UIC Math 313

Analysis I—Class notes

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This is a course about the foundations of real numbers and calculus.

1. Rational numbers

We will assume the rational numbers are familiar and begin by recalling some notation and facts.

$\mathbb{N}$ is the set of positive integers, $\mathbb{N} = \{1, 2, 3, \ldots\}$.

$\mathbb{Z}$ is the set of integers, $\mathbb{Z} = \{0, +1, -1, +2, \ldots\}$.

$\mathbb{Q}$ is the set of rational numbers, $\mathbb{Q} = \{m/n : m \in Z, n \in \mathbb{N}\}$ with $m/n = p/q$ if and only if $mq = np$.

A rational number $r \in \mathbb{Q}$ is positive if $r = m/n$ with both $m, n \in \mathbb{N}$. The rationals satisfy the condition, called trichotomy, that for each $r \in \mathbb{Q}$ either r is positive, $r = 0$, or $-r$ is positive (so r is negative) and only one of these is true. The sets of positive integers and of positive rational numbers are closed under addition and multiplication.

We write $r > 0$ if r is positive and $r > s$ if $r - s$ is positive. Note that if q and n are positive, then $m/n > p/q$ if and only if $qm > pn$, since $\dfrac{m}{n} - \dfrac{p}{q} = \dfrac{qm - pn}{qn}$ which is positive if and only if $qm - pn$ is positive.

Let us accept $\mathbb{Q}$ as a familiar set with addition, multiplication, and the order, $>$. Also we will use argument by induction and the statement that any nonempty set of positive integers has a smallest element. $\mathbb{Q}$ satisfies the following set of axioms.

Axioms for an ordered field

$a + b = b + a$	$ab = ba$	(commutative)
$(a + b) + c = a + (b + c)$	$(ab)c = a(bc)$	(asociative)
$\exists\, 0$ s.t. $a + 0 = a$	$\exists\, 1$ s.t. $a \cdot 1 = a$	(identity)
$\forall\, a\ \exists\, (-a)$ s.t. $a + (-a) = 0$	$\forall\, a \neq 0\ \exists\, a^{-1}$ s.t. $a \cdot a^{-1} = 1$	(inverse)

$1 \neq 0$	(nontriviality)
$a \cdot (b + c) = a \cdot b + a \cdot c$	(distributive)
Exactly one of $a > 0$, $a = 0$, or $-a > 0$ is true.	(trichotomy)
If $a > 0$ and $b > 0$, then $a + b > 0$ and $a \cdot b > 0$.	(closure)

The real numbers, $\mathbb{R}$, are also an ordered field and there are other examples. Anything we prove from these axioms will be true of any ordered field.

2. Some consequences of the axioms

(1) $x \cdot 0 = 0$ and $0 \cdot x = 0$.

Proof. Each equal sign in the following string is a consequence of one of the axioms.

$x \cdot 0 = x \cdot 0 + 0 = x \cdot 0 + (x \cdot 0 + (-(x \cdot 0))) = (x \cdot 0 + x \cdot 0) + (-(x \cdot 0)) = x \cdot (0 + 0) + (-(x \cdot 0)) = x \cdot 0 + (-(x \cdot 0)) = 0$.

(2) $(-x) \cdot y = -(x \cdot y)$ and $x \cdot (-y) = -(x \cdot y)$.

Proof. Now, in addition to the axioms we may use a previously proved statement.

$-(x \cdot y) = -(x \cdot y) + 0 = -(x \cdot y) + x \cdot (y + (-y)) = -(x \cdot y) + (x \cdot y + x \cdot (-y)) = (-(x \cdot y) + x \cdot y) + x \cdot (-y) = 0 + x \cdot (-y) = x \cdot (-y) + 0 = x \cdot (-y)$, and

$(-x) \cdot y = y \cdot (-x) = -(y \cdot x) = -(x \cdot y)$.

(3) $-(-x) = x$.

Proof. $-(-x) = -(-x) + 0 = -(-x) + ((-x) + x) = (-(-x) + (-x)) + x = 0 + x = x + 0 = x$.

(4) $(-x) \cdot (-y) = x \cdot y$, in particular $(-1) \cdot (-1) = 1$.

Proof. $(-x) \cdot (-y) = -((-x) \cdot y) = -(-(x \cdot y)) = x \cdot y$.

(5) $1 > 0$.

Proof. Since $1 \neq 0$, by the first order axiom either $1 > 0$ or $-1 > 0$. If $-1 > 0$, then by the second order axiom , $(-1) \cdot (-1) > 0$. Using (4) this implies $1 > 0$ which, with the assumption that $-1 > 0$, contradicts the first order axiom. Therefore $-1 \not> 0$ and hence $1 > 0$.

If $x > 0$ we say x is positive, If $-x > 0$ we say x is negative. The statement y is greater than x is defined to mean $y - x > 0$; this is written $y > x$ or equivalently $x < y$.

(6) Exactly one of $x < y$, $x = y$, or $x > y$ is true.

Proof. These statements are equivalent to the statements $y - x$ is positive, $y - x = 0$, or $-(y - x)$ is positive. Hence (6) is equivalent the first order axiom. The axiom and (6) are called trichotomy.

(7) If $x < y$ and z is in F, then $x + z < y + z$.

Proof. We need to show that $(y + z) - (x + z)$ is positive when $y - x$ is positive. This follows from a calculation, using the sum axioms, that $(y + z) - (x + z) = y - x$.

(8) If $x < y$ and $y < z$, then $x < z$.

Proof. By our hypothesis $y - x$ and $z - x$ are positive. Then by the second order axiom the sum $(z-y)+(y-x)$ positive. Using the sum axioms we calculate $(z-y)+(y-x) = z-x$. Hence $z - x$ is positive, so $x < z$.

(9) If $x < y$ and $0 < z$, then $xz < yz$.

Proof. By hypothesis, $y-x$ is positive and z is positive, so the product $(y-x)z$ is positive. Using the distributive axiom and (2), we find $(y+(-x))z = (yz)+((-x)z) = (yz)+(-(xz))$. It follows that $xz < yz$.

(10) If $0 < x$ and $x < y$, then $x^2 < y^2$.

Proof. By (9) $x^2 < xy$. By (8) $0 < y$, so by (9) $xy < y^2$. Then by (8) $x^2 < y^2$.

(11) If $0 < x$ and $0 < y$, then $x^2 < y^2 \Rightarrow x < y$.

Proof. We use the method of contradiction. We assume the hypotheses: $0 < x$, $0 < y$, and $x^2 < y^2$. If $x < y$ were false, then either $x = y$ or $y < x$ by (6) If we suppose $x = y$, then $x^2 = y^2$. This contradicts the hypothesis. If we suppose $y < x$, then by (10), with the roles of "x" and "y" interchanged, we have $y^2 < x^2$ which again contradicts the hypothesis. Therefore $x < y$.

(12) If $0 < x$ and $0 < y$, then $x^2 < y^2 \iff x < y$.

Proof. This follows from (10) and (11).

(13) If $xy = 0$ then $x = 0$ or $y = 0$.

This will be an exercise. It can be proved for a field using the existence of multiplicative inverses. It can also be proved for an ordered commutative ring (the integers or polynomials over a field) in which multiplicative inverses may be missing.

3. $\sqrt{2}$

Here is an algebraic version of Pythagoras's geometric proof that the diagonal of a square is not commensurable with its side.

Theorem. There is no rational number whose square is 2.

Proof. Otherwise there is a positive rational number whose square is 2. We could write $\sqrt{2} = m/n$ in such a way that n is the smallest possible denominator in $\mathbb{N}$. Then $2n^2 = m^2$.

We have $m > n$ because $m \le n$ implies $m^2 \le n^2 < 2n^2$, a contradiction. Also $m < 2n$ because $m \ge 2n$ implies $m^2 \ge 4n^2 > 2n^2$, again a contradiction. Hence $m - n < n$. Finally, $(2n - m)^2 = 4n^2 - 4mn + m^2 = 2m^2 - 4mn + 2n^2 = 2(m - n)^2$.

Hence $\sqrt{2} = \dfrac{2n-m}{m-n}$ where the numerator and denominator are positive, but $m-n<n$, contradicting the choice of n as the smallest possible denominator. This shows that $\sqrt{2}$ is not a rational number.

If we regard m/n as only an approximation to $\sqrt{2}$, for example $3/2$ is not so far from $\sqrt{2}$ since $(3/2)^2 = 2 + 1/4$, the argument above replaces $3/2$ by the worse approximation $1/1$. We may reverse this process to get better approximations. Let

$$a = 2n - m, \quad b = m - n.$$

Solving for m and n: $\quad m = a + 2b, \quad n = a + b.$

Beginning with the estimate $1/1$, we get the sequence of rational numbers:

$$1/1,\ 3/2,\ 7/5, \ldots,$$

where a/b is followed by $(a+2b)/(a+b)$. The sequence of squares is $2-1,\ 2+1/4,\ 2-1/25, \ldots$, a sequence alternately below and above 2 and getting closer.

In fact, if $2 - \dfrac{a^2}{b^2} = \dfrac{e}{b^2}$ then $2b^2 - a^2 = e$ and

$$2(a+b)^2 - (a+2b)^2 = 2a^2 + 4ab + 2b^2 - a^2 - 4ab - 4b^2 = a^2 - 2b^2 = -e.$$

So $2 - \left(\dfrac{a+2b}{a+b}\right)^2 = -\dfrac{e}{(a+b)^2}$.

Starting with $a = 1, b = 1$ we have $e = -1$. The sequence is $s_1 = 1/1,\ s_2 = 3/2,\ s_3 = 7/5,\ s_4 = 17/12, \ldots$ for which we have

$$s_1 < s_3 < s_5 < \cdots < \sqrt{2}$$
$$s_2 > s_4 > s_6 > \cdots > \sqrt{2}.$$

and $|s_{2n} - s_{2n-1}| \to 0$ as $n \to \infty$

This gives an explicit way to find rational approximations to $\sqrt{2}$. The reader should carry these computations further.

4. Completeness

The set of real numbers $\mathbb{R}$ is an ordered field containing the rationals. The reals are characterized by an additional property called completeness.

In an ordered field we say $x \le y$ if $x = y$ or $x < y$. It follows from (6) and (8) in §**2** that the relation $\le$ satisfies the axioms:

(1) $x \le x$,

(2) $x \le y$ and $y \le x$ implies $x = y$,

(3) $x \le y$ and $y \le z$ implies $x \le z$, and

(4) for all x and y either $x \le y$ or $y \le x$.

The first three axioms define a partial order and, with the fourth, a total order.

Completeness Axiom. If A and B are nonempty subsets of $\mathbb{R}$ and if $x \in A$ and $y \in B \Rightarrow x \le y$, then there exists a $z \in \mathbb{R}$ such that $x \in A \Rightarrow x \le z$ and $y \in B \Rightarrow z \le y$.

Definition. A *nonempty closed bounded interval* is a subset of $\mathbb{R}$ of the form

$$[a, b] = \{x : a \le x \le b\}$$

where a and b are real numbers with $a \le b$.

