*Exercise 1.30* Determine which of the links of six or fewer crossings in Table 1.1 at the end of the book are and are not tricolorable.

*Exercise 1.31* Show that the link in Figure 1.47 is tricolorable.

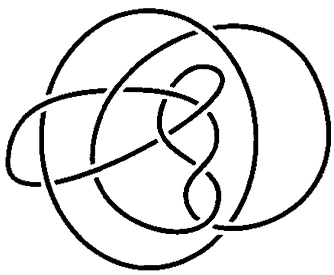

*Figure 1.47* This link is tricolorable.

# 1.6 Knots and Sticks

Suppose we were given a bunch of straight sticks and we were told to glue them together end to end in order to make a nontrivial knot. The sticks can be any length that we want (Figure 1.48). How many sticks will it take to make a nontrivial knot? Try playing with some sticks to see what happens. Certainly, three sticks aren't enough, as they would just form a triangle that lies in a plane. If we looked down at the plane, we would see a projection of the knot with *no* crossings. So it would have to be the unknot.

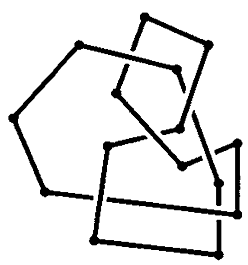

*Figure 1.48* A knot made out of sticks.

How about four sticks? If we view the four sticks from any direction, we will see a projection of the corresponding knot. If two of the sticks are attached to each other at their ends, they cannot cross each other in the projection (since two straight lines can cross at most once, in this case at the point where they are attached to one another). So in the projection, each stick can only cross the one stick that is not attached to either one of its ends. Therefore, there can be at most two crossings in the projection.

But the only knot with a projection of two or fewer crossings is the unknot. (See Exercise 1.2.)

So four sticks aren't enough to make a nontrivial knot. How about five sticks? Let's view the knot so that we are looking straight down one of the sticks. In the projection of the knot that we see, we will only be able to see four of the sticks, since the fifth stick is vertical. For the same reason as in the previous paragraph, the four sticks that we see can have at most two crossings, and so the knot must the unknot.

*Exercise 1.32* Prove that, in fact, a knot with four sticks in the projection can have at most one crossing.

Therefore, it must take at least six sticks to make a knot. In fact, it is possible to make a trefoil knot with six sticks, as shown in Figure 1.49. Although the picture looks believable, how do we know that we could really make a trefoil knot in space out of straight sticks like this? How do we know that the sticks needn't be bent or warped to fit together in this way, and that they only look straight when we see them from this view? We are only looking at a projection of the sticks in this picture.

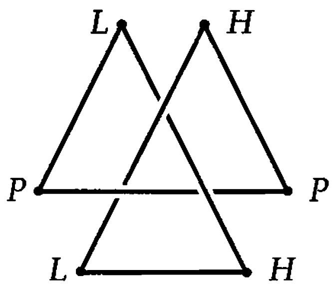


*Figure 1.49* A trefoil knot from six sticks.

One solution is to actually build the knot with real sticks. But we can convince ourselves that this construction works without going to that much trouble. Let the vertices labeled $P$ lie in the $xy$ plane. The vertices labeled $L$ lie low, underneath the plane. The vertices labeled $H$ lie high, above the plane. Then it's clear that such a knot could actually be constructed from sticks.

If we want a hands-on demonstration that five sticks won't suffice to make a knot, we can try it with five "sticks" that we were born with. Namely, think of the first stick as being your left forearm, followed by a stick formed from your left upper arm, followed by a stick that goes from your left shoulder to your right shoulder, followed by a stick formed from your right upper arm, followed by a stick formed from your right lower arm. That's a total of five sticks that are attached end to end (Figure 1.50).

If you can tangle up your arms and then clasp your hands together so that the loop formed from these five sticks is knotted, you will have a knot made from five sticks. Don't hurt yourself, we have already demonstrated that you can't succeed.

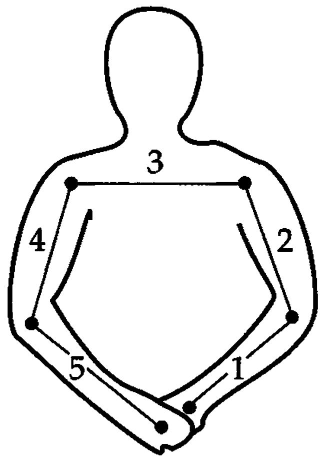


*Figure 1.50* Making knots from your arms?

But supposedly, six sticks are enough to make a knot.

*Exercise 1.33* Take a straight stick (say a yardstick or fireplace poker) as your sixth stick and demonstrate with your arms and this stick that a knot can be made out of six sticks.

What happens if we try to make knots using two people holding hands and their "ten sticks"? What knots can we make?

*Exercise 1.34* How many sticks would it take to make a figure-eight knot?

*Exercise 1.35** Show that the only nontrivial knot you can make with six sticks is the trefoil knot.

*Exercise 1.36* Show that you can make the knot $5_1$ (see the table at the back of the book) or the Whitehead link using only 8 sticks (use $P$'s $L$'s, and $H$'s to demonstrate that your constructions work).

Define the **stick number** ***s(K)*** of a knot $K$ to be the least number of straight sticks necessary to make $K$.

*Exercise 1.37* Show that if $J$ and $K$ are knots, $s(J\#K) \leq s(J) + s(K) - 1$.

### Unsolved Question

Can the inequality in Exercise 1.37 be improved to replace the $-1$ by $-2$ or $-3$? Amazingly, if $J$ and $K$ are trefoil knots (and hence each has stick number 6), then $s(J\#K)=8$, showing that in this very specific example, we have $s(J\#K) \leq s(J) + s(K) - 4$.

*Exercise 1.38** Let $c(K)$ be the least number of crossings in any projection of a knot $K$. Prove that if $K$ is a nontrivial knot, then

$$\frac{5 + \sqrt{(25 + 8(c(K) - 2))}}{2} \leq s(K)$$

(*Hint*: Look straight down one edge and then count crossings to obtain a bound on $c(K)$ in terms of $s(K)$. Then invert the inequality.)

In fact, we also have an upper bound on the stick number of a knot in terms of the minimum crossing number $c(K)$ of the knot. In a paper that appeared in 1991, Seiya Negami, a professor at Yokohama National University in Japan, showed $s(K) \leq 2\,c(K)$. The proof is elementary; however, it depends on some results from graph theory, so we will not discuss it.

### Unsolved Questions

*1.* By Exercise 1.37 and the preceding paragraph, we know that

$$\frac{5 + \sqrt{(25 + 8(c(K) - 2))}}{2} \leq s(K) \leq 2c(K)$$

Either show these are the best bounds we can obtain (by finding examples of knots of any crossing number that have $s(K) = 5 + \sqrt{(25 + 8(c(K) - 2))}/2$ and other examples that have $s(K) = 2c(K)$, or improve these bounds on $s(K)$ to narrow down the possibilities for $s(K)$ still further.

*2.* Does $s(K)$ change if we insist that the sticks that we make the knot out of must all be the same length? (It does not change in the case of the trefoil knot. But it seems unlikely that the same would be true for all knots.)

In Chapter 7, when we talk about applications of knot theory to synthetic chemistry, we will see why one might care about how many sticks it takes to make a knot.