THE METAMATHEMATICS OF RANDOM GRAPHS

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EVENTUAL BEHAVIOR

Definition 1 *A (round robin) tournament is a directed graph with an edge between every pair of points.*

Fix k. Is there a tournament such that for each set of k-players there is another who beats each of them?

Let S_n be the set of all tournaments with n players.

$$|S_n| = 2^{\binom{n}{2}}.$$

Each of these is equally likely.

Call a k-set X *bad* if no element dominates each member of X.
If $Y(T)$ is the number of bad k-sets in a tournament T then

$$E(Y) = \binom{n}{k}(1 - (1/2)^k)^{n-k}.$$

Then $E(Y) \to 0$ and by Markov's inequality $P(Y \geq 1) \to 0$. So a.a., there is such a tournament.

WHAT IS A ZERO-0NE LAW?

Let Ω_n denote the set of graphs on the vertex set $\{0, \ldots, n-1\}$.

Let P_n be a probability measure assigning an element of $[0, 1]$ to each subset of Ω_n.

Let X be a family of sequences X_n of events in Ω_n. Then $(\Omega_n, P_n, \mathrm{X})$ *satisfies a zero-one law* if for each sequence X_n,

$$\lim_{n\to\infty} P_n(X_n) = 0$$

or

$$\lim_{n\to\infty} P_n(X_n) = 1.$$

In the example:
P_n is the uniform probability
X_n is the set of tournaments on n vertices such that every set of k players is dominated by one single player.

EDGE PROBABILITY

We will consider measures that are determined by the 'edge probability' $p(n)$ of two vertices being connected.

Definition 2 *Let B be a graph with $|B| = n$ and $0 < p = p(n) < 1$.*

1. *Let $P^p_n(B) = p^{|e(B)|} \cdot (1-p)^{\binom{n}{2} - e(B)}$.*
2. *For any $X \subset \Omega_n$,*

$$P^p_n(X) = \sum \{P^p_n(B) : B \in X\}.$$

3 PROBABILITY MEASURES

1. $p(n)$ is constant.
2. $p(n)$ is $n^{-\alpha}$ for $0 < \alpha < 1$ and often irrational
3. $p(n) = p_n^l$ is
$$\frac{ln(n)}{n} + \frac{l \cdot ln(ln(n))}{n} + \frac{c}{n}$$
where l is an arbitrary fixed nonnegative integer, and c is a positive constant.

MOTTO

A Logician is a self-conscious mathematician!

LOGIC

$$(\forall x_1), \dots (\forall x_k)(\exists y) \bigwedge y R x_i$$

First order logic is built up from atomic formulas by Boolean operations and quantification over individuals.

k-connected is expressible; connected is not.

Definition 3 *Let B be a graph with $|B| = n$ and $0 < p = p(n) < 1$.*

1. *Let $P_n^p(B) = p^{|e(B)|} \cdot (1-p)^{\binom{n}{2} - e(B)}$.*
2. *For any formula ϕ, let*

$$P_n^p(\phi) = \sum \{P_n^p(B) : B \models \phi, |B| = n\}.$$

[Fagin and (Glebski,Y. and Kogan, V. and Liogon'kii, M.I, and Taimanov, V.A.)]

Theorem 4 *If $p(n) = 1/2$ for each formula ϕ, $\lim_{n\to\infty} P_n^p(\phi)$ is 0 or 1.*

Let T^p denote the collection of almost surely true sentences. That is, the sentences ϕ such that:

$$\lim_{n\to\infty} P_n^p(\phi) = 1.$$

EVENTS

FAMILIES of SEQUENCES of events.

We consider random graphs on finite sets with different background structure.

Two parameters:

1. logic
   (a) first order
   (b) $L_{\omega_1,\omega}$
   (c) the Ramsey quantifier: $L_{\omega,\omega}(Q_{ram,f})$
2. ambient vocabulary: L'
   (a) equality
   (b) successor
   (c) order
   (d) vector space?

$L = L' \cup \{E\}$

ALMOST SURE THEORIES

We consider a family (Ω_n, P_n) and let L represent the first order sentences in a vocabulary τ.

The *almost sure* theory of (Ω_n, P_n, L) is the collection of L-sentences ϕ such that

$$\lim_{n\to\infty} P_n(\phi) = 1.$$

A theory T is complete if for every $L(\tau)$-sentence ψ either $\psi \in T$ or $\neg\psi \in T$.

Thus there is a first order zero-one law for (Ω_n, P_n) just if the almost sure theory is complete.