Definition. A sequence of such intervals, $I_n = [a_n, b_n]$, is nested if $m < n \Rightarrow I_m \supset I_n$, or equivalently if $a_m \le a_n \le b_n \le b_m$.

Cantor Intersection Theorem. If $I_1 \supset I_2 \supset I_3 \supset \cdots$ is a sequence of nonempty closed bounded intervals, then $\bigcap_{n=1}^{\infty} I_n$ is nonempty.

Proof. Let $I_n = [a_n, b_n]$ and let $A = \{a_n : n \in \mathbb{N}\}$, $B = \{b_n : n \in \mathbb{N}\}$. Then $a_m \le a_{m+n} \le b_{m+n} \le b_n$, hence A and B satisfy the hypothesis of the completeness axiom. Hence there exists $z \in \mathbb{R}$ with $a_n \le z \le b_n$, so for all n, $z \in I_n$. Therefore $z \in \bigcap_{n=1}^{\infty} I_n$.

Let $\lambda_n = b_n - a_n$ be the length of I_n. We say $\lim\limits_{n\to\infty} \lambda_n = 0$ if

$$(\forall\, \varepsilon > 0)(\exists\, k)(n \ge k \Rightarrow -\varepsilon < \lambda_n < \varepsilon).$$

Corollary. If $\lim\limits_{n\to\infty} \lambda_n = 0$, then $\bigcap_{n=1}^{\infty} I_n$ contains a unique element.

Proof. If w and z are both in $\bigcap_{n=1}^{\infty} I_n$ with $w < z$, let $\varepsilon = z - w > 0$. Then $(\exists\, k)(n \ge k \Rightarrow -\varepsilon < b_n - a_n < \varepsilon)$. Also $a_n \le w < z \le b_n$, so $\varepsilon = z - w \le z - a_n \le b_n - a_n < \varepsilon$, a contradiction.

Definition. An element $b \in \mathbb{R}$ is an *upper bound* for a nonempty set A if $\forall\, x \in A,\ x \le b$. The nonempty subset $A \subset \mathbb{R}$ is *bounded above* if there exists an upper bound for A, that is $\exists\, b \in \mathbb{R}$ such that $x \in A \Rightarrow x \le b$.

Definition. We say $z \in \mathbb{R}$ is a *least upper bound* for A if

(1) z is an upper bound for A and

(2) if b is any upper bound for A, then $z \le b$.

Lemma. If A has a least upper bound, it is unique.

Proof. If z and z_1 are both least upper bounds for A, then $z \le z_1$ since z_1 is an upper bound and z is a least upper bound. Also $z_1 \le z$ since z is an upper bound and z_1 is a least upper bound. Therefore $z = z_1$.

Theorem. If a nonempty set A is bounded above, then A has a least upper bound.

Proof. Let $B = \{y : y \text{ is an upper bound for } A\}$. Then A and B are nonempty by hypothesis. If $x \in A$ and $y \in B$, then $x \leq y$. By the completeness axiom there exists a $z \in \mathbb{R}$ such that $x \in A \Rightarrow x \leq z$ and $y \in B \Rightarrow z \leq y$. Therefore z is an upper bound for A and, if y is any upper bound for A, then $y \in B$ and hence $z \leq y$. Hence z is the least upper bound of A.

Property of Archimedes. $\mathbb{N}$ is not bounded above in $\mathbb{R}$.

Proof. Assume $\mathbb{N}$ is bounded above and let b be its least upper bound. Then $b-1$ is not an upper bound for $\mathbb{N}$, so there exists $n \in \mathbb{N}$ with $b-1 < n$. But then $b < n+1$, so b is not an upper bound, contradicting the assumption that $\mathbb{N}$ is bounded above.

Corollary. $\forall \varepsilon > 0, \exists n \in \mathbb{N},\ \frac{1}{n} < \varepsilon$.

Proof. Since $\varepsilon > 0$, we have $\varepsilon^{-1} > 0$. By Archimedes property, there is an $n > \varepsilon^{-1}$. Therefore $\varepsilon n > \varepsilon\varepsilon^{-1} = 1$. Thus $\frac{1}{n} < \varepsilon$.

5. Sequences and limits

Definition. A sequence of real numbers is a function s from $\mathbb{N}$ to $\mathbb{R}$; $s(n)$ is usually written s_n and the sequence, $s_1, s_2, s_3, \ldots$ is written $\{s_n\}$.

A sequence $\{s_n\}$ is *bounded above* if the set $\{s_n : n \in \mathbb{N}\}$ is bounded above, that is if $\exists\, b\ \forall\, n\ \ s_n \leq b$.

A sequence $\{s_n\}$ is *increasing* if $s_1 < s_2 < s_3 < \cdots$, that is $\forall\ n \in \mathbb{N},\ s_n < s_{n+1}$, and it is *nondecreasing* if $s_1 \leq s_2 \leq s_3 \leq \cdots$, that is $\forall\, n \in \mathbb{N},\ s_n \leq s_{n+1}$. In either case, by induction $m < n \Rightarrow s_m \leq s_n$.

Examples.

(i) $s_1 = \sqrt{1},\ s_2 = \sqrt{2},\ s_3 = \sqrt{3}, \ldots$.

(ii) $s_1 = 0,\ s_2 = 3/2,\ s_3 = 2/3, \ldots$.

(iii) $s_1 = 1,\ s_2 = 3/2,\ s_3 = 7/4, \ldots$.

Listing the first few term of a sequence is often not helpful. To be clear:

$$\text{(ii) } s_n = 1 + \frac{(-1)^n}{n}, \qquad \text{(iii) } s_n = 2 - \frac{1}{2^{n-1}}.$$

(i) is increasing but not bounded,

(ii) is bounded above, but not increasing—the terms oscillate,

(iii) is bounded above and increasing.

Definition. The sequence $\{s_n\}$ converges to ℓ if

$$(\forall\,\varepsilon > 0)(\exists\,m \in \mathbb{N})(n > m \Rightarrow -\varepsilon < s_n - \ell < \varepsilon).$$

The sequence converges if there is an $\ell \in \mathbb{R}$ such that $\{s_n\}$ converges to ℓ, that is,

$$(\exists\,\ell \in \mathbb{R})(\forall\,\varepsilon > 0)(\exists\,m \in \mathbb{N})(n > m \Rightarrow -\varepsilon < s_n - \ell < \varepsilon).$$

We write $\lim\limits_{n\to\infty} s_n = \ell$ to mean that the sequence $\{s_n\}$ converges to ℓ. We say $\lim\limits_{n\to\infty} s_n$ exists if the sequence converges.

Example. Using this definition we show that the sequence $\{1/n\}$ converges to 0.

Given $\varepsilon > 0$, by the Corollary to Archimedes property, we know there is an integer m with $1/m < \varepsilon$. If $n \geq m$, then $0 < 1/n \leq 1/m < \varepsilon$, hence $-\varepsilon < 1/n - 0 < \varepsilon$. Hence the definition is satisfied.

Uniqueness of limits. If $\lim s_n = \ell$ and $\lim s_n = k$, then $k = \ell$.

Proof. Otherwise we may assume $k < \ell$, relabeling if necessary. Let $\varepsilon = (\ell - k)/2 > 0$. There is an m such that

$$n > m \Rightarrow -\varepsilon < s_n - \ell, \text{ hence } \varepsilon = -\varepsilon + \ell - k < s_n - \ell + \ell - k = s_n - k.$$

Therefore $\lim s_n = k$ is false contradicting the assumption $k < \ell$.

The following result gives convergence without specifying the limit.

Theorem. If a sequence is nondecreasing and bounded above, then the sequence converges.

Proof. Let $A = \{s_n : n \in \mathbb{N}\}$. By hypothesis A is bounded above. Hence A has a least upper bound, call it ℓ. Then $(\forall\,n)(s_n \leq \ell)$. If $\varepsilon > 0$, then $\ell - \varepsilon$ is not an upper bound, so $(\exists\,m)(\ell - \varepsilon < s_m)$. Since the sequence is nondecreasing, if $n \geq m$ then $s_m \leq s_n$. Therefore,

$$n \geq m \Rightarrow \ell - \varepsilon < s_m \leq s_n \leq \ell < \ell + \varepsilon.$$

Hence $n \geq m \Rightarrow -\varepsilon < s_n - \ell < \varepsilon$.

Example. Consider the sequence

$$\sqrt{3},\ \sqrt{3+\sqrt{3}},\ \sqrt{3+\sqrt{3+\sqrt{3}}},\ \ldots$$

defined recursively by $s_1 = \sqrt{3}$ and $s_{n+1} = \sqrt{3 + s_n}$.

This sequence is bounded above: $s_1 = \sqrt{3} < 3$ and, if $s_n < 3$ then $3 + s_n < 6 < 9$, so $s_{n+1} < 3$.

The sequence is increasing. $3 < 3 + \sqrt{3}$, so $s_1 < s_2$. If $s_n < s_{n+1}$, then $3 + s_n < 3 + s_{n+1}$, so $s_{n+2} < s_{n+1}$. Hence the sequence converges. At this point we do not know the limit. Knowing that it exists will lead to a proof that the limit is $(1 + \sqrt{13})/2$.

We will use the definition and three basic properties of absolute values.

DEFINITION. $|a| = \begin{cases} a & \text{if } a \geq 0, \\ -a & \text{otherwise.} \end{cases}$

$$|a+b| \leq |a| + |b|, \qquad |ab| = |a| \cdot |b|, \qquad |c| - |a| \leq |c-a|.$$

The first property, the triangle inequality can be proved by considering the cases where a and b have the same, or different, signs. The second property follows from the closure axiom for $<$. The last property follows from the first by taking $c = a + b$.

THEOREM. If $\lim\limits_{n\to\infty} a_n = a$ and $\lim\limits_{n\to\infty} b_n = b$, then:

(1) $\lim\limits_{n\to\infty} (a_n + b_n) = a + b$.

(2) The set $\{a_n : n \in \mathbb{N}\}$ is bounded above and below.

(3) $\lim\limits_{n\to\infty} a_n b_n = ab$.

(4) If $\forall n\ a_n \neq 0$ and $a \neq 0$, then $\lim\limits_{n\to\infty} \dfrac{1}{a_n} = \dfrac{1}{a}$.

(5) If $\forall n\ a_n \leq b_n$, then $a \leq b$.

(6) If $a = \lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} b_n$ and $a_n \leq c_n \leq b_n$, then $\lim\limits_{n\to\infty} c_n = a$.

Proof (in five parts and an exercise).

(1) We have $|a_n + b_n - (a+b)| = |(a_n - a) + (b_n - b)| \leq |a_n - a| + |b_n - b|$. Given $\varepsilon > 0\ \exists\, m$ s.t. $n \geq m \Rightarrow |a_n - a| < \varepsilon/2$ and $|b_n - b| < \varepsilon/2$. Hence $|(a_n + b_n) - (a+b)| < \varepsilon$.

