BEYOND FIRST ORDER LOGIC II

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WHY GO BEYOND FIRST ORDER LOGIC?

I. Because it's there.

- A. To understand the infinite:
- B. To understand canonical structures

II. To understand first order logic

III. To understand 'Model Theory'

IV. To investigate ordinary mathematical structures

CONTEXT

This is the second of two talks with different emphases discussing some current work in nonelementary contexts. This one will focus more on IB and IV. But we describe the interplay between the 'pure' and 'applied' considerations.

TO UNDERSTAND THE INFINITE!

Most known mathematical results are either

extremely cardinal dependent: about finite or countable structures or at most structures of cardinality the continuum;
or completely cardinal independent: about every structure satisfying certain properties.

Understanding Classes of Models

Model theory has discovered problems that have an intimate relation between the cardinality of structures and algebraic properties of the structures:
i) Stability spectrum and counting models
ii) A general theory of independence
iii) Decomposition theorems for general models

There are structural algebraic, not merely combinatorial features, which are non-trivially cardinal dependent.

TO UNDERSTAND CANONICAL STRUCTURES

A Thesis of Zilber:

Fundamentally important structures like the complex field *wih exponentiation* can be described at least up to categoricity in power in an appropriate logic.

TO UNDERSTAND FIRST ORDER LOGIC

The study of first order logic uses without thinking such methods as:

1. compactness theorem
2. upward and downward Löwenheim-Skolem theorem
3. closure under unions of Elementary Chains
4. Ehrenfeucht-Mostowski models

We can better understand these methods and their use in the first order case by investigating situations where only some of them hold.

TO UNDERSTAND MODEL THEORY

A. What are the *syntactial* and *semantical* components of model theory?

B. Logics vrs classes of models: Robinson, Tarski, Morley, Shelah

C. How does the ability to change vocabulary distinguish model theory from other mathematical disciplines?

For applications, it is often essential to work in a given language where one has control of the definable sets.

For general results, it is often convenient (essential?) to expand the language to make all relations which are definable, definable in a very simple way. This method is much more powerful in the non-elementary context.

INVESTIGATE ORDINARY MATHEMATICS

The following classes require going beyond countable theories in first order logic:

1. Banach Spaces (Krivine, Stern, Henson, Iovino, Bernstein, Usvyatsov, et al)
2. Complex Exponentiation (Zilber)
3. Locally finite groups (Grossberg, Macintyre, Shelah)
4. Compact complex manifolds (Zilber, Pillay, Scanlon, Moosa, Radin)

In extending the study of finding definable groups to the non-elementary context, Hytinnen-Lessmann-Shelah have discovered interesting aspects of Group Representations.

TWO DIRECTIONS

<u>A. Strong Syntax:</u>
Study categoricity for sentences of $L_{\omega_1,\omega}$ or $L_{\omega_1,\omega}(Q)$ with no model theoretic assumptions like upwards Löwenheim-Skolem or the amalgamation property.

<u>B. AEC's with arbitrarily large models:</u>
Study very abstract classes with some strong hypotheses.

We focus on A in this talk but make a few comments about B first.

THE CATEGORICITY SPECTRUM

Theorem 1 (Morley) *A countable* first order *theory T is categorical in one uncountable cardinal if and only if it is categorical in all uncountable cardinalities.*

Is first order crucial? Shelah showed: 'countable' is not.

The study of compact complex manifolds naturally uses uncountable languages.

1. Moosa identified Kähler manifolds as those which admit a countable language.
2. Radin generalized to certain Zariski structures in an uncountable language properties which hold in arbitrary categorical theories only when the language is countable.

Does the theorem generalize to other classes of models?

TWO COUNTEREXAMPLES

Let the vocabulary contain a unary predicate P.

1) $L(Q)$ can say both the set and its compliment are uncountable. This theory is categorical in $\aleph_1$ and nowhere else.

