STABLE AMALGAMATION

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GENERIC STRUCTURES

L is a first order language possibly with function symbols.

$\overline{\mathbf{K}}_0$ is the class of models of a universal theory T.

$\mathbf{K}_0$ is the finitely generated members of $\overline{\mathbf{K}}_0$.

δ is a function from (quantifier free diagrams of) finite subsets of elements of $\overline{\mathbf{K}}_0$ into the real numbers.

STRONG SUBSTRUCTURES

The axioms of $\overline{\mathbf{K}}_0$ forbid any structure containing a sequence $\overline{a}$ with $\delta(\overline{a}) < 0$.

Define relative dimension $\delta(A/B)$ etc.

$A \leq B$, A is a strong substructure of B if

$$\delta(\overline{b}/\overline{a} \cap \overline{b}) \geq 0$$

for all $\overline{a} \in A, \overline{b} \in B$.

Basic Fact. If $\mathbf{K}_0$ has amalgamation for strong substructures there is a countable *generic* (or $\leq$-homogeneous-universal) model.

This structure is always (first-order) homogeneous; it frequently is neither prime nor saturated.

MANY CONSTRUCTIONS

Two major dividing lines:

ab initio vrs. expansion

integer valued versus real valued

Even after that, the choice $\mathbf{K}$ of the finite (dimensional) models is crucial: e.g. finite vrs. infinite rank.

In the *ab initio* case we fix a class $\mathbf{K}$ of finite graphs:

$$\delta(B) = |B| - \alpha e(B)$$

where $e(B)$ is the number of edges of the graph.

CONSTRAINTS ON δ

If $\mathrm{Diag}(\overline{a}) = \mathrm{Diag}(\overline{b})$ then $\delta(\overline{a}) = \delta(\overline{b})$.

(monotonicity) $\delta(\overline{a}/\overline{b}) \geq \delta(\overline{a}/\overline{b}\overline{c})$ whenever $\overline{c} \cap \overline{a} = \overline{c} \cap \overline{b}$.

There are further 'definability conditions' which are automatic and thus invisible in the *ab initio* context that we discuss first.

AB INITIO: RATIONAL α

Theorem. (Hrushovski) There is a strongly minimal set whose geometry is not locally modular which is not biinterpretable with a field.

By similar methods, I obtained:

Theorem. There is an almost strongly minimal nonDesarguesian projective plane.

Problem. Are the strongly minimal sets constructed in these two examples 'really different'?

These methods have been extended:

Theorem. There is an almost strongly minimal generalized n-gon which does interpret a group (odd n, Debonis-Nesin; all n, Tent)

SATURATION AND FINITE RANK

These proofs can be viewed in two steps:

A) Construct an ω-stable structure of infinite rank.
B) Collapse the rank to finite.

The first step is now routine; the second is more difficult.

By a uniform amalgamation we obtain: 'quantifier separation'. This condition implies the generic is saturated, which makes stability and rank computations straightforward.

The uniformity of the amalgamation is virtually automatic in the infinite rank case; not in finite rank.

PRIMITIVE EXTENSIONS

B is *primitive* over A if it is minimal strong over A with $\delta(B/A) = 0$.

Fix a function μ taking pairs $\overline{b}/\overline{a}$ where $\overline{b}$ is primitive over $\overline{a}$ to integers.

Let A, B be finite substructures of $M \in \overline{\mathbf{K}}_0$ with $A \subseteq B$ then $\chi_M(B/A)$ is the number of distinct copies of B over A in M.

Let $\mathbf{K}_\mu$ be the finite graphs such that

$$\chi_M(B/A) \leq \mu(B/A)$$

for every primitive pair A, B.

This is clearly a first order class. Crux of proof: Show $\mathbf{K}_\mu$ has a *uniform* amalgamation property (separation of quantifiers).

AB INITIO: IRRATIONAL α

Let $\mathbf{K}_\alpha$ be the class of finite graphs A with $\delta(A)$ hereditarily positive.

$\mathbf{K}'_\alpha$ is a subclass of $\mathbf{K}_\alpha$ defined by strengthening the lower bound on $\delta(A)$ for $A \in \mathbf{K}'_\alpha$.

Theorem. [Hrushovski] The generic structure for $\mathbf{K}'_\alpha$ is $\aleph_0$ categorical and strictly stable.

Theorem. [Baldwin-Shi] The theory T_α of the generic for $\mathbf{K}_\alpha$ is stable. The theory is not small.