(2) Taking $\varepsilon = 1$, $\exists\, m$ s.t. $n \geq m \Rightarrow |a_n - a| < 1$. Therefore $|a_n| \leq |a| + |a_n - a| < |a| + 1$ when $m \geq n$. Then the set $\{a_n : n \in N\}$ is bounded above by

$$u = \max\{|a_1|, \ldots, |a_m|, |a| + 1\}$$

and below by $-u$.

(3) We have

$$|a_n b_n - ab| = |a_n b_n - a_n b + a_n b - ab| \leq |a_n|\,|b_n - b| + |a_n - a|\,|b|.$$

Let $M > 0$ be a bound for $|a_n|$, $|b_n|$, and $|b|$. This means

$$\forall\, n,\ |a_n| < M,\ |b_n| < M, \text{ and } |b| < M.$$

Now given $\varepsilon > 0$, $\exists\, m$ s.t. $n \geq m \Rightarrow |a_n - a| < \frac{\varepsilon}{2M}$ and $|b_n - b| < \frac{1}{a_n}$.

Then $|a_n b_n - ab| < M \cdot \frac{\varepsilon}{2M} + \frac{\varepsilon}{2M} \cdot M = \varepsilon$.

(4) We first prove that $\{1/a_n\}$ is bounded. Let $\varepsilon = |a|/2 > 0$. Then $\exists\, m$ s.t. $n \geq m \Rightarrow |a_n - a| < |a|/2$. Hence $|a| - |a_n| < |a|/2$, so that $|a|/2 < |a_n|$ or, equivalently, $1/|a_n| < 2/|a|$ for $n \geq m$. Let

$$M = \max\left\{\frac{1}{|a_1|}, \ldots, \frac{1}{|a_m|}, \frac{2}{|a|}\right\}.$$

Then $1/|a_n| < M$ for all n. Next

$$\left|\frac{1}{a_n} - \frac{1}{a}\right| = \left|\frac{a_n - a}{a a_n}\right| = \frac{1}{|a|}|a_n - a|\frac{1}{|a_n|} \leq \frac{1}{|a|}|a_n - a|M.$$

Now, given $\varepsilon > 0$,

$$\exists\, m \text{ s.t. } n \geq m \Rightarrow |a_n - a| < \frac{\varepsilon|a|}{M}.$$

Hence $|1/a_n - 1/a| < \varepsilon$.

(5) This proof is by contradiction. Assume $a > b$ and let $\varepsilon = (a-b)/2 > 0$. Then

$$\exists\, m \text{ s.t. } n \geq m \Rightarrow |a_n - a| < \varepsilon/2 \text{ and } |b_n - b| < \varepsilon/2.$$

Then $|b - a + a_n - b_n| = |a_n - a + b - b_n| < (a-b)/2$.

Since $b - a < 0$ by assumption and $a_n - b_n \leq 0$ by hypothesis, $b - a + a_n - b_n < 0$ and its absolute value is $a - b - a_n + b_n < (a-b)/2$ by the line above. This implies $(a-b)/2 < a_n - b_n \leq 0$, a contradiction, therefore $a \leq b$.

(6) This result, known as the pinching theorem, is left as an exercise.

6. Series

The finite geometric series is the sum

$$s_n = 1 + x + x^2 + \cdots + x^{n-1}.$$

Lemma 1. $(x-1)s_n = x^n - 1$.

Proof. $s_1 = 1$, so the statement is true when $n = 1$. If we assume the statement is true for n, then $(x-1)s_{n+1} = (x-1)(s_n + x^n) = (x-1)s_n + (x-1)x^n = x^n - 1 + x^{n+1} - x^n = x^{n+1} - 1$. So by induction the statement is true for all n.

Lemma 2. If $x \geq 1$, then $s_n \geq n$ and $x^n \geq 1 + (x-1)n$.

Proof. If $x \geq 1$, then each $x_j \geq 1$, so from the definition $s_n \geq n$. The second statement follows from Lemma 1.

Lemma 3. If $x > 1$, then the sequence $\{x^n\}$ is not bounded above.

Proof. If $x > 1$, then $\forall\, b\ \exists\, m$ s.t. $m > b/(x-1)$. Then $n \geq m \Rightarrow x^n > b$.

Lemma 4. If $|x| < 1$, then $\lim\limits_{n\to\infty} x^n = 0$.

Proof. If $x = 0$, then each $x^n = 0$. Assume $0 < |x| < 1$. Then $1/|x| > 1$. Given $\varepsilon > 0$, by Lemma 3, $\exists\, m$ s.t. $n \geq m \Rightarrow |1/x|^n > 1/\varepsilon$, hence $|x^n| < \varepsilon$.

THEOREM. If $|x|<1$ then $\lim_{n\to\infty} s_n = \dfrac{1}{1-x}$.

Proof.

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{1-x^n}{1-x} = \frac{1}{1-x}\left(1-\lim_{n\to\infty} x^n\right) = \frac{1}{1-x}.$$

This result is also written $\sum_{n=0}^{\infty} x^n = \dfrac{1}{1-x}$; $s_n = \sum_{i=0}^{n-1} x^i$ is called a partial sum of the infinite series $\sum_{n=0}^{\infty} x^n$. In general, given a sequence $\{a_n\}$, $s_m = \sum_{n=1}^{m} a_n$ is called the mth partial sum of the series $\sum_{n=1}^{\infty} a_n$.

DEFINITION. If the sequence $\{s_m\}$ converges to ℓ, we say the infinite series $\sum_{n=1}^{\infty} a_n$ converges to ℓ.

THEOREM. If $\sum_{n=1}^{\infty} a_n = a$ and $\sum_{n=1}^{\infty} b_n = b$, then $\sum_{n=1}^{\infty}(a_n+b_n) = a+b$ and $\sum_{n=1}^{\infty} ca_n = ca$.

Proof. The first part follows from part 1 of the Theorem of §**4** applied to the sequence of partial sums $\{s_m\}$ and $\{t_m\}$. For the second part let $\{t_m\}$ be the constant sequence, $t_m = c$ and use part 3 of the Theorem.

Two widely known geometric series are the decimal expansion $0.9999\cdots$ which is

$$\frac{9}{10}+\frac{9}{10^2}+\frac{9}{10^3}\cdots = \frac{9}{10}\times\sum_{n=0}^{\infty}\frac{1}{10}^{n} = \frac{9}{10}\times\frac{1}{1-1/10} = 1$$

and the sum $\dfrac{1}{2}+\dfrac{1}{4}+\dfrac{1}{8}+\cdots = 1$.

7. Series of nonnegative terms

If $a_n \geq 0$ we say $\sum_{n=0}^{\infty} a_n$ is a series of nonnegative terms. In this case the sequence of partial sums $\{s_m\}$ is nondecreasing. If $\{s_m\}$ is bounded above, then the series converges by §**5**.

For example consider the series $\sum_{n=1}^{\infty} \dfrac{1}{n(n+1)}$. Computing the first several partial sums suggests $s_m = 1-1/(m+1)$ which can be proved by induction. Hence $\{s_m\}$ is increasing and bounded above by 1. Thus the series converges. In fact we know that $\lim s_m = 1$, so the series converges to 1.

Now consider the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$. Explicit partials sums are not revealing. Comparing partials sums with the previous series, we have

$$1 + \frac{1}{2 \cdot 2} + \frac{1}{3 \cdot 3} + \cdots + \frac{1}{m \cdot m} < 1 + \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{(m-1) \cdot m} = 1 + 1 - \frac{1}{m} < 2.$$

Again this implies this sum converges to a limit between 1 and 2. The limit can be shown to be $\pi^2/6$.

For the series $\sum_{n=0}^{\infty} \frac{1}{n!} = 1 + 1 + \frac{1}{2} + \frac{1}{2 \cdot 3} + \cdots$, the nth term, $\frac{1}{n!} < \frac{1}{(n-1)n}$. Comparing terms we have

$$1 + 1 + \frac{1}{2!} + \cdots + \frac{1}{n!} \leq 1 + 1 + \frac{1}{1 \cdot 2} + \cdots + \frac{1}{(n-1)n} < 3.$$

Hence this series converges to a number, e, between 2 and 3.

To prove from the definition that a series converges we need to know the limit ℓ of the partial sums. Often the limit is not known but, if it exists, is some new object of mathematical interest about which we would like to know more. In that case we need some theoretical argument to show convergence. The result used above, called the comparison theorem, can be stated as follows:

THEOREM. If $a_n \geq 0$ and if $\exists\, b$ s.t. $\forall\, m$, $\sum_{n=1}^{m} a_n < b$, then $\sum_{n=1}^{\infty} a_n$ converges.

8. Limits of functions and continuity

DEFINITION. $\lim_{x \to a} f(x) = \ell$ means

$$(\forall\, \varepsilon > 0)(\exists\, \delta > 0)(0 < |x - a| < \delta \ \Rightarrow\ |f(x) - \ell| < \varepsilon).$$

The set $\{x : f \text{ is defined at } x\}$ is called the domain of f, written $\operatorname{dom} f$. If $x \notin \operatorname{dom} f$ then $|f(x) - \ell| < \varepsilon$ is false. Hence for $\lim_{x \to a} f(x)$ to exist, there must be some $\delta > 0$ with $\{x : 0 < |x - a| < \delta\} \subset \operatorname{dom} f$. The value of f at a, or even whether f is defined at a, does not matter.

UNIQUENESS OF LIMITS. If $\lim_{x \to a} f(x) = \ell$ and $\lim_{x \to a} f(x) = k$, then $k = \ell$.

Proof. If $k \neq \ell$, then let $2\varepsilon = |\ell - k| > 0$. Then

$$(\exists\, \delta > 0)(0 < |x - a| < \delta \Rightarrow |f(x) - 0\ell| < \varepsilon, |f(x) - k| < \varepsilon), \quad \text{hence}$$
$$2\varepsilon = |\ell - k| = |\ell - f(x) + f(x) - k| \leq |\ell - f(x)| + |f(x) - k| < 2\varepsilon,$$

a contradiction.. Hence $k = \ell$.

THEOREM. Let $\lim_{x\to a} f(x) = \ell$ and $\lim_{x\to a} g(x) = m$. Then:

(1) $\lim_{x\to a}(f(x) + g(x)) = \ell + m$.

Proof. Given $\varepsilon > 0$, $(\exists\,\delta_1 > 0)(0 < |x-a| < \delta_1 \;\Rightarrow\; |f(x) - \ell| < \varepsilon/2)$ and

$$(\exists\,\delta_2 > 0)(0 < |x-a| < \delta_2 \;\Rightarrow\; |g(x) - m| < \varepsilon/2).$$

Take $\delta = \min\{\delta_1, \delta_2\}$. Then $0 < |x-a| < \delta$ implies

$$|f(x) + g(x) - \ell - m| \le |f(x) - \ell| + |g(x) - m| < \varepsilon.$$

(2) $(\exists\,\delta > 0)(0 < |x-a| < \delta \;\Rightarrow\; |f(x)| < |\ell| + 1)$.