2) With an additional binary relation we can say

$$2^{|P(M)|} \geq |M|.$$

The class of reducts is categorical in κ only if $\kappa = \beth_\alpha$ for some limit ordinal α.

What kinds of classes do we mean?

PCΓ CLASSES

A class $\boldsymbol{K}$ of τ-structures is called PC if it is the collection of reducts to τ of the models of a first order theory T' in some $\tau' \supseteq \tau$.

A class $\boldsymbol{K}$ of τ-structures is called $PC\Gamma$ if it the collection of reducts to τ of the models of a first order theory T' in some $\tau' \supseteq \tau$ which omit all types in a specified collection Γ of types in finitely many variables over the empty set.

ABSTRACT ELEMENTARY CLASSES

Definition 2 *A class of L-structures,* $(\boldsymbol{K}, \leq)$*, is said to be an* abstract elementary class: AEC *if both* $\boldsymbol{K}$ *and the binary relation* $\leq$ *are closed under isomorphism and satisfy the following conditions.*

- **A1**. *If* $M \leq N$ *then* $M \subseteq N$.
- **A2**. $\leq$ *is a partial order on* $\boldsymbol{K}$.
- **A3**. *If* $\langle A_i : i < \delta \rangle$ *is* $\leq$*-increasing chain:*
  1. $\bigcup_{i<\delta} A_i \in \boldsymbol{K}$;
  2. *for each* $j < \delta$, $A_j \leq \bigcup_{i<\delta} A_i$
  3. *if each* $A_i \leq M \in \boldsymbol{K}$ *then* $\bigcup_{i<\delta} A_i \leq M$.
- **A4**. *If* $A, B, C \in \boldsymbol{K}$, $A \leq C$, $B \leq C$ *and* $A \subseteq B$ *then* $A \leq B$.
- **A5**. *There is a Löwenheim-Skolem number* $\mathrm{LS}(\boldsymbol{K})$ *such that if* $A \subseteq B \in K$ *there is a* $A' \in \boldsymbol{K}$ *with* $A \subseteq A' \leq B$ *and* $|A'| < \mathrm{LS}(\boldsymbol{K}) + |\mathrm{A}|$.

THE PRESENTATION THEOREM

Theorem 3 *Every AEC is a PC*Γ*.*

$H(\kappa)$, the Hanf number for AEC's in vocabularies of size κ, is the least cardinal such that:

Every AEC in vocabulary of size κ with a model of size $H(\kappa)$ has arbitrarily large models.

The presentation theorem implies:

$$H(\kappa) = \beth_{\omega_1}((2^\kappa)^+) = H_1.$$

i.e. the Hanf number for omitting types in PCΓ classes.
$H_2 = H(H_1)$.

We also get Ehrenfeucht-Mostowski models and omitting types theorems in AEC if there is a model at least H_1.

AMALGAMATION AND GALOIS TYPES

If the AEC $(\boldsymbol{K}, \leq)$ has the amalgamation property, then a monster model $\mathcal{M}$ exists. One can define:

The Galois type of a over $M \in \boldsymbol{K}$ is the orbit of a in $\mathcal{M}$ under automorphisms of $\mathcal{M}$ fixing M.

Theorem 4 *Model-homogenous and universal for $\leq$ is the same as Galois saturated.*

TAMENESS

In general, Galois equivalence refines having the same syntactic type even in AEC's where syntactic types make sense.

Definition 5 ***K*** *is* (χ, χ_1)*-tame if two distinct Galois types over a model of cardinality* χ*, have different restrictions to a submodel of cardinality* χ_1*.*

Here, Galois types are defined only over models. Recently, Hytinnen has some very interesting ideas for defining types over arbitrary sets in any AEC.

Any superstable theory with OTOP is $\aleph_0$-tame but not excellent.

CONTEXT

Conjecture: Let X be the class of cardinals in which a *reasonably defined* class is categorical.
Not both X and the complement of X are cofinal.

(Note: So, PC-classes are not 'reasonable'.)