Since the generic is not ω-saturated; stability is much harder than in the $\aleph_0$-categorical case.

RANDOM GRAPHS

Let B be a graph with $|B| = n$. Let

$$P_n(B) = n^{-\alpha|e(B)|} \cdot (1 - n^{-\alpha})^{\binom{n}{2} - e(B)}.$$

Let α be irrational $0 < \alpha < 1$.

Key Point: The expected value of embedding B over A into a random graph of size n is $n^{\delta(B/A)}$.

Theorem. [Spencer-Shelah] T^α, the sentences ϕ with $\lim_{n\to\infty} P_n(\phi) = 1$ is complete.

Theorem. [Baldwin-Shelah] The almost sure theory T^α is the theory T_α.

Theorem. [Baldwin] The theory T'_α is the almost sure theory for the relativized measure on $\mathbf{K}'_\alpha$; thus the Hrushovski $\aleph_0$-categorical stable theory has the finite model property.

Baldwin and Shelah independently have shown that the 0-1 law holds for expansions of successor (for edge probability $n^{-\alpha}$). Formally, this looks like an 'expansion' result but the techniques are more in the *ab initio* line.

THE RAMSEY QUANTIFIER

Consider the quantifier $(Q_{ram,f})$ defined by $Q^n_f \overline{x}\phi(\overline{x}, \overline{y})$ which holds in a finite model $|A|$ if there is a homogeneous subset for ϕ of cardinality at least $f(|A|)$.

Theorem. If f is unbounded, the logic $L_{\omega,\omega}(Q_{ram,f})$ is almost everywhere equivalent to first order logic on graphs with respect to either the uniform distribution or edge probability $n^{-\alpha}$.

Proof Sketch.

1. Baldwin-Kueker: The Ramsey quantifier is eliminable from T in the $\aleph_0$ interpretation if T is either $\aleph_0$-categorical or does not have the finite cover property.

2. Baldwin-Shelah: The almost sure theory T^α does not have the finite cover property.

EXPANSIONS: RATIONAL α

Hrushovski created the theory of fusions and proved that any two strongly minimal sets (on same universe) satisfying certain regularity conditions could be 'fused' into a common strongly minimal expansion.

Holland provided a common framework for the *ab initio* strongly minimal set construction and fusions; she proved the resultant theories were model complete.

Holland and I have extended this framework to include expansions of higher rank:

Bicolored fields

L is $(+, 0, -, *, 1, B)$.

$\delta(\overline{a})$ is twice the transcendence degree of $\overline{a}$ minus the number of black points.

$(\mathbf{K}_0, \leq)$ has the amalgamation property. (Take linearly disjoint copies of the fields and paint all new points in the join white.)

Under what conditions is the generic ω-saturated? How do we make the rank of T finite?

Unattributed work below is due to Baldwin/Holland.

DEFINABILITY CONSTRAINTS

For any integer k and $\overline{a}, \overline{b} \subseteq C \in \mathbf{K}$ with $\delta(\overline{a}/\overline{b}) = k$, there is a *$\delta$-formula for $\overline{a}$ over $\overline{b}$*, $\varphi(\overline{x}; \overline{y}) \in \mathrm{Diag}(\overline{a}; \overline{b})$, such that the following hold for any $\overline{a}', \overline{b}' \subseteq B \in \mathbf{K}$:
if $B \models \varphi(\overline{a}'; \overline{b}')$, then $\delta(\overline{a}'/\overline{b}') \leq k$.

Uniqueness

For any $N \in \mathbf{K}$ and $\overline{a}, \overline{a'}, \overline{b} \subseteq N$,
if $N \models \phi(\overline{a}; \overline{b}) \wedge \phi(\overline{a'}; \overline{b})$
and $\delta(\overline{a}/\overline{b}) = \delta(\overline{a'}/\overline{b}) = k$
and $\overline{b}\overline{a} \leq \langle \overline{b}\overline{a} \rangle$, $\overline{b}\overline{a'} \leq \langle \overline{b}\overline{a'} \rangle$,
then $Diag(\overline{b}\overline{a}) = Diag(\overline{b}\overline{a'})$.

In particular, under these conditions,
$\langle \overline{b}\overline{a} \rangle \simeq \langle \overline{b}\overline{a'} \rangle$ via $\overline{b}\overline{a} \mapsto \overline{b}\overline{a'}$.