Proof. Let $\varepsilon = 1$ then $(\exists\,\delta > 0)(0 < |x-a| < \delta \;\Rightarrow\; |f(x) - \ell| < 1)$. Hence $|f(x)| = |\ell + f(s) - \ell| \le |\ell| + |f(x) - \ell| < |\ell| + 1$.

(3) $\lim_{x\to a}(f(x)g(x)) = \ell m$.

Proof. Taking the minimum of three values for δ we find $\delta > 0$ such that $0 < |x-a| < \delta$ implies $|f(x)| < |\ell|+1$, and $|g(x)-m| < \varepsilon/(2(|\ell|+1))$, and $|f(x)-\ell| < \varepsilon/(2(|m|+1))$. Then

$$\begin{aligned}
|f(x)g(x) - \ell m| &= |f(x)g(x) - f(x)m + f(x)m - \ell m| \\
&\le |f(x)|\,|g(x) - m| + |f(x) - \ell|\,|m| \\
&< (|\ell| + 1)\frac{\varepsilon}{2(|\ell| + 1)} + \frac{\varepsilon}{2(|m| + 1)}|m| \\
&< \varepsilon.
\end{aligned}$$

(4) If $\ell \ne 0$, $(\exists\,\delta > 0)(0 < |x-a| < \delta \;\Rightarrow\; |f(x)| > |\ell|/2)$.

Proof. Since $|\ell|/2 > 0$, $(\exists\,\delta > 0)(0 < |x-a| < \delta \;\Rightarrow |f(x) - \ell| < |\ell|/2)$. Then $|\ell| - |f(x)| \le |\ell - f(x)| < |\ell|/2$, so $|\ell|/2 < |f(x)|$.

(5) If $\ell \ne 0$, then $\lim_{x\to a} \dfrac{1}{f(x)} = \dfrac{1}{\ell}$.

Proof. Given $\varepsilon > 0$, take $\delta > 0$ so that $0 < |x-a| < \delta$ implies $|f(x)| > |\ell|/2$ and $|f(x) - \ell| < \varepsilon\ell^2/2$. Then $\left|\dfrac{1}{f(x)} - \dfrac{1}{\ell}\right| = \dfrac{|f(x) - \ell|}{|\ell|\,|f(x)|} < \varepsilon$.

DEFINITION. f is continuous at a if $\lim_{x\to a} f(x) = f(a)$

Hence f is continuous at a if $(\forall\,\varepsilon > 0)(\exists\,\delta > o)(|x-a| < \delta \;\Rightarrow\; |f(x) - f(a)| < \varepsilon)$.

Proposition. A constant function, $f(x) = c$, and the identity function, $f(x) = x$, are continuous at every $a \in R$.

Proof. For $f(x) = c$, given $\varepsilon > 0$, for any a and any x, $|f(x) - f(a)| = |c - c| = 0 < \varepsilon$. For $f(x) = x$ we may take $\delta = \varepsilon$ since then, if $|x - a| < \delta$ we have $|x - a| < \varepsilon$.

Corollary. Any polynomial function is continuous at all $a \in R$. Any ratio of polynomials is continuous at any a which is not a root of the denominator.

Proposition. If $\lim_{n\to\infty} s_n = \ell$ and f is continuous at ℓ, then $\lim_{n\to\infty} f(s_n) = f(\ell)$.

Proof. Given $\varepsilon > 0$, $(\exists\,\delta > 0)(|x - a| < \delta \;\Rightarrow\; |f(x) - f(\ell)| < \varepsilon)$ and $(\exists\,k)(n \geq k \;\Rightarrow\; |s_n - \ell| < \delta)$. Hence $n \geq k$ implies $|f(s_n) - f(\ell)| < \varepsilon$. Therefore $\lim_{n\to\infty} f(s_n) = f(\ell)$.

Proposition. If $f(a) = b$, f is continuous at a, and g is continuous at b, then $g \circ f$ is continuous at a.

Proof. Since g is continuous at b, given $\varepsilon > 0$, $\exists\,\delta > 0$ such that $|y - b| < \delta \;\Rightarrow\; |g(y) - g(b)| < \varepsilon$. Then since f is continuous at a and $\delta > 0$, $\exists\,\gamma > 0$ such that $|x - a| < \gamma \;\Rightarrow\; |f(x) - f(a)| < \delta$.

Hence $|g(f(x)) - g(f(a))| < \varepsilon$ provided $|x - a| < \gamma$ and therefore $g \circ f$ is continuous at a.

9. Functions continuous on an interval

Intermediate Value Theorem. If f is continuous on $[a, b]$ and $f(a)f(b) < 0$, then there exists $z \in [a, b]$ such that $f(z) = 0$.

Proof. Assume that $f(a) < 0$ and $f(b) > 0$. (In the opposite case, replace f by $-f$.) Let $I_0 = [a, b]$. We construct recursively a nested sequence of nonempty, bounded, closed intervals. If $I_n = [a_n, b_n]$ has been constructed with $f(a_n) < 0$ and $f(b_n) > 0$, let $x = (b_n - a_n)/2$ be its midpoint. Define I_{n+1} by

$$a_{n+1} = a_n,\; b_{n+1} = x \text{ if } f(x) > 0,$$
$$a_{n+1} = x,\; b_{n+1} = b_n \text{ if } f(x) < 0.$$

(In the case $f(x) = 0$, we may take $z = x$ and the proof is complete.) By the Cantor intersection theorem there exists $z \in \bigcap_{n=1}^{\infty} I_n$.

We claim that $f(z) = 0$. If $f(z) > 0$, then there is a $\delta > 0$ such that

$0 < |x - z| < \delta \;\Rightarrow\; -f(z) < f(x) - f(z) < f(z)$, and therefore $f(x) > 0$.

Further there exists n such that $b_n - a_n = (b - a)/2^n < \delta$. Now $a_n \leq z \leq b_n$, so $0 \leq z - a_n \leq b_n - a_n \leq \delta$, so $f(a_n) > 0$ which contradicts the construction of a_n.

Similarly, $f(z) < 0$ leads to a contradiction. Therefore $f(z) = 0$.

Example. Given $c > 0$, let $f(x) = x^2 - c$ and take $b = \max\{c, 2\}$. Then $f(0) = -c < 0$ and, in either case, $f(b) > 2 > 0$. By the intermediate value theorem, there exists $z > 0$ with $z^2 = c$.

Definition. A function f is *bounded above* on $[a, b]$ is there exists k such that $a \leq x \leq b \;\Rightarrow\; f(x) \leq k$.

Boundedness Theorem. If f is continuous on $[a, b]$, then f is bounded above on $[a, b]$.

Lemma. Suppose $a < c < b$. If f is bounded above on $[a, c]$ and on $[c, b]$, then f is bounded above on $[a, b]$.

Proof. By hypothesis, $(\exists\, k_1)(a \le x \le c \;\Rightarrow\; f(x) \le k_1)$ and $(\exists\, k_2)(c \le x \le b \;\Rightarrow\; f(x) \le k_2)$. Then if $a \le x \le b$, either $x \le c$ and $f(x) \le k_1$ or $c \le x$ and $f(x) \le k_2$. Therefore $f(x) \le \max\{k_1, k_2\}$.

Proof of theorem. Say f is not bounded on $I_0 = [a, b]$. We construct recursively a nested sequence of nonempty, bounded, closed intervals on which f is unbounded. If $I_n = [a_n, b_n]$ has been constructed and f is unbounded on I_n, let $x = (b_n - a_n)/2$ be its midpoint. By the lemma, f is unbounded on at least one of the intervals $[a, x]$ or $[x, b]$. Let I_{n+1} be one of these on which f is unbounded.

By the Cantor intersection theorem there exists $z \in \bigcap_{n=1}^{\infty} I_n$. Since f is continuous at z, there exists $\delta > 0$ such that $|x - z| < \delta \;\Rightarrow\; f(x) \le f(z) + 1$. But there exists n such that $b_n - a_n = (b - a)/2^n < \delta$ and $a_n \le z \le b_n$. If $a_n \le x \le b_n$, then $|x - z| \le b_n - a_n < \delta$ so $f(x) \le f(z) + 1$. Hence f is bounded above on I_n, contradicting the construction. Therefore f is bounded above on $[a, b]$.

Extreme Value Theorem. If f is continuous on $[a, b]$, then there exist numbers m, M, c_1, c_2 such that

$$\begin{gathered} m \le f(x) \le M \quad \text{for} \quad x \in [a, b], \\ f(c_1) = m, \quad f(c_2) = M, \quad \text{where} \quad c_1, c_2 \in [a, b]. \end{gathered}$$

Proof. We prove this for M and c_2 and apply this result to $-f$ to get m and c_1. By the previous theorem the set $\{f(x) : x \in [a, b]\}$ is bounded. Hence there is a least upper bound M. Suppose for all $x \in [a, b]$, $f(x) \neq M$. Then $f(x) < M$ and hence the function

$$g(x) = \frac{1}{M - f(x)} \text{ for } x \in [a, b]$$

is continuous. Again by the previous theorem, there is a bound k such that

$$0 < \frac{1}{M - f(x)} \le k \text{ and hence } M - f(x) \ge \frac{1}{k} \text{ or } f(x) \le M - \frac{1}{k} < M.$$

This contradicts the fact that M is the least upper bound. Therefore there is a c_2 with $f(c_2) = M$.

10. Inverse functions

Let A and B be subsets of R and let f be a function with domain A which takes values in B, $f(x) \in B$. We write $f : A \longrightarrow B$ and call f a function, or a *map*, from A to B.

Definitions. The *image* of f, $\operatorname{im} f = f(A) = \{y : \exists\, x \in A, f(x) = y\}$;

f is *surjective* or *onto* B if $\operatorname{im} f = B$;
f is *injective* or *one-to-one* if $f(u) = f(x) \ \Rightarrow \ u = x$;
The *identity* map $1_A : a \longrightarrow A$ is defined by $1_A(x) = x$ for all $x \in A$;
A map $g : B \longrightarrow A$ is an *inverse* of f if $g \circ f = 1_A$ and $f \circ g = 1_B$.

Notice that $f(x) = y$ if and only if $g(y) = x$. If $f : A \longrightarrow B$ has an inverse, the inverse is unique, and if $g : B \longrightarrow A$ is an inverse of f, then f is an inverse of g.

Proposition. If $(g \circ f) = 1_A$, then f is one-to-one and g is onto.

Proof. If $f(u) = f(x)$, then $u = (g \circ f)(u) = g(f(u)) = g(f(x)) = (g \circ f)(x) = x$, so f is one-to-one. If $x \in A$, let $y = f(x)$. Then $g(y) = g(f(x)) = x$, so g is onto.

Proposition. $f : A \longrightarrow B$ has an inverse if and only if f is one-to-one and onto.

Proof. If f has an inverse, g, then $(g \circ f) = 1_A$, so f is one-to-one. Also $f \circ g = 1_B$, so f is onto by the previous proposition. Conversely, given $y \in B$, if f is onto, $\exists\, x \in A$ with $f(x) = y$. Since f is one-to-one, this x is unique. Thus g is well-defined if we set $g(y) = x$.

