Constructing ω-stable Structures: Rank 2 Fields

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Generic Structures

L is a first order language possibly with function symbols.

$\overline{\mathbf{K}}_0$ is the class of models of a universal theory T.

$\mathbf{K}_0$ is the finitely generated members of $\overline{\mathbf{K}}_0$.

δ is a function from (quantifier free diagrams of) finite subsets of elements of $\overline{\mathbf{K}}_0$ into the integers.

Strong substructures

The axioms of $\overline{\mathbf{K}}_0$ forbid any structure containing a sequence $\overline{a}$ with $\delta(\overline{a}) < 0$.

Define relative dimension $\delta(A/B)$ etc.

$A \leq B$, A is a strong substructure of B if $\delta(\overline{b}/\overline{a} \cap \overline{b})$ for all $\overline{a} \in A, \overline{b} \in B$.

Basic Fact. If $\mathbf{K}_0$ has amalgamation for strong substructures there is a countable *generic* (or $\leq$-homogeneous-universal) model.

Examples

1. ab initio
   (a) 'flat' strongly minimal set
   (b) nonDesarguesian projective plane
2. fusions

Holland provided a common framework for 1a) and 2) and proved the resultant theories were model complete. We extend the framework to include 1b) and ...

Bicolored fields (Poizat)

L is $(+, 0, -, *, 1, B)$.

$\delta(\overline{a})$ is twice the transcendence degree of $\overline{a}$ minus the number of black points.

$(\mathbf{K}_0, \leq)$ has the amalgamation property. (Take linearly disjoint copies of the fields and paint all new points in the join white.)

Under what conditions is the generic ω-saturated? How do we make the rank of T finite?

Constraints on δ

Partial list:

1. If $\mathrm{Diag}(\overline{a}) = \mathrm{Diag}(\overline{b})$ then $\delta(\overline{a}) = \delta(\overline{b})$.
2. (monotonicity) $\delta(\overline{a}/\overline{b}) \geq \delta(\overline{a}/\overline{b}\overline{c})$ whenever $\overline{c} \cap \overline{a} = \overline{c} \cap \overline{b}$.
3. (definability) For any integer k and $\overline{a}, \overline{b} \subseteq C \in \mathbf{K}$ with $\delta(\overline{a}/\overline{b}) = k$, there is a *$\delta$-formula for $\overline{a}$ over $\overline{b}$*, $\varphi(\overline{x}; \overline{y}) \in \mathrm{Diag}(\overline{a}; \overline{b})$, such that the following hold for any $\overline{a}', \overline{b}' \subseteq B \in \mathbf{K}$: if $B \models \varphi(\overline{a}'; \overline{b}')$, then $\delta(\overline{a}'/\overline{b}') \leq k$.

Uniqueness

For any $N \in \mathbf{K}$ and $\overline{a}, \overline{a'}, \overline{b} \subseteq N$,
if $N \models \phi(\overline{a}; \overline{b}) \wedge \phi(\overline{a'}; \overline{b})$
and $\delta(\overline{a}/\overline{b}) = \delta(\overline{a'}/\overline{b}) = k$
and $\overline{b}\overline{a} \leq \langle \overline{b}\overline{a} \rangle$, $\overline{b}\overline{a'} \leq \langle \overline{b}\overline{a'} \rangle$,
then $Diag(\overline{b}\overline{a}) = Diag(\overline{b}\overline{a'})$.

In particular, under these conditions,
$\langle \overline{b}\overline{a} \rangle \simeq \langle \overline{b}\overline{a}' \rangle$ via $\overline{b}\overline{a} \mapsto \overline{b}\overline{a'}$.

Notation

From the definability conditions, we are able to construct *complete δ-formulas* such that $\phi(\overline{a}; \overline{m})$ describes upper bounds (which are achieved by generic solutions) on $\delta(\overline{b}/\overline{c})$ for any $\overline{b}\overline{c}$ contained in $\overline{am}$.

For bicolored fields, to say $\delta(\overline{a}) \leq k$ is to say $t.d.(\overline{a})$ is at most $(k + \#\text{ black pts})/2$.

Intrinsic Extensions

$\overline{a}$ is a *minimal intrinsic extension* of $\overline{b}$ if $\delta(\overline{a}/\overline{b}) < 0$ but for every $\overline{a}'$ properly contained in $\overline{a}$, $\delta(\overline{a}'/\overline{b}) \geq 0$,

We denote by $I(\overline{y})$ the collection of all formulas $\forall \overline{x} \neg \phi(\overline{x}; \overline{y})$, where for some $\overline{a}$ and $\overline{b}$, $\overline{a}$ is a minimal intrinsic extension of $\overline{b}$ and $\phi(\overline{x}, \overline{y})$ is a complete δ-formula for $\overline{a}$ over $\overline{b}$.