STRATEGY: Find a collection Σ of axioms that are

1. almost surely true
2. complete

PROVING COMPLETENESS

TECHNIQUES:

1. categoricity
2. ‘quantifier elimination’
3. Ehrenfeucht-Games
4. Determined Theories

THE RANDOM GRAPH

The Rado universal graph is the unique countable model of the following extension axioms.

Axioms ϕ_k :

$$(\forall v_0 \ldots v_{k-1} w_0 \ldots w_{k-1})(\exists z)\ \wedge_{i<k} (Rzv_i \bigwedge \neg Rzw_i)$$

A variant on our initial probability arguments shows each extension axiom has probability 1.

And a back and forth argument shows the theory is categorical in $\aleph_0$; hence complete.

ALMOST EVERYWHERE EQUIVALENCE

Definition. The logics L and L$'$ are *almost everywhere equivalent* with respect to the probability measure P if there exists a collection C of finite models such that $P(C) = 1$ and for every sentence θ of L there is a sentence θ' of L$'$ such that θ and θ' are equivalent on C (and conversely).

Theorem. (Hella, Kolaitis, Luosto) FO and $L^{\omega}_{\infty,\omega}$ are almost everywhere equivalent with respect to the uniform distribution.

THE RAMSEY QUANTIFIER

Consider the quantifier $(Q_{ram,f})$ defined by $Q^n_f \mathbf{x}\phi(\mathbf{x},\mathbf{y})$ which holds in a finite model $|A|$ if there is a homogeneous subset for ϕ of cardinality at least $f(|A|)$.

Theorem. If f is unbounded, the logic $L_{\omega,\omega}(Q_{ram,f})$ is almost everywhere equivalent to first order logic on graphs with respect to either the uniform distribution or edge probability $n^{-\alpha}$.

Proof Sketch.

1. Baldwin-Kueker: The Ramsey quantifier is eliminable from T in the $\aleph_0$ interpretation if T is either $\aleph_0$-categorical or does not have the finite cover property.

2. Baldwin-Shelah: The almost sure theory T^α does not have the finite cover property.

But I am ahead of myself, what is T^α and why is it complete?

THE GREAT COINCIDENCE?

Theorem 5 (Spencer-Shelah-1988) *If α is irrational, for each formula ϕ, $\lim_{n\to\infty} P_n^\alpha(\phi)$ is 0 or 1.*

Theorem 6 (Hrushovski late 80's)

1. *There is an $\aleph_0$ categorical strictly stable theory.*
2. *There is a strongly minimal set which is neither 'trivial', nor 'vector-space like' nor 'field-like'.*

These results depend on the same fundamental ideas.

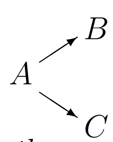
RETHINKING THE RANDOM GRAPH

The Rado random graph is the unique countable model of T^p.

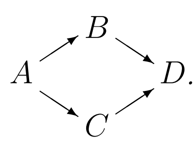
Definition 7 *Let $\boldsymbol{K}_0^p$ be the collection of all finite graphs (including the empty graph) and write $A \prec_{\boldsymbol{K}} B$ if A is subgraph of B.*

Note:

Definition 8 *The class $(\boldsymbol{K}_0, \prec_{\boldsymbol{K}})$ satisfies the* amalgamation property *(**AP**) if for any situation:*



there exist a $D \in \boldsymbol{K}_0$ and strong embeddings, such that the following diagram is commutative.



GENERIC STRUCTURES

Definition 9 *The countable model M is $(\boldsymbol{K}_0, \prec_{\boldsymbol{K}})$*-generic *if*

1. *If $A \leq M, A \leq B \in \boldsymbol{K}_0$, then there exists $B' \leq M$ such that $B \cong_A B'$,*

2. *For every finite $A \subseteq M$ there is a finite B with $A \subseteq B \prec_{\boldsymbol{K}} N$.*

The Rado graph is $(\boldsymbol{K}_0^{.5}, \prec_{\boldsymbol{K}})$-generic.

Theorem 10 *Any two countable $(\boldsymbol{K}_0, \prec_{\boldsymbol{K}})$-generic structures are isomorphic.*

PREDIMENSIONS

Fix a base language L and expand it by a new binary relation, R. Call the new language L^+.

R is symmetric and irreflexive. For any finite B, $e(B)$ is number of 'edges' of B.