Our goal is to prove that if f is continuous and has a inverse, then the inverse is continuous.

A more general type of interval than the nonempty, closed, bounded intervals used so far is given by the following

Definition. An *interval* is a set I such that, if $a,\ b \in I$ and $a \leq x \leq b$, then $x \in I$.

Theorem. If f is defined and continuous on an interval I and f is one-to-one, then f is either increasing or f is decreasing on I.

Proof. The proof is an application of the intermediate value theorem. Let $a, b, c, d \in I$, $a < b$, $c < d$, and suppose that $f(a) < f(b)$. (If not, replace f by $-f$.) We must prove that $f(c) < f(d)$. We define a family of intervals $[x_t, y_t]$ for $0 \leq t \leq 1$ with $x_0 = a,\ y_0 = b$ and $x_1 = c,\ y_1 = d$.

Set $x_t = (1-t)a + tc$ and $y_t = (1-t)b + td$. Note that $y_t - x_t = (1-t)(b-a) + t(d-c) > 0$ for $0 \leq t \leq 1$, so $x_t < y_t$.

Define $h : [0,1] \longrightarrow R$ by $h(t) = f(x_t) - f(y_t)$; h is a continuous function of t since it is a composition of continuous functions.

Since f is one-to-one, $h(t) \neq 0$. Now $h(0) = f(a) - f(b) < 0$. Then, by the intermediate value theorem applied to h on the interval $[0,1]$, we must have $f(c) < f(d)$, hence f is increasing.

Lemma. If f is continuous on an interval I, then $J = f(I)$ is an interval.

Proof. Given $u,\ v \in J$, $\exists\, a,\ b \in I$ such that $f(u) = a$, $f(b) = v$. Since I is an interval, $[a,b] \subset I$ so f is defined on $[a,b]$. If $y \in [u,v]$, then $f(a) \leq y \leq f(b)$, so by the intermediate value theorem $\exists\, x \in [a,b]$ with $f(x) = y$. Therefore $y \in J$ and hence J is an interval.

Lemma. If $f : I \longrightarrow J$ and $g : J \longrightarrow I$ are inverse functions and f is increasing, then g is also increasing.

Proof. Given $u, v \in J$ with $u < v$, let $g(u) = a,\ g(v) = b$. If $a \geq b$ then $u = f(a) \geq f(b) = v$, a contradiction, hence $a < b$ and g is increasing.

THEOREM. If $f : I \longrightarrow J$ and $g : J \longrightarrow I$ are inverse functions and f is continuous, then g is also continuous.

Proof. Assume that f is increasing (and apply this result to $-f$ if f is decreasing). Given $y \in J$ we must show the $\lim\limits_{v \to y} g(v) = g(y)$. Given $\varepsilon > 0$ we need $\delta > 0$ such that

$$|v - y| < \delta \;\Rightarrow\; |g(v) - g(y)| < \varepsilon,$$

or equivalently

$$y - \delta < v < y + \delta \;\Rightarrow\; g(y) - \varepsilon < g(v) < g(y) + \varepsilon.$$

Let $y = f(x)$ and $v = f(u)$. Since $x - \varepsilon < x < x + \varepsilon$ and f is increasing we have

$$f(x - \varepsilon) < f(x) < f(x + \varepsilon).$$

Let $\delta = \min\{f(x + \varepsilon) - f(x),\, f(x) - f(x - \varepsilon)\}$. If $y - \delta < v < y + \delta$, then

$$f(x-\varepsilon) = y - \big(f(x) - f(x-\varepsilon)\big) \le y - \delta < v < y + \delta \le y + \big(f(x+\varepsilon) - f(x)\big) = f(x+\varepsilon).$$

Since g is increasing and $x = g(y)$ we have

$$g(y) - \varepsilon < g(v) < g(y) + \varepsilon.$$

EXAMPLES. The power functions $f_n(x) = x^n$, $n > 0$, are increasing on R if n is odd and on $[0, \infty)$ if n is even. The inverse functions are $\sqrt[n]{x}$. The compositions $x^{m/n}$ are continuous. For nonrational exponents a further limit argument, or the definition $x^a = \exp(a \ln x)$ for $x > 0$ is needed.

Let $f(x) = \sqrt{3 + x}$. The sequence defined in §**5** by $s_1 = \sqrt{3}$ and $s_{n+1} = f(s_n)$ was shown there to converge to a real number $\ell \in [1, 3]$. Since f is continuous on $[0, \infty)$, $f(\ell) = f(\lim s_n) = \lim f(s_n) = \lim s_{n+1} = \ell$. By the quadratic formula, $\ell = (1 + \sqrt{13})/2$.

11. Differentiation

DEFINITION. The *derivative* of f at a is

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}.$$

If this limit exists, f is said to be *differentiable* at a.

EXAMPLES. If $f(x) = c$ for some real number c, then $f'(a) = 0$ for any a.

Recall the finite geometric series from §**6.** In

$$(x - 1)(1 + x + x^2 + \cdots + x^{n-1}) = x^n - 1,$$

replace x with x/a and multiply by a^n to get

$$(x - a)(a^{n-1} + a^{n-2}x + \cdots + x^{n-1}) = x^n - a^n. \qquad (*)$$

Set

$$\varphi_a(x) = \sum_{i=0}^{n-1} a^{n-1-i} x^i.$$

Then φ_a is continuous at a (and for all x). If $x \neq a$, then $\varphi_a(x) = (x^n - a^n)/(x-a)$. Hence $\lim_{x\to a} \dfrac{x^n - a^n}{x-a} = \lim_{x\to a} \varphi_a(x) = na^{n-1}$. Thus if $f(x) = x^n$, $f'(a) = na^{n-1}$.

Next divide equation $(*)$ by $a^n x^n$ for $a, x \neq 0$ to get

$$(x-a)(a^{-1}x^{-n} + a^{-2}x^{-n+1} + \cdots + a^{-n}x^{-1}) = a^{-n} - x^{-n}.$$

Thus for $a, x \neq 0$,

$$(x^{-n} - a^{-n})/(x-a) = -(ax)^{-1}\varphi_{a^{-1}}(x^{-1}) = \psi_a(x)$$

and ψ_a is continuous at $x = a \neq 0$. Thus if $f(x) = x^{-n}$, $f'(a) = -na^{-n-1}$.

Finally, replace x by $x^{1/n}$ and a by $a^{1/n}$ in equation $(*)$ to find

$$\frac{x^{1/n} - a^{1/n}}{x-a} = \frac{1}{\varphi_{a^{1/n}}(x^{1/n})}$$

and deduce that the derivative of $f(x) = x^{1/n} = \sqrt[n]{x}$ is $f'(a) = \frac{1}{n}a^{\frac{1}{n}-1}$.

In the three cases above we introduced functions which were equal to the difference quotient for $x \neq a$ and which were continuous at a. Constantin Carathéodory gave an equivalent reformulation of the definition of the derivative in terms of such a function. His definition leads to shorter proofs and makes the use of continuity more explicit.

Carathéodory's Theorem. f is differentiable at a if and only if there is a function $\varphi_a(x)$ which is continuous at a such that

$$f(x) = f(a) + (x-a)\varphi_a(x).$$

If f is differentiable, then $f'(a) = \varphi_a(a)$.

Proof. Assume such a function $\varphi_a(x)$ exists. Then $(f(x) - f(a))/(x-a) = \varphi_a(x)$ for $x \neq a$ and $\lim_{x\to a}(f(x) - f(a))/(x-a) = \lim_{x\to a} \varphi_a(x) = \varphi_a(a)$.

Conversely, assume f is differentiable at a and $f'(a) = \ell$ Define

$$\varphi_a(x) = \begin{cases} (f(x) - f(a))/(x-a) & \text{if } x \neq a, \\ \ell & \text{if } x = a. \end{cases}$$

Then $\lim_{x\to a} \varphi_a(x) = \lim_{x\to a}(f(x) - f(a))/(x-a) = \ell$, so $\varphi_a(x)$ is continuous at a.

Carathéodory's definition is the subject of an article by Stephen Huhn in the *American Mathematical Monthly* 98 (Jan. 1991) pp. 40–44.

Theorem. If f is differentiable at a then f is continuous at a.

Proof. By Carathéodory, $f(x) = f(a) + (x - a)\varphi_a(x)$ where $\varphi_a(x)$ is continuous at a. Then by the results of §**8**, f is continuous at a.

Sum rule. If f and g are differentiable at a, then $f + g$ is differentiable at a and

$$(f + g)'(a) = f'(a) + g'(a).$$

Proof. We have $f(x) = f(a) + (x - a)\varphi_a(x)$ and $g(x) = g(a) + (x - a)\psi_a(x)$, hence $(f + g)(x) = (f + g)(a) + (x - a)(\varphi_a(x) + \psi_a(x))$. By §**8**, $\varphi_a + \psi_a$ is continuous at a and $(\varphi_a + \psi_a)(a) = \varphi_a(a) + \psi_a(a)$.

Product rule. If f and g are differentiable at a, then fg is differentiable at a and

$$(fg)'(a) = f'(a)g(a) + f(a)g'(a).$$

Proof. As above,

$$\begin{aligned}(fg)(x) &= \big(f(a) + (x - a)\varphi_a(x)\big)\big(g(a) + (x - a)\psi_a(x)\big)\\ &= (fg)(a) + (x - a)\big(\varphi_a(x)g(a) + f(a)\psi_a(x) + (x - a)\varphi_a(x)\psi_a(x)\big)\end{aligned}$$

where $\varphi_a(x)g(a) + f(a)\psi_a(x) + (x - a)\varphi_a(x)\psi_a(x)$ is continuous at a and at $x = a$ is equal to $f'(a)g(a) + f(a)g'(a)$.

Corollary (Linearity). $(kf + \ell g)' = kf' + \ell g'$ for $k, \ell \in R$.

Chain rule. If f is differentiable at a, $f(a) = b$, and g is differentiable at b, then the composition $g \circ f$ is differentiable at a and $(g \circ f)'(a) = g'(b)f'(a)$.

Proof.

$$\begin{aligned}(g \circ f)(x) = g(f(x)) &= g(b) + (f(x) - b)\psi_b(f(x))\\ &= g(b) + \big(f(a) + (x - a)\varphi_a(x) - b\big)\psi_b(f(x))\\ &= g(b) + (x - a)\varphi_a(x)\psi_b(f(x)).\end{aligned}$$

Since f is continuous at a, $\psi_b \circ f$ is continuous at a so $(\psi_b \circ f)\varphi_a$ is continuous at a and $\psi_b(f(a))\varphi_a(a) = g'(b)f'(a)$.

Proposition. Let f be differentiable at a and $f(a) \neq 0$. Then

$$(1/f)'(a) = -f'(a)/(f(a))^2.$$

Proof. If $g(y) = y^{-1}$, g is differentiable at $b = f(a)$ and $g'(b) = -b^{-2}$. Then $(g \circ f)(x) = 1/f(x)$ and $(g \circ f)'(a) = -(f(a))^{-2}f'(a)$.