We know this conjecture for first order theories and for excellent classes in $L_{\omega_1,\omega}$. But is open even for general sentences in $L_{\omega_1,\omega}$. So it is reasonable to investigate it first with quite strong hypotheses.

Of course, it is only interesting when $\boldsymbol{K}$ has arbitrarily large models – EM methods are applicable.

EVENTUAL CATEGORICITY

Theorem 6 *(Shelah) Assume the AEC* $\boldsymbol{K}$ *has*

1. *ap and jep*
2. *is categorical in a successor cardinal* $\lambda > H_2$

A. *for some* $\chi < H(\tau)$ *and any* $\chi_1 < \lambda$*,* $\boldsymbol{K}$ *is* (χ_1, χ)*-tame and*

B. $\boldsymbol{K}$ *is categorical in every* μ *with* $H_2 \leq \mu \leq \lambda$*.*

Note we get (χ_1, χ)-tame for small χ_1 and smaller but not minute χ. I.e. $\chi \sim H_1$.

Theorem 7 *(Grossberg-VanDieren) If we add the hypothesis:*
for some $\chi < H(\tau)$ and ANY χ_1, $\boldsymbol{K}$ is (χ_1, χ)-tame

then if $\boldsymbol{K}$ is categorical in two successive cardinals above χ_1, then $\boldsymbol{K}$ is categorical in all larger cardinals.

In particular, if $\boldsymbol{K}$ is categorical in one successor cardinal $> H_2$, $\boldsymbol{K}$ is categorical in every θ with $\theta \geq H_2$.

Jep is assumed for convenience.

AP is a very significant assumption

Categorical Structures

I. $(\mathrm{C}, =)$

IIa. $(\mathrm{C}, +, =)$
IIb. $(\mathrm{C}, \times, =)$
not quite vector spaces over Q.

III. $(\mathrm{C}, +, \times, =)$
Algebraically closed fields - Steinitz

IV. $(\mathrm{C}, +, \times, =, \exp)$
Clearly not first order categorical. But maybe the only obstruction is $(\mathrm{Z}, +, \times, =)$.

GEOMETRIES

Definition. A pregeometry is a set G together with a dependence relation

$$cl : \mathcal{P}(G) \to \mathcal{P}(G)$$

satisfying the following axioms.

A1. $cl(X) = \bigcup \{cl(X') : \; X' \subseteq_{fin} X\}$
A2. $X \subseteq cl(X)$
A3. $cl(cl(X)) = cl(X)$
A4. If $a \in cl(Xb)$ and $a \notin cl(X)$, then $b \in cl(Xa)$.

If points are closed the structure is called a geometry.

STRONGLY MINIMAL I

M is *strongly minimal* if every first order definable subset of any elementary extension M' of M is finite or cofinite.

$a \in \mathrm{acl}(\mathrm{X})$ if there is a first order formula with finitely solutions over X which is satisfied by a.

Exercise: If f takes X to Y is an elementary isomorphism, f extends to an elementary isomorphism from acl(X) to acl(Y).

STRONGLY MINIMAL II

Lemma. A complete theory T is strongly minimal if and only if it has infinite models and

1. algebraic closure induces a pregeometry on models of T;
2. any bijection between acl-bases for models of T extends to an isomorphism of the models

Theorem. A strongly minimal theory is categorical in any uncountable cardinality.

QUASIMINIMALITY I

Trial Definition M is *'quasiminimal'* if every first order ($L_{\omega_1,\omega}$?) definable subset of M is countable or co-countable.

$a \in \mathrm{acl}'(\mathrm{X})$ if there is a first order formula with countably solutions over X which is satisfied by a.

Exercise ? If f takes X to Y is an elementary isomorphism, f extends to an elementary isomorphism from $\mathrm{acl}'(\mathrm{X})$ to $\mathrm{acl}'(\mathrm{Y})$.