Definition 11 *Define predimensions on finite structures as follows.*

1. *Fix an real number α, $0 < \alpha < 1$ and let*

$$\delta_\alpha(B) = |B| - \alpha e(B).$$

2. *Let $\boldsymbol{K}_\alpha$ be all finite graphs B such that for all $A \subseteq B$, $\delta_\alpha(A) \geq 0$.*

3. *For any M, and finite $A \subseteq M$, $d_M(A) = \inf(\delta_\alpha(B)$ for $A \subseteq B \subseteq_\omega M$.*

Definition 12 *For $M \subseteq N$, we say that M is* **strong** *in N, and write $M \leq N$, if for all finite $X \subseteq M$,*

$$d_N(X) = d_M(X).$$

STRONG SUBSTRUCTURES

Axiom Group A Let $A, B, C \in \boldsymbol{K}$.
A1. $A \leq A$.
A2. If $A \leq B$ then $A \subseteq B$.
A3. If $A, B, C \in \boldsymbol{K}_0$, then

$$A \leq B \leq C \Longrightarrow A \leq C.$$

A4. If $A, B, C \in \boldsymbol{K}_0$, $A \prec_{\boldsymbol{K}} C$, $B \prec_{\boldsymbol{K}} C$ and $A \subseteq B$ then $A \prec_{\boldsymbol{K}} B$.
A5. $\emptyset \in \boldsymbol{K}$ and $\emptyset \leq A$ for all $A \in \boldsymbol{K}$.

THE FIRST EXPLANATION

Theorem 13 *If* $(\boldsymbol{K}_0, \prec_{\boldsymbol{K}})$ *is a collection of finite relational structures that satisfies A1-A5 and has the amalgamation property then there is a countable* $\boldsymbol{K}_0$*-generic model* M*.*

Lemma 14 *The class* $\boldsymbol{K}_\alpha$ *satisfies* **A1-A5** *and has the amalgamation property.*

1. *If* $\alpha = .5$ *the generic model is an* $\aleph_1$*-categorical non-Desarguesian projective plane (Baldwin).*
2. *If* α *is irrational the theory* T_α *of the generic model is a strictly stable first order theory (Baldwin-Shi).*

Problem: (A La Cameron) What are the automorphisms of the generic structure?

But emphasis on the 'generic' model is misplaced. In order to prove 0-1 laws we must identify T_α as an almost sure theory.

DETERMINED THEORIES

The theory T is *determined* if there is a family of functions F^n_M with the following property. For any formula $\phi(x_1 \ldots x_r)$ there is an integer ℓ_ϕ, such that for any $M, M' \models T$ and any r-tuples $\mathbf{a} \in \mathbf{M}$ and $\mathbf{a}' \in \mathbf{M}'$ if $F^{\ell_\phi}_M(\mathbf{a}) \approx \mathbf{F}^{\ell_\phi}_{\mathbf{M}'}(\mathbf{a}')$ by an isomorphism taking $\mathbf{a}$ to $\mathbf{a}'$, then $M \models \phi(\mathbf{a})$ if and only if $M' \models \phi(\mathbf{a}')$.

Theorem. If T is determined and for each $M, M' \models T$ and each n, $F^n_M(\emptyset) \approx F^n_{M'}(\emptyset)$ then T is complete.

SOME DETERMINED THEORIES

We will describe in a moment the notion of a semigeneric structure. The following theories are determined:

1. The semigeneric structures with respect to the class $\boldsymbol{K}_\alpha$. (Expansions of equality)
2. The semigeneric structures with respect to the class $\boldsymbol{K}_\alpha^S$. (Expansions of successor)
3. The semigeneric structures with respect to the class $\boldsymbol{K}_\alpha^V$. (Expansions of vector spaces over finite fields)
4. The theory T^ℓ of Spencer and Thoma.

The axioms of 1,2, and 4 can be proved to be almost surely true (for the appropriate probability measure).