Quotient rule. Let f and g be differentiable at a with $g(a) \neq 0$. Then

$$\left(\frac{f}{g}\right)'(a) = \frac{f'(a)g(a) - f(a)g'(a)}{g(a)^2}.$$

Proof. Write $\left(\dfrac{f}{g}\right)(x)$ as the product $f(x) \cdot (1/g(x))$ and use the previous proposition and the product rule.

Inverse Function Theorem. If $f : I \longrightarrow J$ and $g : J \longrightarrow I$ are inverses and f is differentiable at $a \in I$ with $f'(a) \neq 0$, then g is differentiable at $b = f(a)$ and $g'(b) = 1/f'(g(b))$.

Proof. We have $f(x) = f(a)+(x-a)\varphi_a(x)$ where φ_a is continuous at a and $f'(a) = \varphi_a(a)$. Since g is continuous at b and φ_a is continuous at $g(b)$, $\varphi_a \circ g$ is continuous at b and $\varphi_a(g(b)) = f'(a) \neq 0$. Hence there exists $\varepsilon > 0$ such that when $|y - b| < \varepsilon$, $\varphi_a(g(y)) \neq 0$. Let $y = f(x)$ and $x = g(y)$. Then

$$y = b + (g(y) - g(b))\varphi_a(x), \quad \text{hence} \quad g(y) = g(b) + (y - b)/\varphi_a(g(y)).$$

Therefore $g'(b) = 1/f'(g(b))$.

Theorem. If $f'(a) > 0$, then there is a $\delta > 0$ such that

$$\begin{aligned} a - \delta < x < a \ &\Rightarrow \ f(a) < f(x) \\ a < x < a + \delta \ &\Rightarrow \ f(x) < f(a). \end{aligned}$$

Proof. Again $f(x) = f(a)+(x-a)\varphi_a(x)$, φ_a is continuous at a, and $\varphi_a(a) > 0$. Therefore there is a $\delta > 0$ such that $|x - a| < \delta \ \Rightarrow \ \varphi_a(x) > 0$. If $a < x < a + \delta$ then $x - a > 0$ and $f(a) < f(a) + (x - a)\varphi_a(x) = f(x)$, while if $a - \delta < x < a$ then $x - a < 0$ and $f(a) > f(a) + (x - a)\varphi_a(x) = f(x)$.

Definition. A *critical point* for f is a number c such that $f'(c) = 0$.

The critical point test for extreme values is the following:

Corollary. Let f be defined on (a, b), $a < c < b$, and f be differentiable at c. If $f(x) \leq f(c)$ for all $x \in (a, b)$ or if $f(x) \geq f(c)$ for all $x \in (a, b)$, then $f'(c) = 0$.

Thus if f is defined on $[a, b]$ and $c \in [a, b]$ is an extreme point of f, then c is either an end point of $[a, b]$, a point where f is *not* differentiable, or a critical point of f.

Proof. By contradiction, applying the theorem to either f or $-f$.

The conclusion of the previous theorem is sometimes described by saying that f is *increasing at a*. This does not imply that f is increasing on any interval. An example of a function which is increasing at 0, but not in any interval $(0, \delta)$ or $(-\delta, 0)$ for any $\delta > 0$ is

$$f(x) = \begin{cases} 0, & \text{if } x = 0; \\ x/2 + x^2 \sin(1/x), & \text{otherwise.} \end{cases}$$

This function is differentiable at 0 and in fact on R, but f' is *not* continuous at 0.

12. Functions differentiable on an interval

Rolle's Theorem. If f is continuous on $[a,b]$ and differentiable on (a,b), and if $f(a) = f(b)$, then $\exists\,\xi \in (a,b)$ such that $f'(\xi) = 0$.

Proof. Since f is continuous on $[a,b]$, f has an extreme value on $[a,b]$ by (§**9**). If a maximum point or a minimum point occurs at $\xi \in (a,b)$, then by the critical point test, $f'(\xi) = 0$. If both the maximum and minimum values occur at the end points then, since $f(a) = f(b)$, f is constant on $[a,b]$ and $f'(\xi) = 0$ for any $\xi \in (a,b)$.

Mean Value Theorem. If f is continuous on $[a,b]$ and differentiable on (a,b), then $\exists\,\xi \in (a,b)$ such that $(f(b) - f(a))/(b-a) = f'(\xi)$.

Proof. Let $h(x) = f(x) - \dfrac{f(b)-f(a)}{b-a}(x-a)$. Then $h(a) = f(a)$ and $h(b) = f(a)$, so h satisfies the hypotheses of Rolle's theorem. Therefore $\exists\,\xi \in (a,b)$ such that $h'(\xi) = 0$. But $h'(\xi) = f'(\xi) - (f(b) - f(a))/(b-a)$.

Corollary 1. If f is defined on an interval I and $f'(x) = 0$ for all $x \in I$, then f is constant on I.

Proof. Take $a, b \in I$ and apply the mean value theorem.

Corollary 2. If $f'(x) = g'(x)$ on I then there is a constant c such that $g(x) = f(x) + c$.

Proof. Apply Corollary 1 to $f - g$.

Corollary 3. If $f'(x) > 0$ on I then f is strictly increasing on I. If $f'(x) < 0$ on I then f is strictly decreasing on I.

Proof. Let $a, b \in I$ with $a < b$. By the mean value theorem, there exists $\xi \in (a,b)$ with $(f(b) - f(a))/(b-a) = f'(\xi) > 0$, hence $f(b) > f(a)$. For the second part, apply the first part to $-f$.

Corollary 4. If f and g are continuous on $[a,b]$, $f'(x) \geq g'(x)$ on (a,b), and $f(a) \geq g(a)$, then $f(x) \geq g(x)$ for all $x \in [a,b]$.

Proof. For $x = a$ the conclusion is part of the hypothesis. For $x \in (a,b]$ there is a $\xi \in (a,x)$ such that $f(x) - g(x) \geq (f-g)(x) - (f-g)(a) = (f-g)'(\xi) \cdot (x-a) \geq 0$.

Corollary 5. If $f'(a) = 0$ and $f''(a) > 0$, then $\exists\,\delta > 0$ such that for $x \in (a-\delta, a+\delta)$ and $x \neq a$, $f(x) < f(a)$.

Proof. The existence of $f''(a)$ implies that there is a $\delta > 0$ such that $f'(x)$ exists for $|x - a| < \delta$ and, by §**10**, that

$$a - \delta < x < a \;\Rightarrow\; 0 < f'(x) \quad\text{and}\quad a < x < a + \delta \;\Rightarrow\; f'(x) < 0.$$

By Corollary 3, f is increasing on $(a-\delta, a)$ and decreasing on $(a, a+\delta)$, and therefore $f(x) < f(a)$ for $0 < |x - a| < \delta$.

Cauchy Mean Value Theorem. Let f and g be continuous on $[a, b]$ and differentiable on (a, b). Then $\exists\, \xi \in (a, b)$ such that

$$(f(b) - f(a))g'(\xi) = (g(b) - g(a))f'(\xi).$$

Note. If $g'(\xi) \neq 0$ and $g(b) \neq g(a)$ we have

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(\xi)}{g'(\xi)}.$$

Proof. Let $h(x) = f(x)(g(b) - g(a)) - g(x)(f(b) - f(a))$. Then h is continuous on $[a, b]$, differentiable on (a, b), and $h(a) = h(b)$, so Rolle's theorem implies the result.

l'Hôpital's Rule. Assume $\exists\, \delta > 0$ such that when $0 < |x - a| < \delta$, $f'(x)$ exists, $g'(x)$ exists, and $g'(x) \neq 0$. If

$$\lim_{x \to a} f(x) = 0, \qquad \lim_{x \to a} g(x) = 0, \qquad \text{and} \tag{1}$$

$$\lim_{x \to a} f'(x)/g'(x) = \ell. \tag{2}$$

Then

$$\lim_{x \to a} f(x)/g(x) = \ell.$$

Proof. The first three assumptions are implicit in condition (2). If f or g is not defined or is not continuous at a we may redefine them at a by setting $f(a) = 0$ and $g(a) = 0$. Then f and g are continuous on $(a - \delta, a + \delta)$ by (1) and none of the limits as x goes to a are changed. Further $g(x) \neq 0$ for $|x - a| < \delta$, since, if $g(x) = 0$, then by the MVT $\exists\, \xi$ such that $|\xi - a| < |x - a| < \delta$ and $g'(\xi) = 0$, a contradiction.

Given $\varepsilon > 0$, by (2), $\exists\, \delta_1,\ 0 < \delta_1 \le \delta$ such that

$$\forall\, \xi,\ 0 < |\xi - a| < \delta_1 \ \Rightarrow\ \left| \frac{f'(\xi)}{g'(\xi)} - \ell \right| < \varepsilon.$$

If $0 < |x - a| < \delta_1$, by the Cauchy MVT, $\exists\, \xi$ such that $|\xi - a| < |x - a| < \delta_1$ and

$$\frac{f(x)}{g(x)} = \frac{f'(\xi)}{g'(\xi)}, \quad \text{therefore} \quad \left| \frac{f(x)}{g(x)} - \ell \right| < \varepsilon.$$

Hence $\lim_{x \to a} f(x)/g(x) = \ell$.

13. Integration

Recall from §**4** that if A is a nonempty set that is bounded above, then A has a least upper bound. This bound is also called the supremum of A or $\sup A$. By way of review, we discuss the greatest lower bound in detail. This requires essentially the same argument as for the least upper bound, but it was not given in §**4**.

Let B be a nonempty set that is bounded below. Let $A \subset R$ be the set of all lower bounds for B, $A = \{a \ : \ b \in B \ \Rightarrow \ a \leq b\}$. Since we assumed that B is bounded below, A is not empty. By the completeness axiom, $\exists\, z \in R$ such that:

$$(1) \ \ a \in A \ \Rightarrow \ a \leq z \quad \text{ and } \quad (2) \ \ b \in B \ \Rightarrow \ z \leq b.$$

By (2) and the definition of A, $z \in A$ and by (1), z is a greatest lower bound for B.

If z_1 were another greatest lower bound for B, then $z, z_1 \in A$. This implies $z \leq z_1$ and $z_1 \leq z$ and therefore $z_1 = z$, so the greatest lower bound is unique. It is called the infimum of A or $\inf A$. Note that $\inf A \leq \sup A$.

Definition. A partition P of the interval $[a, b]$ is a finite set of points, $P = \{x_0, x_1, \ldots, x_n\}$ where the $x_i \in R$ are subscripted so that

$$a = x_0 < x_1 < \ldots < x_{i-1} < x_i < \ldots < x_n = b.$$

Definition. The function f is bounded on $[a, b]$ if

$$\exists\, m, M \in R \text{ s.t. } x \in [a, b] \ \Rightarrow \ m \leq f(x) \leq M.$$

Thus the set of values of f on $[a, b]$ is bounded.

If f is bounded on $[a, b]$ it will be bounded on any subset of $[a, b]$.