Let $\overline{b} \in N \in \overline{\mathbf{K}}_0$ with $\overline{b} \leq \langle \overline{b} \rangle \leq N$ and $\lg(\overline{y}) = \lg(\overline{b})$: $I^*(\overline{b})$ is $I(\overline{y}) \cup \mathrm{Diag}(\overline{b})$.

Note that if $\overline{b} \subseteq B$, then $B \models I(\overline{b})$ if and only if $\overline{b} \leq B$.

0.1 Lemma. *If $A, B \in \mathbf{K}$ and A is an elementary submodel of B, then $A \leq B$.*

Syntactic Amalgamation

Definition. If $A \subseteq B \in \mathbf{K}$ and $\overline{b} \subseteq B$, then $\overline{b}$ is *minimal strong* over A if $\overline{b} \not\subseteq A$, $A \leq A\overline{b}$ and for any nonnull, proper subtuple $\overline{c}$ from $\overline{b}$, $A\overline{c} \not\leq A\overline{b}$.

Definition. We say that T has *syntactic amalgamation* if for any

$$\overline{a} \leq \overline{a}\overline{b} \leq \langle \overline{a}\overline{b} \rangle \in \mathbf{K}$$

with $\overline{b}$ minimal strong over $\langle \overline{a} \rangle$,
for each $\phi(\overline{x}, \overline{y}) \in \mathrm{tp_{qf}}(\overline{\mathrm{a}}, \overline{\mathrm{b}}) \cup \mathrm{I}(\overline{\mathrm{y}})$,

$$T \cup \mathrm{tp_{qf}}(\overline{\mathrm{a}}) \cup \mathrm{I}(\overline{\mathrm{a}}) \vdash (\exists \overline{\mathrm{y}})\phi(\overline{\mathrm{a}}, \overline{\mathrm{y}}).$$

Saturation Implies

Theorem. If the generic model M is ω-saturated then $T = \mathrm{Th(M)}$ has the syntactic amalgamation property.

Proof. $\mathrm{tp_{qf}(\overline{b}) \cup I(\overline{y})}$ is realized by $\overline{b}$ in a model N just if $\overline{b} \leq \langle \overline{b} \rangle$ and there is a strong embedding of $\langle \overline{b} \rangle$ into N.

Now suppose $\overline{a} \leq \overline{a}\overline{b} \leq \langle \overline{a}\overline{b} \rangle$ and $\overline{a} \leq M$. Then there is a strong embedding $\overline{b}$ into M over $\overline{a}$. Thus, M satisfies $(\exists \overline{y})\phi(\overline{a}, \overline{y})$ for each $\phi(\overline{x}, \overline{y}) \in \mathrm{tp_{qf}(\overline{a}, \overline{b}) \cup I(\overline{y})}$.

If M is saturated then in $T = \mathrm{Th(M)}$, for each $\phi(\overline{x}, \overline{y}) \in \mathrm{tp_{qf}(\overline{a}, \overline{b}) \cup I(\overline{x}, \overline{y})}$,

$$T \cup \mathrm{tp_{qf}(\overline{a}) \cup I(\overline{a}) \vdash (\exists \overline{y})}\phi\mathrm{(\overline{a}, \overline{y})}.$$

Saturation Implied

Theorem. If T has syntactic amalgamation then the generic M is saturated.

Moreover, T has syntactic amalgamation iff $(\mathbf{K}_0, \delta)$ has syntactic amalgamation in the following sense:

For any $\overline{a} \leq \overline{a}\overline{b} \leq \langle \overline{a}\overline{b} \rangle \in \mathbf{K}$ with $\overline{b}$ minimal strong over $\langle \overline{a} \rangle$, the following holds:

For any formula $\tau(\overline{x}; \overline{y})$ in $\mathrm{tp_{qf}}(\overline{\mathrm{a}}, \overline{\mathrm{b}}) \cup \mathrm{I}(\overline{\mathrm{x}}, \overline{\mathrm{y}})$, there exist $\sigma(\overline{x}) \in I(\overline{x})$ and $\rho(\overline{x}) \in \mathrm{tp_{qf}}(\overline{\mathrm{a}})$ such that whenever $\overline{a}' \subseteq C \in \mathbf{K}$ and $C \models (\sigma \wedge \rho)(\overline{a}')$, there is $D \in \mathbf{K}$ with $C \leq D$ and $\overline{b}' \in D$ such that $D \models \tau(\overline{a}'; \overline{b}')$.