INTRINSIC CLOSURE

Definition 15 *For $A, B \in S(\boldsymbol{K}_0)$, we say B is an* intrinsic *extension of A and write $A \leq_i B$ if $\delta(B/A') < 0$ for any $A \subseteq A' \subset B$.*

Definition 16 *For any $M \in \boldsymbol{K}$, any $m \in \omega$, and any $A \subseteq M$,*

$$\mathrm{cl}_M^m(A) = \bigcup\{B : A \leq_i B \subseteq M \& |B - A| < m\}.$$

Definition 17 *If $B \cap C = A$ we write $B \otimes_A C$ for the structure with universe $B \cup C$ and no relations other than those on B or C.*

SEMIGENERICITY

Definition 18 *The countable model* M *is* $(\boldsymbol{K}_0, \prec_{\boldsymbol{K}})$*-semigeneric, or just semigeneric, if*

1. $M \in \boldsymbol{K}$
2. *If* $A \prec_{\boldsymbol{K}} B \in \boldsymbol{K}_0$ *and* $g : A \mapsto M$*, then for each finite* m *there exists an embedding* $\hat{g}$ *of* B *into* M *which extends* g *such that*
   (a) $\mathrm{cl}^m_M(\hat{g}B) = \hat{g}B \cup \mathrm{cl}^m(A)$
   (b) $M|\mathrm{cl}^m_M(gA)\hat{g}B = \mathrm{cl}^m_M(gA) \otimes_A \hat{g}B$

Lemma 19 *There exist formulas* $\phi^m_{A,B,C}$ *such that the structure* $N \in \boldsymbol{K}$ *is semigeneric, if and only if for each* $A \prec_{\boldsymbol{K}} B$ *and* $C \in \mathcal{D}_A$ *and each* $m < \omega$*,* $N \models \phi^m_{A,B,C}$

Theorem 20 *If* $A \prec_{\boldsymbol{K}} B$ *and* $A \leq_i C$ *with* $|\hat{C}| < m$ *then*

$$\lim_{n\to\infty} P_n(\phi^m_{A,B,C}) = 1.$$

Under appropriate hypotheses we can prove all the semigeneric models are elementarily equivalent.

MAIN THEOREM

Definition 21 *We denote by* Σ_α *the conjunction of a) the sentences axiomatizing* $(\boldsymbol{K}_0, \leq_s)$*-semigenericity and b) the sentences asserting that if* $\mathbf{a} \in \mathrm{icl}_{\mathrm{M}}(\emptyset)$ *then* $\neg R(\mathbf{a})$ *(for any* $R \in L$*-*L'*) and describing the* L'*-structure of* $\mathrm{icl}_{\mathrm{M}}(\emptyset)$*.*

Theorem 22 *If* T_α *is the theory of the semigeneric models of* Σ_α *then* T_α *is a complete theory, axiomatized by* Σ_α*. Moreover,* T_α *is nearly model complete and stable. And* T_α *is* not *finitely axiomatizable.*

Two cases:

1. L' has only equality.
2. L' has successor.

The first case gives the 0-1 law for $n^{-\alpha}$ α irrational.

The second gives the same laws for the random graph over successor.

QUANTIFIER COMPLEXITY

Nearly model complete means every formula is equivalent to a Boolean combination of existential formulas.

As given, the axioms for semigenericity are $\forall\exists\forall$.

Lemma 23 *(Baldwin-Laskowski) The theory T_α is* not *π_2-axiomatizable.*

THE FUNDAMENTAL CONNECTION

L' is the ambient vocabulary: successor

L includes the graph relation R.

$\delta(B)$ is the number of components of $(B, S) - \alpha e$ where e is the number of edges in the graph.

Definition 24 *Let $A \subseteq B$ be L-structures. Fix an L'-isomorphism f from A into the L'-structure (n, S, I, F), and $M \in \Omega_n$, i.e. M is an L-structure expanding (n, S, I, F). Let N_f be a random variable such that $N_f(M)$ is the number of extensions of f to (L-L')-homomorphism over A mapping B onto M.*

Lemma 25 *For all sufficiently large n and all $f : A \to n$, the expectation*

$$\mu_f = E(N_f) \sim n^{\delta(B/A)}.$$

TECHNICAL GOAL

Theorem 26 *Fix L-structures $A \subseteq B$ with $A \leq_s B$. Let V be the event (which depends on c_1): for every L'-isomorphism $f : A \to n$,*

$$n^{v-r}(\ln n)^{-(v+1)} < N_f < c_1 n^{v-r}. \qquad (1)$$

Then, for some choice of c_1

$$\lim_{n\to\infty} P_n(V) = 1.$$

The upper bound is proved exactly as in Spencer-Shelah; the lower bound is a new argument avoiding the second moment method.

LIMIT LAWS

Consider a family (Ω_n, P_n) and let L represent the first order sentences in a vocabulary τ.

(Ω_n, P_n, L) obeys limit laws if for each L-sentences ϕ

$$\lim_{n \to \infty} P_n(\phi)$$

exists.