Definition. If f is bounded on $[a, b]$, the *upper* and *lower sums* of f with respect to a partition of $[a, b]$ are defined by

$$\begin{aligned} U(f, P) &= \sum_{i=1}^{n} M_i(x_i - x_{i-1}), \\ L(f, P) &= \sum_{i=1}^{n} m_i(x_i - x_{i-1}), && \text{where} \\ m_i &= \inf\{f(x) : x \in [x_{i-1}, x_i]\}, && \text{and} \\ M_i &= \sup\{f(x) : x \in [x_{i-1}, x_i]\}, && \text{for } 1 \leq i \leq n. \end{aligned}$$

Lemma 1. $L(f, P) \leq U(f, P)$.

Proof. This follows from $m_i \leq M_i$.

Lemma 2. If the partition Q contains one more point than the partition P, then

$$L(f, P) \leq L(f, Q) \leq U(f, Q) \leq U(f, L).$$

Proof. Say the extra point is u and $t_{i-1} < u < t_i$. Then if

$$\begin{aligned} m_i' &= \inf\{f(x) : x \in [x_{i-1}, u]\} \le m_i && \text{and} \\ m_i'' &= \inf\{f(x) : x \in [u, x_i]\} \le m_i && \text{we have} \\ m_i'(u - x_{i-1}) &+ m_i''(x_i - u) \le m_i(x_i - x_{i-1}). \end{aligned}$$

This with Lemma 1 gives the result first inequality. The second is Lemma 1. The third is analogous to the first, replacing m by M and reversing the inequalities.

We say the partition Q is a refinement of P if Q is obtained from P by inserting a finite number of points. As finite sets, $P \subset Q$. A finite number of applications of lemma 2 shows that $L(f, P) \le L(f, Q) \le U(f, Q) \le U(f, P)$. If P and Q are each partitions of $[a, b]$ then $P \cup Q$ contains any point in either P or Q and is a common refinement of both.

Lemma 3. If P and Q are partitions of $[a, b]$, then $L(f, P) \le U(f, Q)$.

Proof. $L(f, P) \le L(f, P \cup Q) \le U(f, P \cup Q) \le U(f, Q)$.

Fixing an interval $[a, b]$ and a function f bounded on $[a, b]$, Lemma 3 says that any upper sum $U(f, Q)$ is a an upper bound for the set of all lower sums, so

$$\sup\{L(f, P) : P \text{ a partition of } [a, b]\} \le U(f, Q).$$

Now $\sup\{L(f, P)$ is a lower bound for the set of all upper sums. This proves the

Theorem 1. $\sup\{L(f, P)\} \le \inf\{U(f, P)\}$.

Definition. Let f be bounded on $[a, b]$, then f is *integrable* on $[a, b]$ if

$$\sup\{L(f, P)\} = \inf\{U(f, P)\},$$

and this number is, by definition, the *integral* of f over $[a, b]$, written $\displaystyle\int_a^b f(x)\,dx$.

The integral defined this way is called the Darboux integral.

Examples.

(a) If $f(x) = c$, then $\int_a^b f(x)\,dx = c(b - a)$.

(b) If $f(x) = \begin{cases} 0, & x \text{ irrational,} \\ 1, & x \text{ rational;} \end{cases}$ then f is not integrable.

Theorem 2. Let f be bounded on $[a, b]$, then f is integrable if and only if for all $\varepsilon > 0$ there is a partition P of $[a, b]$ such that $U(f, P) - L(f, P) < \varepsilon$.

Proof. Let P be such a partition for a given $\varepsilon > 0$. Since $L(f, P) \le \sup\{L(f, Q)\} \le \inf\{U(f, Q)\} \le U(f, P)$, we have $\sup\{L(f, Q)\} - \inf\{U(f, Q)\} < \varepsilon$. Since this is true for any $\varepsilon > 0$, $\sup\{L(f, Q)\} = \inf\{U(f, Q)\}$.

If f is integrable, then $\sup\{L(f, Q)\} = \inf\{U(f, Q)\}$, so for any $\varepsilon > 0$ there are partitions P' and P'' such that $\sup\{L(f, Q)\} - L(f, P') < \varepsilon/2$ and $U(f, P'') - \sup\{L(f, Q)\} < \varepsilon/2$. Hence $U(f, P'') - L(f, P') < \varepsilon$. Then, taking $P = P' \cup P''$, we have $U(f, P) - L(f, P) \le U(f, P'') - L(f, P') < \varepsilon$.

14. Integrability

THEOREM 1. If the function f is bounded and monotone on the interval $[a, b]$, then f is integrable on $[a, b]$.

Proof. Suppose f is nondecreasing on $[a, b]$, that is, if $a \le u \le v \le b$ then $f(u) \le f(v)$. Given $\varepsilon > 0$ choose $n > (f(b)-f(a))(b-a)/\varepsilon$ and let P_n be the partition of $[a, b]$ with division points $x_i = a + i(b-a)/n$ for $i = 0 \ldots n$. If $x_{i-1} \le x \le x_i$, then $f(x_{i-1}) \le f(x) \le f(x_i)$, hence $m_i = f(x_{i-1})$ and $M_i = f(x_i)$. Then

$$U(f, P_n) - L(f, P_n) = \Sigma_{i=1}^n (f(x_i) - f(x_{i-1}))(b-a)/n = (f(b) - f(a))(b-a)/n < \varepsilon.$$

By Theorem 13.2, this implies that f is integrable.

EXAMPLES.

(1) The step function:

$$f(x) = \begin{cases} 0, & \text{if } x < 0; \\ 1, & \text{if } x \ge 0. \end{cases}$$

(2) The function $f(t) = 1/t$ is decreasing for $t > 0$ so f is integrable on $[1, x]$ or $[x, 1]$ for $x > 0$ and we can define

$$\log x = \int_1^x \frac{1}{t}\, dt \quad \text{for} \quad 1 \le x \quad \text{or} \qquad -\int_x^1 \frac{1}{t}\, dt \quad \text{for} \quad 0 < x \le 1.$$

THEOREM 2. If the function f is continuous on the interval $[a, b]$, then f is integrable on $[a, b]$.

Proof. Since f is continuous on $[a, b]$, f is bounded and, for any partition P, the upper and lower sums are defined. Given $\varepsilon > 0$ we must show there is a partition P with $U(f, P) - L(f, P) < \varepsilon$. If for each interval in the partition we have

$$M_i - m_i < \mu = \varepsilon/(b-a),$$

then

$$U(f, P) - L(f, P) = \Sigma_{i=1}^n (M_i - m_i)(x_i - x_{i-1}) < \mu(b-a) = \varepsilon.$$

Suppose $[a, b]$ is divided into two intervals, $[a, c]$ and $[c, b]$ and that P_1 is a partition of $[a, c]$ and P_2 is a partition of $[c, b]$. If $M_i - m_i < \mu$ for intervals in the partition P_1 and for intervals in the partition P_2, then this inequality will hold for intervals in the partition $P_1 \cup P_2$ of $[a, b]$.

If we assume there is no partition of $[a, b]$ for which this inequality is true for each interval, then either there is no such partition of $[a, c]$ or else there is no such partition of $[c, b]$. Taking $c = (a+b)/2$ we get an interval $[a_1, b_1]$ equal to either $[a, c]$ or $[c, b]$ for which there is no such partition. Inductively we get a nested sequence of closed intervals

$$[a, b] \supset [a_1, b_1] \supset [a_2, b_2] \supset \cdots$$

such that for each $[a_k, b_k]$ there is no partition for which the inequality holds. Further the lengths of the intervals tend to zero since $b_k - a_k = 2^{-k}(b-a)$.

The intersection of a nested sequence of closed intervals is nonempty; let c be a point in the intersection. Since f is continuous at c there is a δ such that $|f(x) - f(c)| < \mu/2$ for any x with $|x - c| < \delta$. If $b_k - a_k < \delta$ then $c \in [a_k, b_k]$ and, if $x \in [a_k, b_k]$ then $|x - c| < \delta$, hence $|f(x) - f(c)| < \mu/2$.

Since f is continuous on $[a_k, b_k]$, $m = \inf\{f(u) : a_k \leq u \leq b_k\} = f(x_1)$ and $M = \sup\{f(u) : a_k \leq u \leq b_k\} = f(x_2)$ for some $x_1, x_2 \in [a_k, b_k]$. Then

$$M - m = f(x_2) - f(x_1) \leq |f(x_2) - f(c)| + |f(c) - f(x_1)| < \mu.$$

If we take the partition $P = \{a_k, b_k\}$ of $[a_k, b_k]$, then P satisfies the inequality $M - m < \mu$ for the only interval in P which contradicts the way $[a_k, b_k]$ was chosen. Therefore there is a partition of $[a, b]$ with $M_i - m_i < \mu$ for each interval and f is integrable.

15. Properties of integrals

Theorem 1. If $a < c < b$, then f is integrable on $[a, b]$ if and only if f is integrable on both $[a, c]$ and $[c, b]$. Further, when these conditions hold,

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^a f(x)\,dx. \tag{1}$$

Proof. Suppose f is integrable on both $[a, c]$ and $[c, b]$. If $\varepsilon > 0$ there are partitions P_1 of $[a, c]$ and P_2 of $[c, b]$ such that

$$U(f, P_i) - L(f, P_i) \leq \varepsilon/2 \text{ for } i = 1, 2.$$

Then $Q = P_1 \cup P_2$ is a partition of $[a, b]$ and

$$\begin{aligned} L(f, Q) &= L(f, P_1) + L(f, P_2) \\ U(f, Q) &= U(f, P_1) + U(f, P_2), \end{aligned}$$

hence $U(f, Q) - L(f, Q) \leq \varepsilon$. Therefore f is integrable on $[a, b]$. Also

$$\begin{aligned} L(f, P_1) \leq \int_a^c f(x)\,dx \leq U(f, P_1) \\ L(f, P_2) \leq \int_c^b f(x)\,dx \leq U(f, P_2), \end{aligned}$$

so

$$L(f, Q) \leq \int_a^c f(x)\,dx + \int_c^a f(x)\,dx \leq U(f, Q).$$

Since $\int_a^b f(x)\,dx$ also lies between $L(f, Q)$ and $U(f, Q)$, $|\int_a^b f(x)\,dx - \int_a^c f(x)\,dx + \int_c^a f(x)\,dx| < \varepsilon$. This is true for any $\varepsilon > 0$, proving (1).

Now assume f is integrable on $[a, b]$ and let Q be a partition with

$$U(f, Q) - L(f, Q) < \varepsilon. \tag{2}$$

If $c \notin Q$, replace Q by $Q \cup \{c\}$. By §13 Lemma 2, (2) still holds. Let $P_1 = Q \cap [a, c]$ and $P_2 = Q \cap [c, b]$. Then

$$\big(U(f, P_1) - L(f, P_1)\big) + \big(U(f, P_2) - L(f, P_2)\big) = U(f, Q) - L(f, Q) < \varepsilon.$$

The two differences on the left are nonnegative, hence each is less than ε. Therefore f is integrable on $[a, b]$ and by the first part of the proof, (1) holds.