Finite Rank

$\overline{a}$ is *primitive* over A if it is minimal strong over A with $\delta(\overline{a}/A) = 0$.

Note that in bicolored fields $\overline{a}$ is *primitive* over A means all points in $\overline{a}$ are black and $t.d.(\overline{a}/A)$ is $1/2\ \lg(\overline{a})$ and no subset of $\overline{a}$ satisfies this condition.

Theorem. There is a rank 2 field.

This answers a question raised by Berline-Lascar in 1986.

Codes

A *code* $\mathbf{c}$ is a complete description of a primitive. Here is a partial description of a code in the bicolored field case.

It includes field formulas $\phi_{\mathbf{c}}(\overline{x};\overline{u})$ and $\theta_{\mathbf{c}}(\overline{u})$.

Important integers are $m(\mathbf{c})$; $k(\mathbf{c})$, the rank of $\phi_{\mathbf{c}}$ (in $\overline{x}$), $n(\mathbf{c})$; the length of $\overline{x}$, $q(\mathbf{c}) = m(\mathbf{c}) \cdot (n(\mathbf{c}) + 1)$.

All the x_i are black.

1. Whenever $\theta_{\mathbf{c}}(\overline{b})$ holds, $\phi_{\mathbf{c}}(\overline{x};\overline{b})$ has Morley degree 1, and $d(\overline{g}/\overline{b}) = k$ for a generic realization $\overline{g}$ of $\phi_{\mathbf{c}}(\overline{x};\overline{b})$.
2. $\overline{y}$ is a canonical parameter for $\phi_{\mathbf{c}}(\overline{x},\overline{y})$.
3. If $\overline{e}_i$ for $i < m(\mathbf{c})$ are independent realizations of $\phi_{\mathbf{c}}(\overline{x};\overline{b})$ then $\overline{b}$ is algebraic in the $\overline{e}_i$.

$\mathbf{K}^{\mu}$: Amalgamation

For any function μ from codes to natural numbers we define $\mathbf{K}^{\mu}$ to be the members of $\mathbf{K}_0$ which have less than $\mu(\mathbf{c})$ independent realizations over the same canonical parameter of each code $\mathbf{c}$.

Poizat proved for bicolored fields, $(\mathbf{K}^{\mu}, \leq)$ has the amalgamation property if

$$\mu(\mathbf{c}) \geq q(\mathbf{c})$$

(Use linear disjointness and identify maximal isomorphic strong substructures).

K^μ: Syntactic Amalgamation

Theorem. If μ is finite-one and for each $\mathbf{d}$, $\mu(\mathbf{d}) \geq m(\mathbf{d}) + q(\mathbf{d})$ then K^μ has the syntactic amalgamation property.

Lemma 1. For any pair of codes $\mathbf{c}, \mathbf{d}$ and natural number, there is a field formula $\chi_{\mathbf{c},\mathbf{d},k}$ such that

$$\models \chi_{\mathbf{c},\mathbf{d},k}(\overline{a}')$$

if and only if $\theta_{\mathbf{c}}(\overline{a}')$ and for any generic realization $\overline{g}$ of $\phi_{\mathbf{c}}(\overline{a}', \overline{y})$, there are less than k independent realizations of the code d in $\overline{a}'\overline{g}$.

Suppose $\overline{g}$ is field generic for $\phi_{\mathbf{c}}(\overline{a}', \overline{y})$ over C and D is the algebraically closed field generated by $C\overline{g}$ with all elements of $D - C\overline{g}$ white. We write $C[\overline{g}]$ for D.

Main Lemma. Let $\overline{a} \leq \overline{a}\overline{b} \leq \langle \overline{a}\overline{b} \rangle$ and $\overline{b}$ be primitive over $\overline{a}$ with complete δ-formula $\phi_{\mathbf{c}}$. Fix any $\overline{a}'$ of the same length as $\overline{a}$ with $\overline{a}' \subset C$. Suppose the code $\mathbf{d} \neq \mathbf{c}$ is realized by $\langle \overline{e}_1, \ldots e_H \rangle$ in $C[\overline{g}]$ with

$$H \geq m(\mathbf{d}) + q(\mathbf{d}).$$

Then,

1. $H \leq 3n(\mathbf{c})$ and
2. if $C \models \chi_{\mathbf{c},\mathbf{d},q(\mathbf{d})}(\overline{a}')$ then $\overline{a}'$ is not strong in $\overline{a}'(C \cap \overline{e}_i)$ for some i.