QUASIMINIMAL EXCELLENCE

A class $(\boldsymbol{K}, \mathrm{cl})$ is *quasiminimal excellent* if it admits a combinatorial geometry which satisfies on each $M \in \boldsymbol{K}$

1. there is a unique type of a basis,
2. a technical homogeneity condition:
   $\aleph_0$-homogeneity over $\emptyset$ and over models.
3. and the 'excellence condition' which follows.

If $(\boldsymbol{K}, cl)$ is quasiminimal excellence then cl is given by the trial definition on the previous slide.

The ω-homogeneity yields by an easy induction:

Lemma 8 *If* $\mathrm{cl}(X)$ *and* $\mathrm{cl}(Y)$ *are countable then an isomorphism between* X *and* Y *extends to an isomorphism of* $\mathrm{cl}(X)$ *and* $\mathrm{cl}(Y)$

QUASIMINIMAL EXCELLENCE

In the following definition it is essential that $\subset$ be understood as *proper* subset.

Definition 9 *1. For any* Y, $\mathrm{cl}^-(Y) = \bigcup_{X \subset Y} \mathrm{cl}(X)$.

2. We call C *(the union of)* an n-dimensional independent system *if* $C = \mathrm{cl}^-(Z)$ *and* Z *is an independent set of cardinality* n.

To visualize a 3-dimensional independent system think of a cube with the empty set at one corner A and each of the independent elements z_0, z_1, z_2 at the corners connected to A. Then each of $\mathrm{cl}(z_i, z_j)$ for $i < j < 3$ determines a side of the cube: $\mathrm{cl}^-(Z)$ is the union of these three sides; $\mathrm{cl}(Z)$ is the entire cube.

EXCELLENCE

The class $(\boldsymbol{K}, \mathrm{cl})$ is quasiminimal excellent if it satisfies the following additional condition.

Let $C = \mathrm{cl}^{-}(Z)$ be an n-dimensional independent system. For any $\overline{a} \in \mathrm{cl}(Z)$, there is a finite $X \subset C$:

$$\mathrm{tp}(\overline{a}/X) \models \mathrm{tp}(\overline{a}/C).$$

EXCELLENCE IMPLIES CATEGORICITY

Theorem 10 *Let $\boldsymbol{K}$ be a quasiminimal excellent class and suppose $H, H' \in \boldsymbol{K}$ satisfy the countable closure condition.*
Let A, A' be cl*-independent subsets of H, H' with* $\mathrm{cl}(A) = H$, $\mathrm{cl}(A') = H'$ *respectively and ψ a bijection between A and A'.*
Then ψ extends to an isomorphism of H and H'.

Thus $\boldsymbol{K}$ is categorical in every uncountable cardinality.

EXCELLENCE IMPLIES CATEGORICITY: PROOF

Fix a countable subset A_0 of A; without loss of generality, we can assume ψ is the identity on A_0 and work over $G = \text{cl}(A_0)$. So from now on monomorphism means monomorphism over G and $\text{cl}(X)$ means $\text{cl}(A_0 X)$.

Note that ψ is a monomorphism and so is $\psi_0 = \psi | A_0$. By Lemma 8 and induction, for any X with $|X| \leq \aleph_1$, $\psi | X$ extends to a isomorphism from $\text{cl}(X)$ to $\text{cl}(X)$.

$H = \lim_{X \subset A; |X| < \aleph_0} \mathrm{cl}(X)$.

The theorem follows if:
for each finite X we can choose $\psi_X : \mathrm{cl}(X) \to H'$ so that $X \subset Y$ implies $\psi_X \subset \psi_Y$.

We prove this by induction on $|X|$. Suppose $|Y| = n+1$ and we have appropriate ψ_X for $|X| < n+1$. We will prove two statements by induction.

1. $\psi_Y^- : \mathrm{cl}^-(Y) \to H'$ defined by $\psi_Y^- = \bigcup_{X \subset Y} \psi_X$ is a monomorphism.