Spence and Thoma consider:
$p^*(n) = p_n^l$ is

$$\frac{ln(n)}{n} + \frac{l \cdot ln(ln(n))}{n} + \frac{c}{n}$$

where l is an arbitrary fixed nonnegative integer, and c is a positive constant.

They prove limit laws for this probability by Ehrenfeucht games.

DETERMINED THEORIES AND LIMIT LAWS

Baldwin and Mazzucco prove the almost sure theory for p^* is determined for an appropriate notion of closure. In contrast to the T_α case the closure of the empty set is not empty. Using determined theories we obtain:

Theorem 27 *There are a family of easily described sentences* σ_s^l. *Let* $lim_{n\rightarrow\infty} p_n^l(\sigma_s^l) = q_s^l$. *For any L-sentence* θ, *there exists a finite set I of nonnegative integers such that* $lim_{n\rightarrow\infty} p_n^l(\theta) = \Sigma_{i\in I} q_i^l$ *or* $lim_{n\rightarrow\infty} p_n^l(\theta) = 1 - \Sigma_{i\in I} q_i^l$.

TWO ALMOST SURE THEORIES

THE RANDOM GRAPH –uniform distribution

1. unstable; prototypical theory with independence property
2. $\aleph_0$-categorical
3. has the finite cover property
4. elimination of quantifiers
5. $L^{\omega}_{\infty,\omega}$ almost equivalent to first order.
6. $\forall\exists$-axiomatizable

THE RANDOM GRAPH –edge probability $n^{-\alpha}$, α irrational.

1. stable
2. not $\aleph_0$-categorical; not small
3. does not have the finite cover property
4. nearly model complete, not model complete
5. $L^{\omega}_{\infty,\omega}$ is not almost equivalent to first order (McArthur-Spencer).
6. $\forall\exists\forall$ axiomatizable.

Urysohn Space

Let $\boldsymbol{K}_0$ be the set of finite metric spaces in the language containing binary relations R_q for each positive rational q. Cameron pointed out that if $\mathbb{Q}$ is the homogeneous universal (i.e Fraïssé limit) for $\boldsymbol{K}_0$ then the completion of Q is the Urysohn space.

Vershik's version specifies a set of constant a_i and the distances between a_i and a_j.

Note that in either case, we need the *prime* model of the theory of the generic. So the infinitary logic of the model theory talks enters again – by omitting all nonprincipal types.

A PROBABILITY MODEL

Fix a slow growing (Blass) function $f(n)$ and let L_n contain the R_q with the denominator of q less than $f(n)$ and $0 \leq q \leq 1$.

Let Ω_n be the set of L_n structures with universe n that satisfy the universal axioms of metric spaces.

Let P_n be the uniform measure on Ω_n.

Let $\boldsymbol{K}_0$ be the class of substructures of models in $\bigcup \Omega_n$.

Claim 28 *$\mathbb{Q}$ is the Fraïssé limit of $\boldsymbol{K}_0$ under substructure.*

Conjecture 29 *The extension axioms for finite metric spaces are almost surely true with respect to (Ω_n, P_n).*

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SUMMARY

I. Model Theory

- **A.** $(\boldsymbol{K}_0, \prec_{\boldsymbol{K}})$ generic structures
- **B.** Applications
  - **1.** New Strongly Minimal Set (Hrushovski)
  - **2.** $\aleph_0$-categorical strictly stable theory (Hrushovski)
  - **3.** $\aleph_1$-categorical nonDesarguesian projective plane (Baldwin)
  - **4.** Strictly stable theories T^α (Baldwin-Shi)
  - **5.** Algebraic Constructions: Baudish, Baldwin-Holland, Chapuis, Nesin, Poizat, Tent, Zilber
  - **6.** Other model theoretic phenomena, Ikeda, Pourmahdian-Wagner

II. Random Graphs

- **A.** $0 - 1$ laws
- **B.** $0 - 1$ laws for $p(n) = n^{-\alpha}$: T_α
  - **1.** Graphs (Spencer-Shelah; Baldwin-Shelah)
  - **2.** Arbitrary finite relational language imposed on successor (Baldwin and Shelah independently)

III. The theory T_α is complete, stable, nearly model complete, and decidable but not finitely axiomatizable. This has consequences for 0-1 laws in extended logics.

IV. The method of determined theories works for limit laws as well as 0-1 laws.

A few relevant references follow.
[1] [3] [2] [4] [5]

Most papers are on my homepage:
http://www2.math.uic.edu/ jbaldwin/model.html

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