From Lemma to Theorem

Proof. Fix $\overline{a} \leq \overline{a}\overline{b} \leq \langle \overline{a}\overline{b} \rangle \in \mathbf{K}$ with $\overline{b}$ minimal strong over $\langle \overline{a} \rangle$.

For any formula $\tau(\overline{x}; \overline{y})$ in $\mathrm{tp_{qf}}(\overline{\mathrm{a}}, \overline{\mathrm{b}}) \cup \mathrm{I}(\overline{\mathrm{a}}, \overline{\mathrm{b}})$, we need formulas $\sigma(\overline{x}) \in I(\overline{x})$ & $\rho(\overline{x}) \in \mathrm{tp_{qf}}(\overline{\mathrm{a}})$ such that:

Whenever $\overline{a}' \subseteq C \in \mathbf{K}$ and

$$C \models (\sigma \wedge \rho)(\overline{a}'),$$

there is $D \in \mathbf{K}$ with $C \leq D$ and $\overline{b}' \in D$ such that $D \models \tau(\overline{a}'; \overline{b}')$.

In fact $D = C[\overline{g}]$, where $\overline{g}$ is generic for the field part of a complete δ-formula for $\overline{b}/\overline{a}$.

We omit two easy cases when $\delta(\overline{x}/\overline{a}) > 0$.

We suppose $\overline{a} \leq \overline{a}\overline{b} \leq \langle\overline{a}\overline{b}\rangle$ and $\overline{b}$ is primitive over $\overline{a}$ with code $\mathbf{c}$ and complete δ-formula $\phi_{\mathbf{c}}$. Let $\langle P_1, G_1\rangle \ldots \langle P_M, G_M\rangle$ list all witnesses for all codes $\mathbf{d}$ with $\mu(\mathbf{d}) < 3 \cdot n(\mathbf{c})$.

Let $\langle\overline{a}, H_1\rangle \ldots \langle\overline{a}, H'_M\rangle$ list all (isomorphism types) of finite structures such that H_i is isomorphic to a subset of some G_j and H_i is a minimal intrinsic extension of $\overline{a}$.

Refining τ

Suppose some $\overline{e} \in C[\overline{g}]$ satisfies $\tau(\overline{a}', \overline{g}', \overline{e})$. Then the $\overline{e}$ are black and the transcendence degree of $\overline{e}$ over $\overline{a}'\overline{g}'$ is less than twice the length of $\overline{e}$. However, $\overline{g}'$ and C are independent in the field sense over $\overline{a}'$. So the transcendence degree of $\overline{e}$ over $\overline{a}'$ is less than twice the length of $\overline{e}$. That is, $\overline{e}$ is intrinsic over $\overline{a}$. So for some $\tau' \in I(\overline{y})$, $C[\overline{g}] \models \tau'(\overline{a}')$.

Choosing σ and ρ

Let $\sigma(\vec{x})$ be the conjunction of $\neg(\exists\overline{y})\tau'(\overline{x},\overline{y})$ with the formulas $\neg(\exists\overline{y})\beta_i(\overline{x},\overline{y})$;
β_i is a complete δ formula for $\overline{h}_i$ over $\overline{a}'$.

Let $\rho(\overline{x})$ be the assertion, 'there are less than $\mu(\mathbf{c})$ independent over $\overline{x}$ realizations of the code $\mathbf{c}$' and the conjunction of the formulas

$$\chi_{\mathbf{c},\mathbf{d},q(\mathbf{d})}(\overline{x})$$

(for codes $\mathbf{d}$ with $\mu(\mathbf{d}) < 3 \cdot n(\mathbf{c})$).

The quantifier eliminability of acf and definability of rank show ρ is in $\mathrm{tp}_{\mathrm{qf}}(\overline{\mathrm{a}})$.

Le Coup de Grace

Suppose $C \models \sigma(\overline{a}') \wedge \rho(\overline{a}')$. We claim that $C[\overline{g}] \in K^{\mu}$. If not, some code $\mathbf{d}$ provides a counterexample. There are more than

$$\mu(\mathbf{d}) \geq m(\mathbf{d}) + q(\mathbf{d})$$

independent realizations of $\mathbf{d}$ over the same canonical parameter in $C[\overline{g}]$. By the choice of σ, $\mathbf{d}$ cannot be $\mathbf{c}$. By item 1) of the main lemma, $\mu(\mathbf{d}) < 3 \cdot n(\mathbf{c})$. By item 2) of the main lemma, there is a realization e_i of d with $\overline{a}' \not\leq (\overline{a}' \cap C)$. But then $C \models \neg\rho(\overline{a}')$.