2. ψ_Y^- extends to ψ_Y defined on $\mathrm{cl}(Y)$.

The first step is done by induction and ω-homogeneity using Lemma 8. The exchange axiom is used to guarantee that the maps ψ'_Y for $Y' \supset Y$ agree where more than one is defined. The second is follows by Excellence and induction using Lemma 8 and the fact that $\mathrm{cl}(Y)$ is countable.

CATEGORICITY

Theorem 11 *Suppose the quasiminimal excellent (I-IV) class* $\boldsymbol{K}$ *is axiomatized by a sentence* Σ *of* $L_{\omega_1,\omega}$*, and the relations* $y \in \mathrm{cl}(x_1, \ldots x_n)$ *are* $L_{\omega_1,\omega}$*-definable.*

Then, for any infinite κ *there is a unique structure in* $\boldsymbol{K}$ *of cardinality* κ *which satisfies the countable closure property.*

NOTE BENE: The categorical class could be axiomatized in $L_{\omega_1,\omega}(Q)$. But, the categoricity result does not depend on any such axiomatization.

COVERS OF THE MULTIPLICATIVE GROUP OF C

Consider a short exact sequence:

$$0 \to Z \to H \to F^* \to 0. \tag{1}$$

H is a torsion-free divisible abelian group (written additively), F is an algebraically closed field, and exp is the map from H to F^*.

We can code this sequence as a structure:

$$(H, +, E, S),$$

where $E(h_1, h_2)$ iff $\exp(h_1) = \exp(h_2)$
and
$S(h_1, h_2, h_3)$ iff $\exp(h_1) + \exp(h_2) = \exp(h_3)$.

AXIOMATIZING COVERS

Theorem. There is an $L_{\omega_1,\omega}$-sentence Σ such that there is a 1-1 correspondence between models of Σ and sequences 1.

$L_{\omega_1,\omega}$ comes in to say the kernel is 1-generated (and therefore countable).

Theorem. Σ is quasiminimal excellent.

THUMBTACK LEMMA

Definition 12 *A* multiplicatively closed divisible subgroup *associated with $a \in C^*$, is a* **choice** *of a multiplicative subgroup isomorphic to* $\mathbb{Q}$ *containing a .*

Theorem 13 (Zilber's thumbtack lemma)
For any $b_1, \ldots b_\ell \subset C^*$, there exists an m such that $b_1^{\frac{1}{m}} \in b_1^{\mathbb{Q}}, \ldots b_\ell^{\frac{1}{m}} \in b_\ell^{\mathbb{Q}} \subset C^*$, determine the isomorphism type *of $b_1^{\mathbb{Q}}, \ldots b_\ell^{\mathbb{Q}} \subset C^*$ over F.*

CATEGORICITY IN $L_{\omega_1,\omega}$

Fundamental Fact I. To study categoricity in power of sentences in $L_{\omega_1,\omega}$ it suffices to study the class of atomic models of a complete first order theory – an atomic AEC.

Fundamental Fact II. Under weak CH, categoricity in an uncountable power for an atomic class implies ω-stability (for atomic types over models).

Fundamental Fact III. In an ω-stable atomic class, *we may assume* that *nonsplitting* defines a dependence relation which satisfies the properties of non-forking in an ω-stable theory.

Proved by Shelah, 25 years ago, using work of Keisler and the Lopez-Escobar theorem.

The *we may assume* is crucial. By expanding the language we obtain stronger and stronger properties on the class to be analyzed without changing the spectrum function.