Notation

From the definability conditions, we are able to construct *complete δ-formulas* such that $\phi(\overline{a};\overline{m})$ describes upper bounds (which are achieved by generic solutions) on $\delta(\overline{b}/\overline{c})$ for any $\overline{b}\overline{c}$ contained in $\overline{am}$. This makes it possible to 'define' such notions as intrinsic extension.

For bicolored fields, to say $\delta(\overline{a}) \leq k$ is to say $t.d.(\overline{a})$ is at most $(k + \#$ black pts$)/2$.

We write $\langle X \rangle$ for the substructure generated by X.

Intrinsic Extensions

$\overline{a}$ is a *minimal intrinsic extension* of $\overline{b}$ if $\delta(\overline{a}/\overline{b}) < 0$ but for every $\overline{a}'$ properly contained in $\overline{a}$, $\delta(\overline{a}'/\overline{b}) \geq 0$,

We denote by $I(\overline{y})$ the collection of all formulas $\forall \overline{x} \neg \phi(\overline{x}; \overline{y})$, where for some $\overline{a}$ and $\overline{b}$, $\overline{a}$ is a minimal intrinsic extension of $\overline{b}$ and $\phi(\overline{x}, \overline{y})$ is a complete δ-formula for $\overline{a}$ over $\overline{b}$.

Let $\overline{b} \in N \in \overline{\mathbf{K}_0}$ with $\overline{b} \leq \langle \overline{b} \rangle \leq N$ and $\lg(\overline{y}) = \lg(\overline{b})$: $I^*(\overline{b})$ is $I(\overline{y}) \cup \mathrm{Diag}(\overline{b})$.

Note that if $\overline{b} \subseteq B$, then $B \models I(\overline{b})$ if and only if $\overline{b} \leq B$.

Lemma. If $A, B \in \mathbf{K}$ and A is an elementary submodel of B, then $A \leq B$.

SEPARATION OF QUANTIFIERS

Definition. If $A \subseteq B \in \mathbf{K}$ and $\overline{b} \subseteq B$, then $\overline{b}$ is *minimal strong* over A if $\overline{b} \not\subseteq A$, $A \leq A\overline{b}$ and for any nonnull, proper subtuple $\overline{c}$ from $\overline{b}$, $A\overline{c} \not\leq A\overline{b}$.

Definition. We say that T has *separation of quantifiers* if for any

$$\overline{a} \leq \overline{a}\overline{b} \leq \langle \overline{a}\overline{b} \rangle \in \mathbf{K}$$

with $\overline{b}$ minimal strong over $\langle \overline{a} \rangle$,
for each $\phi(\overline{x}, \overline{y}) \in \mathrm{tp_{qf}}(\overline{\mathrm{a}}, \overline{\mathrm{b}}) \cup \mathrm{I}(\overline{\mathrm{y}})$,

$$T \cup \mathrm{tp_{qf}}(\overline{\mathrm{a}}) \cup \mathrm{I}(\overline{\mathrm{a}}) \vdash (\exists \overline{\mathrm{y}})\phi(\overline{\mathrm{a}}, \overline{\mathrm{y}}).$$

SATURATION: INFINITE RANK

Theorem. The generic model M is ω-saturated if and only if $T = \mathrm{Th(M)}$ has separation of quantifiers

If there are no further restriction on $\mathbf{K}$, ordinary amalgamation yields separation of quantifiers. Using linear disjointness and identifying maximal isomorphic strong substructures it is straightforward to get amalgamation. Thus,

Theorem. [Poizat] There is a field of rank $\omega \cdot 2$.

CODES

Note that in bicolored fields $\overline{a}$ is *primitive* over A means all points in $\overline{a}$ are black and $t.d.(\overline{a}/A)$ is $1/2$ $\lg(\overline{a})$ and no subset of $\overline{a}$ satisfies this condition.

A *code* $\mathbf{c}$ is a complete description of a primitive pair $\mathbf{a}/\mathbf{b}$. It includes field formulas $\phi_{\mathbf{c}}(\overline{x};\overline{u})$ and $\theta_{\mathbf{c}}(\overline{u})$ which assert:

All the x_i are black.

Whenever $\theta_{\mathbf{c}}(\overline{b})$ holds, $\phi_{\mathbf{c}}(\overline{x};\overline{b})$ has Morley degree 1, and $\delta(\overline{g}/\overline{b}) = k$ for a generic realization $\overline{g}$ of $\phi_{\mathbf{c}}(\overline{x};\overline{b})$.

If $\overline{e}_i$ for $i <$ some fixed $m(\mathbf{c})$ are independent realizations of $\phi_{\mathbf{c}}(\overline{x};\overline{b})$ then $\overline{b}$ is algebraic in the $\overline{e}_i$.