Unsaturated generics

Lemma. For a particular (any) μ which is not finite to one, there are a code $\mathbf{c}$ and an infinite family of codes $\mathbf{d_i}$ and field formulas $\gamma_i(v_1, \ldots v_r, \overline{w})$ such that if $M \models \theta_{\mathbf{c}}(\overline{b}, \overline{c}, \overline{d}, \overline{h})$ and $\gamma_i(v_1 \ldots v_r, \overline{b}, \overline{c}, \overline{d}, \overline{h})$ is realized in M, then in any generic realization of the code $\mathbf{c}$ over $\overline{b}, \overline{c}, \overline{d}, \overline{h}$ there are r independent realizations of $\mathbf{d_i}$ over the same canonical parameter.

Corollary. The generic M of this K^{μ} is not saturated.

Proof of nonsaturation

Let $T = \mathrm{Th}(\mathrm{M})$. We show

$$T \cup \mathrm{tp}_{\mathrm{qf}}(\overline{\mathrm{b}}, \overline{\mathrm{c}}, \overline{\mathrm{d}}, \overline{\mathrm{h}})$$

$$\cup I(\overline{w}) \cup \theta_{\mathbf{c}}(\overline{w}) \cup \neg\exists\overline{u}\phi_{\mathbf{c}}(\overline{u}, \overline{w})$$

is consistent. But this type is omitted in the generic so M is not saturated.

For $\mathrm{tp}_{\mathrm{qf}}(\overline{\mathrm{b}}, \overline{\mathrm{c}}, \overline{\mathrm{d}}, \overline{\mathrm{h}}) \cup \mathrm{I}(\overline{\mathrm{w}})$ does not imply $\exists\overline{u}\phi_{\mathbf{c}}(\overline{u}, \overline{w})$.

As, any finite subset $\Gamma_0(\overline{w})$ of

$$\mathrm{tp}_{\mathrm{qf}}(\overline{\mathrm{b}}, \overline{\mathrm{c}}, \overline{\mathrm{d}}, \overline{\mathrm{h}}) \cup \mathrm{I}(\overline{\mathrm{w}})$$

omits one (almost all) of the formulas

$$\gamma'_j : \neg(\exists\overline{v})\gamma_j(\overline{v}, \overline{w}).$$

If γ'_p is omitted there is a model C of $\Gamma_0(\overline{b}, \overline{c}, \overline{d}, \overline{h})$ and $(\exists \overline{v})\gamma_p(\overline{v}, \overline{b}, \overline{c}, \overline{d}, \overline{h})$. But, since for all i, $\mu(\mathbf{d_i}) < r$,

$$(\exists \overline{v})\gamma_p(\overline{v}, \overline{w}) \to \neg \exists \overline{u} \phi_{\mathbf{c}}(\overline{u}, \overline{w}).$$

(Any realization of the code $\mathbf{c}$ would give rise to r independent realizations of $\mathbf{d_i}$ over the same canonical parameter.)

Not even Small

A theory is *small* if for each n there are only countably many n-types over the empty set.

By diagonalizing the previous construction we can show:

Corollary. There exist μ (which are not finite to one) such that the theory T of the generic is not small.

STABLE ?

On to Bad Fields

A 'bad field' is an algebraically closed field k with a distinguished subset H which is a subgroup of the torsion free elements of the multiplicative group of the field.

Let $[X]$ denote the algebraically closed field generated by X.

For $k \models T$ and $X \subset_\omega k$,

$$\delta(X) = 2 \cdot d_f([X]) - d_{vs}([X] \cap H(k)).$$

0.2 Definition. The sequence $\overline{a}$ is *normal* if $\overline{a} \cap H$ is a vector space basis for $[\overline{a}] \cap H$.

Definability Constraints

Theorem For any normal sequence $\overline{a}, \in H$, there is a formula $\phi(\overline{x})$ which is a δ-formula for $\overline{a}$ over $\emptyset$.

The argument depends on the following theorem of Zil'ber.

Theorem Fix a variety $W = W(\overline{x}, \overline{y})$ defined over $\overline{Q}$. There is a $\sigma \in \Sigma$ such that: for any $\overline{b}$, and $W = W(\overline{b})$ if $S \subset T_W$ is a torus with

$$d_f(W) - d_f(W \cap S) < d_f(T_W) - d_f(S) \quad (1)$$

then for some $\overline{e} \in \sigma$ and some c, $W \cap S \subseteq T_{\overline{e},c}$.