INDEPENDENT SYSTEMS

Notation 14 *An independent (λ, n)-system is a family of models $\langle M_s : s \subset n \rangle$ such that:*

1. *Each $M_s \in \boldsymbol{K}$ has cardinality λ.*
2. *If $s \subseteq t$, $M_s \leq M_t$.*
3. *For each s, $A_s = \bigcup_{t \subset s} M_t$ is atomic.*
4. *For each s, $M_s \underset{A_s}{\downarrow} B_s$ where $B_s = \bigcup_{t \not\subset s} M_t$.*

Definition 15 $\boldsymbol{K}$ *satisfies the (λ, n)-existence property if there is a primary (i.e. strictly constructible) model over $\bigcup_{t \subset n} M_t$ for every independent (λ, n)-system.*

EXCELLENCE IN GENERAL

Definition 16 *The atomic AEC* $\boldsymbol{K}$ *is* excellent *if*

1. $\boldsymbol{K}$ *is* ω*-stable;*
2. *For every* n, $\boldsymbol{K}$ *satisfies the* $(\aleph_0, n)$*-existence property.*

CRUCIAL STEPS

Theorem 17 *If* $\boldsymbol{K}$ *is excellent then*
$\boldsymbol{K}$ *has the* (λ, n)*-existence property for all* n *and* λ*.*

.

Lessmann has shown that this last property is the 'active ingredient' for deducing categoricity in all powers from excellence.

Theorem 18 *If for all* $\mu < \lambda$*, the* $(\mu, 2)$*-existence property holds then for any model* M *of cardinality* λ *and any* $\overline{a}$ *such that* $M\overline{a}$ *is atomic, there is a primary model* N *over* $M\overline{a}$*.*

Corollary 19 $\boldsymbol{K}$ *has arbitrarily large models.*

QUASIMINIMALITY IN ATOMIC CLASSES

Definition 20 *The type p over $A \subseteq M \in \boldsymbol{K}$ is* big *if for any $M' \supseteq A$ there exists an N' with $M' \leq N'$ and with a realization of p in $N' - M'$.*

Definition 21 *The type $p \in S_{\mathrm{at}}(A)$ is* quasiminimal *if p is big and for any M containing A, p has a unique extension to a type over M which is not realized in M.*

Lemma 22 *Let $\boldsymbol{K}$ be excellent. For any $M \in \boldsymbol{K}$, there is a $\overline{c} \in M$ and a formula $\phi(x, \overline{c})$ which is quasi-minimal.*

GEOMETRY AGAIN

Definition 23 *Let* $\overline{\mathrm{c}} \in M \in \boldsymbol{K}$ *and suppose* $\phi(x, \overline{\mathrm{c}})$ *generates a quasiminimal type over* M*. For any elementary extension* N *of* M *define* cl *on the set of realizations of* $\phi(x, \overline{\mathrm{c}})$ *in* N *by* $a \in \mathrm{cl}(A)$ *if* $\mathrm{tp}(a/A\overline{\mathrm{c}})$ *is not big.*

Theorem 24 *This closure defines a homogeneous pregeometry on the quasiminimal set.*

PARADISE REGAINED

Theorem 25 *Suppose* $\boldsymbol{K}$ *is *-excellent. The following are equivalent.*

1. $\boldsymbol{K}$ *is categorical in some uncountable cardinality.*
2. $\boldsymbol{K}$ *has no two cardinal models.*
3. $\boldsymbol{K}$ *is categorical in every uncountable cardinal.*

Recall:

Theorem 26 *[Shelah]*

1. *(For* $n < \omega$, $2^{\aleph_n} < 2^{\aleph_{n+1}}$*) A complete* $L_{\omega_1,\omega}$*-sentence which has few models in* $\aleph_n$ *for each* $n < \omega$ *is excellent.*
2. *(ZFC) An excellent class has models in every cardinality.*
3. *(ZFC) Suppose that* ϕ *is an excellent* $L_{\omega_1,\omega}$*-sentence. If* ϕ *is categorical in one uncountable cardinal* κ *then it is categorical in all uncountable cardinals.*

1. Weak GCH and categoricity up $\aleph_\omega$ implies excellence.
2. Categoricity up to $\aleph_n$ does not suffice.

Corollary 27 *Suppose* $\boldsymbol{K}$ *is *-excellent. If* $\boldsymbol{K}$ *is not* $\aleph_1$ *categorical, then* $\boldsymbol{K}$ *has at least* $n+1$ *models of cardinality* $\aleph_n$ *for each* $n < \omega$*.*

Study $(\mathbf{C}, +, \cdot, \exp)$

Zilber's program:

The most ambitious aim of the pseudo-analytic model program is to realize $(\mathrm{C}, +, \cdot, \exp)$ as a model of an $L_{\omega_1,\omega}$-sentence discovered by the Hrushovski construction. This program has two parts.