K^{μ}: RANK TWO FIELDS

For any function μ from codes to natural numbers we define $\mathbf{K}^{\mu}$ to be the members of $\mathbf{K}_0$ which have less than $\mu(\mathbf{c})$ independent realizations over the same canonical parameter of each code $\mathbf{c}$.

Theorem. [Baldwin-Holland] If μ is finite-one and for each $\mathbf{d}$, $\mu(\mathbf{d}) \geq m(\mathbf{d}) + q(\mathbf{d})$ then K^{μ} has separation of quantifiers property.

The requirement that μ be finite to one is not needed in the ab initio case but is required for fusions.

Theorem. [Baldwin-Holland] There is a rank two field.

This answers a question raised by Berline-Lascar in 1986.

UNSATURATED GENERICS

Lemma. For a particular μ which is not finite to one, the generic M of K^{μ} is not saturated.

A theory is *small* if for each n there are only countably many n-types over the empty set.

By diagonalizing the previous construction we can show:

Corollary. There exist μ (which are not finite to one) such that the theory T of the generic is not small.

Problem. Is this theory stable?

ON TO BAD FIELDS

A 'bad field' is an algebraically closed field k such that

1. There is a distinguished subset H which is a subgroup of the torsion free elements of the multiplicative group of the field.
2. (k, H) has finite Morley rank.

Let $[X]$ denote the algebraically closed field generated by X.

For $k \models T$ and $X \subset_\omega k$,

$$\delta(X) = 2 \cdot d_f([X]) - d_{vs}([X] \cap H(k)).$$

WHICH δ?

The choice of what subset generated by X is intersected with H in the definition of δ is subtle. Suppose the natural definition from the finite rank field case is carried over without change:

$$\delta(X) = 2 \cdot d_f([X]) - d_{vs}(X \cap H(k)).$$

Let c, cd, cd^2 be white, d, d^2 black.

$\delta(d/cd, cd^2) = 2(2) - 1 - (2(2) - 0) = -1;$

while

$\delta(d/cd, cd^2, d^2) = 2(2) - 1 - (2(2) - 1) = 0.$

Monotonicity fails.

SPECIALIZATION FAILS

Let a_1, a_2, a_3, a_4, a_5 be generic for the variety defined by the equations:

$$x_1 \cdot x_2 - x_2 \cdot x_3 + 2 = 0;$$

$$x_4 \cdot x_5 - x_2 \cdot x_3 + 2 = 0.$$

Then $\delta(\mathbf{a}) = 2(3) - 5 = 1.$

Now specialize $\mathbf{a}$ to $\mathbf{b}$ by imposing the equation

$$x_2 \cdot x_3 = 1.$$

This implies the equations

$$x_1 \cdot x_2 = 1;$$

$$x_4 \cdot x_5 = 1;$$

So $\delta(\mathbf{b}) = 2(2) - 2 = 2.$

SOME GEOMETRY

Let $\overline{e}$ be a matrix and $\overline{c}$ a vector. A torus $T_{\overline{e},\overline{c}}$ is the solution of a system of equation

$$\overline{x}^{\overline{e}_i} = \overline{c}_i.$$

For an irreducible variety W, T_W is the minimal torus containing W.

The following lemma of Zil'ber/Hrushovski (adapted Baldwin-Holland-Marker) is crucial.

Lemma. Fix a variety $W = W(\overline{x}, \overline{y})$ defined over $\overline{Q}$. There is a finite set σ of matrices such that: for any $\overline{b}$, and $W = W(\overline{b})$ if $S \subset T_W$ is a torus with

$$d_f(W) - d_f(W \cap S) < d_f(T_W) - d_f(S) \quad (1)$$

then for some $\overline{e} \in \sigma$ and some $\overline{c}$,

$$W \cap S \subseteq T_{\overline{e},\overline{c}} \subset T_W.$$

TOWARDS DEFINING δ

The sequence $\overline{a}$ is *normal* if $\overline{a} \cap H$ is a vector space basis for $[\overline{a}] \cap H$.

Lemma. [Baldwin-Holland] For any normal sequence $\overline{a}, \in H$, there is a formula $\phi(\overline{x})$ which is a δ-formula for $\overline{a}$ over $\emptyset$.

Many more complications arise in trying to prove specializations of $\mathbf{ab}$ can be forbidden to increase $\delta(\mathbf{a}/\mathbf{b})$.