Theorem 2. If f is integrable on $[a, b]$ and $k \in R$ then kf is integrable on $[a, b]$ and

$$\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx.$$

Proof. For $k > 0$, $\inf\{kf(x) : u \le x \le v\} = k\inf\{f(x) : u \le x \le v\}$, hence $L(kf, P) = kL(f, P)$ and similarly for the upper sum. The result follows. For $k < 0$, we use $0 = \int_a^b (-k)f(x) + kf(x)\,dx = \int_a^b (-k)f(x)\,dx + \int_a^b kf(x)\,dx = (-k)\int_a^b f(x)\,dx + \int_a^b kf(x)\,dx$ by Theorem 1 and the argument above with $-k > 0$ taking the role of k.

Theorem 3. If f and g are integrable on $[a, b]$, then $f + g$ is integrable on $[a, b]$ and

$$\int_a^b f + g\,dx = \int_a^b f\,dx + \int_a^b g\,dx.$$

Proof. Let I be some interval, $[x_{i-1}, x_i]$, in a partition of $[a, b]$. For any bounded function f, let $m_f = \inf\{f(x) : x \in I\}$. Then for $x \in I$, $m_f + m_g \le f(x) + g(x)$, hence $m_f + m_g \le m_{f+g}$. Similarly for the supremum, $M_f + M_g \ge M_{f+g}$.

Now for any $\varepsilon > 0$, let P be a partition of $[a, b]$ with both $U(f, P) - L(f, P) \le \varepsilon/2$ and $U(g, P) - L(g, P) \le \varepsilon/2$. Then

$$L(f, P) + L(g, P) \le L(f + g, P) \le U(f + g, P) \le U(f, P) + U(g, P).$$

Hence $U(f+g, P) - L(f+g, P) < \varepsilon$. Therefore $f+g$ is integrable. Since both $\int_a^b f+g\,dx$ and $\int_a^b f\,dx + \int_a^b g\,dx$ lie between these lower and upper sums, $|\int_a^b f\,dx + \int_a^b g\,dx - \int_a^b f+g\,dx| < \varepsilon$. This is true for all $\varepsilon > 0$ proving the equality.

Proposition. If $m \le f(x) \le M$ for $a \le x \le b$ and f is integrable, then

$$m(b - a) \le \int_a^b f(x)\,dx \le M(b - a).$$

Proof. Let $P = \{a, b\}$, the partition with just one interval. Then $m(b - a) \le L(f, P) \le \int_a^b f(x)\,dx \le U(f, P) \le M(b - a)$.

Corollary. If f and g are integrable on $[a, b]$ and $f(x) \le g(x)$, then

$$\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx.$$

16. The fundamental theorem of calculus

In the case $b < a$ define $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$ whenever the second integral exists. Then §15 Theorem 1 holds with any permutation of a, b, c. In particular $\int_a^x f(t)\,dt = \int_a^b f(t)\,dt + \int_b^x f(t)\,dt$. If f is integrable on an interval $[u, v]$, and $a, x \in [u, v]$, we define the function

$$F(x) = \int_a^x f(t)\,dt.$$

Fundamental Theorem of Calculus. If f is integrable on $[u, v]$, $u < b < v$, $[a, b] \subset [u, v]$, and f is continuous at b, then F is differentiable at b and $F'(b) = f(b)$.

Proof. Since $F(x)$ and $\int_b^x f(t)\,dt$ differ by the constant $\int_a^b f(t)\,dt$, it suffices to prove the theorem in the special case $a = b$; we let

$$F(x) = \int_b^x f(t)\,dt = -\int_x^b f(t)\,dt.$$

Since f is continuous at b by hypothesis,

$$(\forall\,\varepsilon > 0)(\exists\,\delta > 0)(|t - b| < \delta \Rightarrow |f(t) - f(b)| < \varepsilon),$$

hence

$$|t - b| \le |x - b| < \delta \Rightarrow f(b) - \varepsilon < f(t) < f(b) + \varepsilon.$$

By the proposition of §**15**, if $b < x$, then

$$(f(b) - \varepsilon)(x - b) \le \int_b^x f(t)\,dt \le (f(b) + \varepsilon)(x - b),$$

$$-\varepsilon \le \frac{F(x)}{x - b} - f(b) \le \varepsilon \text{ for } |x - b| < \delta. \tag{1}$$

On the other hand, if $x < b$, then

$$(f(b) - \varepsilon)(b - x) \le \int_x^b f(t)\,dt \le (f(b) + \varepsilon)(b - x),$$

Since $\int_x^b f(t)\,dt = -\int_b^x f(t)\,dt = -F(x)$ and $b - x = -(x - b)$, this again gives (1).

Hence for any $\varepsilon > 0$ there is a $\delta > 0$ such that (1) holds for any x with $|x - b| < \delta$, hence

$$F'(x) = \lim_{x \to b} \frac{F(x)}{x - b} = f(b).$$

The fundamental theorem proves the existence of antiderivatives—functions F such that F' is a given function f. Of course knowing a lot of derivatives may allow one to guess such an F. But for $f(x) = 1/x$ or $1/(1+x^2)$ or e^{-x^2}, guessing is not a real option unless you are already familiar with an answer. On the other hand, finding an antiderivative for a given f permits one to evaluate a definite integral of f.

Corollary. If $G'(x) = f(x)$ and f is continuous on $[a, b]$, then

$$\int_a^b f(x)\,dx = G(b) - G(a).$$

Proof. Since f is continuous, it is integrable. Let $F(x) = \int_a^x f(t)\,dt$. Then $F'(x) = f(x) = G'(x)$, so $(F-G)'(x) = 0$. By Corollary 2 of §12 there is a $c \in R$ with $F(x) = G(x) + c$. Then $F(b) = F(b) - F(a) = G(b) + c - G(a) - c = G(b) - G(a)$.

The following alternative proof gives the result of the Corollary with slightly weaker hypotheses.

Theorem. If $G'(x) = f(x)$ snd f is integrable on $[a, b]$, then

$$\int_a^b f(x)\,dx = G(b) - G(a).$$

Proof. Given $\varepsilon > 0$ let $P = \{x_0, \dots, x_n\}$ be a parrtition of $[a, b]$ with $U(f, P) - L(f, P) < \varepsilon$. Then

$$\begin{aligned}
G(b) - G(a) &= \sum_{i=1}^{n} G(x_i) - G(x_{i-1}) \\
&= \sum_{i=1}^{n} f(\xi_i)(x_i - x_{i-1}) \qquad \text{where } \xi_i \in (x_{i-1}, x_i) \\
&\le \sum_{i=1}^{n} M_i(x_i - x_{i-1}) = U(f, P) \quad \text{ and} \\
L(f, P) &= \sum_{i=1}^{n} m_i(x_i - x_{i-1}) \\
&\le f(\eta_i)(x_i - x_{i-1}) = G(b) - G(a).
\end{aligned}$$

Therefore

$$\begin{gathered}
L(f, P) \le G(b) - G(a) \le U(f, P) \qquad \text{and} \\
\left| \int_a^b f(x)\,dx - G(b) + G(a) \right| < \varepsilon.
\end{gathered}$$

17. Darboux and Riemann integrals

Let $P = \{x_0, \ldots, x_n\}$ be a partition of $[a, b]$ and let $t_* = \{t_1, \ldots, t_n\}$ satisfy

$$x_{i-1} \leq t_i \leq x_i.$$

DEFINITION. $R(f, P, t_*) = \Sigma_{i=1}^n f(t_i)(x_i - x_{i-1})$ is called a *Riemann sum*.

DEFINITION. $\mathrm{mesh}(P) = \max\{x_1 - x_0, \ldots, x_n - x_{n-1}\}$.

DEFINITION. f is *Riemann integrable* on $[a, b]$ if there is a number A such that

$$(\forall \varepsilon > 0)(\exists\, \delta > 0)(\forall P, t_*)(\mathrm{mesh}(P) < \delta \Rightarrow A - \varepsilon < R(f, P, t_*) < A + \varepsilon).$$

The Riemann integral of f is A.

THEOREM. A bounded function f is Darboux integrable if and only if it is Riemann integrable. The two integrals are equal.

Proof. If f is Darboux integrable on $[a, b]$, then for all $\varepsilon > 0$ there is a partition P with $U(f, P) - L(f, P) < \varepsilon$ and $L(f, P) \leq \int_a^b f \leq U(f, P)$.

Say $|f(x)| \leq B$ on $[a, b]$ and $P = \{x_0, \ldots, x_m\}$. Take $\delta = \min\{\mathrm{mesh}(P), \varepsilon/(Bm)\}$. Let $Q = \{y_0, \ldots, y_n\}$ be any partition of $[a, b]$ with $\mathrm{mesh}(Q) < \delta$.

If $[y_{j-1}, y_j] \not\subset [x_{i-1}, x_i]$ for any i, then $(\exists\, k, 1 \leq k \leq m - 1)(y_{j-1} < x_k < y_j)$. There are at most $m - 1$ such subintervals in Q. For the rest we have $[y_{j-1}, y_j] \subset [x_{i-1}, x_i]$ for some i depending on j.

Then

$$U(f, Q) \leq U(f, P) + (m - 1)\delta B < U(f, P) + \varepsilon,$$
$$L(f, Q) \geq L(f, P) - (m - 1)\delta B > L(f, P) - \varepsilon.$$

Hence

$$U(f, Q) - L(f, Q) < 3\varepsilon.$$

Also

$$L(f, Q) \leq R(f, Q, t_*) \leq U(f, Q).$$

So

$$|R(f, Q, t_*) - \int_a^b f| < 3\varepsilon.$$

Hence f is Riemann integrable and the integrals are equal.

If f is Riemann integrable there is a number A such that

$$(\forall \varepsilon > 0)(\exists\, \delta > 0)(\forall P, t_*)(\mathrm{mesh}(P) < \delta \Rightarrow A - \varepsilon < R(f, P, t_*) < A + \varepsilon)$$

Given $\varepsilon > 0$, fix such a $P = \{x_0, \ldots, x_m\}$. Let $M_i = \sup\{f(x) : x_{i-1} \le x \le x_i\}$. Since M_i is a *least* upper bound, there is a $t_i \in [x_{i-1}, x_i]$ such that $f(t_i) > M_i - \dfrac{\varepsilon/m}{x_i - x_{i-1}}$, hence $M_i(x_i - x_{i-1}) < f(t_i)(x_i - x_{i-1}) + \varepsilon/m$. This and a similar argument for m_i gives

$$U(f,P) < R(f,P,t_*) + \varepsilon < A + 2\varepsilon,$$
$$L(f,P) > R(f,P,s_*) - \varepsilon > A - 2\varepsilon$$

So $U(f,P) - L(f,P) < 4\varepsilon$ and f is Darboux integrable. Now from the first part of the proof we know the integrals are equal.

Note that we have not assumed f is continuous in this proof. We have shown that the set of integrable functions is the same in the two theories and that the integrals coincide.