A. Expand $(\mathrm{C}, +, \cdot)$ by a unary function which behaves like exponentiation using a Hrushovski-like dimension function. Prove some $L_{\omega_1,\omega}$-sentence Σ is categorical and has quantifier elimination.

B. Prove $(\mathrm{C}, +, \cdot, \exp)$ is a model of the sentence Σ found in Objective A.

A: PSEUDO-EXPONENTIATION

$\boldsymbol{K}$ is the class of algebraically closed fields F equipped with a unary function E such that:

1. E is a surjective map from F to F^*, which a homomorphism between the additive and multiplicative group.
2. $\ker E = \mathrm{Z}$.
3. Schanuel's conjecture holds for E:
   If $x_1, \ldots x_n$ are linearly independent
   $$\mathrm{td}(\mathrm{x_1, \ldots x_n, E(x_1, \ldots E(x_n) \geq n.})$$
4. $F \in \boldsymbol{K}$ is *strongly exponentially algebraically closed*:
   For any 'suitable' variety V defined over a finite $C \subset F$, there is a **generic** over C realization of V in F.

A QUASIMINIMAL EXCELLENT CLASS

For $X \subset_{<\omega} A$,

$$\delta_A(X) = \mathrm{td}(\mathrm{span}_{\mathbb{Q}}\mathrm{X} \cup (\exp(\mathrm{span}_{\mathbb{Q}}\mathrm{X})) - \mathrm{ld}(\mathrm{span}_{\mathbb{Q}}\mathrm{X}).$$

Now define a combinatorial geometry.

Definition 28 *1. For $M \in \boldsymbol{K}$, $A \subseteq M$, A finitely generated,*

$$d_M(A) = \inf\{\delta(B) : A \subset B \subseteq M, B \in \boldsymbol{K}_0\}.$$

2. For A, b contained M, $b \in \mathrm{cl}(A)$ if $d_M(bA) = d_M(A)$.

Extend to closures of infinite sets by imposing finite character.

This extends the Hrushovski construction because $\boldsymbol{K}$ is axiomatized in $L_{\omega_1,\omega}$ not first order.

DONE

$F \in \boldsymbol{K}$ is *strongly exponentially algebraically closed* if for any 'suitable' variety V defined over a finite $C \subset F$, there is a **generic** over C realization of V in F.

Theorem 29 *$\boldsymbol{K}$ $L_{\omega_1,\omega}$-axiomatizable and is quasiminimal excellent.*

The members of $\boldsymbol{K}$ with countable closure are categorical in all uncountable powers. This class is $L_{\omega_1,\omega}(Q)$-axiomatizable.

This concludes the proof of Objective A.

AND TO DO??

Objective B is reduced to proving the hypotheses of the following theorem.

Theorem 30 *If the Schanuel conjecture holds in* C *and if the strong exponential closure axioms hold in* C*, then*

1. $(C, +, \cdot, \exp) \in \boldsymbol{K}$.
2. $(C, +, \cdot, \exp)$ *has the countable closure property.*

REPRISE

I. Some natural mathematical structures are better axiomatized in non-elementary logics.

II. The 'pure' analysis, like stability theory in the 80's, is well-ahead of the applications. One can hope that this general analysis will yield applications akin to the applications of orthogonality and geometric stability to Diophantine analysis in the 90's.

III. In any case, the model theoretic analysis stimulates interesting mathematical questions.
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