relation $<$, a unary predicate T, the language of L, binary predicates $B_0(x,t)$,$B_1(x,t)$,$B_\#(x,t)$, $S(x,y)$ and $C_q(x,t)$, which encode the action M,

We provide below the key points of the coding. Various boundary conditions are implicit. Also, to clarify specifying that Q contains the code for $(P(A)$, different notations for 0's and 1's preceding or following are used. Of course, M only sees one kind of 0 or 1.

Let θ assert:

i) P and Q partition the universe.

ii) The symbols of L apply only to elements of P.

iii) $<$ is a linear order (separately) of each of P and Q, T is the terminal element of P.

iv) $B_0(x,t)$,$B_1(x,t)$, $S(x,y)$ and $C_q(x,t)$, encode the action M on input $c(A)$ which halts just if $c(A)$ is the code of an element of K.

v) sentences asserting that the initial segment of Q is binary for the length of A. For $Q(x)$ and $P(t)$;

(a) $B_0'(0,t)$, $C_0'(x,t)$.
(b) $t < T \wedge B_0'(x+1,t) \wedge C_0'(x,t+1) \to B_0'(x+1,t+1) \wedge C_0'(x+1,t+1)$
(c) $t < T \wedge B_0'(x+1,t) \wedge C_1'(x,t+1) \to B_1'(x+1,t+1) \wedge C_0'(x+1,t+1)$
(d) $t < T \wedge B_1'(x+1,t) \wedge C_0'(x,t+1) \to B_1'(x+1,t+1) \wedge C_0'(x+1,t+1)$
(e) $t < T \wedge B_1'(x+1,t) \wedge C_1'(x,t+1) \to B_0'(x+1,t+1) \wedge C_1'(x+1,t+1)$
(f) $t = T \to C_h'(x,T) \wedge B_\#(x+1,t)$,

vi) for $P(x_i)$ and $Q(f(x_1,x_2,x_3))$:

(a) $B_\#(x,T) \to S(x, f(0,0,0)$;
   1. $S(f(x_1,x_2,x_3), f(x_1,x_2,S(x_3))$ if $x_3 < T$
   2. $S(f(x_1,x_2,x_3), f(x_1,S(x_2),0)$ if $x_3 = T$, etc.

(b) $B_1(f(x_0,x_1,x_2)$ iff $R(x_0,x_1,x_2)$.
(c) $Q(x) \to x < \# \vee (\exists\overline{x})x = f(\overline{x})$.

Now, $B \models \theta$ and B finite implies that $Q(B)$ is a computation of M accepting $c(P(B)|L)$, so $P(B)|L) \in K$ as required. And clearly each member of K can be expanded to a finite model of θ.

Note that much of the effort was to code the cardinality of $P(A)$ in binary. This is totally irrelevant to the theorem at hand; we could use the 1-ary representation. It could be important for finer complexity computations.