TOPOLOGICAL INVARIANTS OF KNOTS AND LINKS*

BY

J. W. ALEXANDER

1. **Introduction.** The problem of finding sufficient invariants to determine completely the knot type of an arbitrary simple, closed curve in 3-space appears to be a very difficult one and is, at all events, not solved in this paper. However, we do succeed in deriving several new invariants by means of which it is possible, in many cases, to distinguish one type of knot from another. There exists one invariant, in particular, which is quite simple and effective. It takes the form of a polynomial $\Delta(x)$ with integer coefficients, where both the degree of the polynomial and the values of its coefficients are functions of the curve with which it is associated. Thus, for example, the invariant $\Delta(x)$ of an unknotted curve is 1, of a trefoil knot $1-x+x^2$, and so on. At the end of the paper, we have tabulated the various determinations of the invariant $\Delta(x)$ for the 84 knots of nine or less crossings listed as distinct in the tables of Tait and Kirkman. It turns out that with this one invariant we are able to distinguish between all the tabulated knots of eight or less crossings, of which there are 35. Repetitions of the same polynomial begin to appear when we come to knots of nine crossings.

The invariants found in this paper are all intimately related to the so-called *knot group*, as defined by Dehn. This is, of course, what one would expect; for many, if not all, of the topological properties of a knot are reflected in its group. The knot group would undoubtedly be an extremely powerful invariant if it could only be analyzed effectively; unfortunately, the problem of determining when two such groups are isomorphic appears to involve most of the difficulties of the knot problem itself.

In §11, we indicate, very briefly, how the results obtained for knots may be generalized to systems of knots, or links. We also establish the connection between the new invariants derived below and the invariants of the n-sheeted Riemann 3-spreads (generalized Riemann surfaces), associated with a knot.

2. **Knots and their diagrams.** In order to avoid certain troublesome complications of a point-theoretical order we shall always think of a *knot* as a simple, closed, sensed polygon in 3-space. A knot will, thus, be composed of a finite number of *vertices* and *sensed edges*. We shall allow ourselves to operate on a knot in the following three ways:

* Presented to the Society, May 7, 1927; received by the editors, October 13, 1927.

(i) To subdivide an edge into two sub-edges by creating a new vertex at a point of the edge.

(ii) To reverse the last operation: that is to say, to amalgamate a pair of consecutive collinear edges, along with their common vertex, into a single edge.

(iii) To change the shape of the knot by continuously displacing a vertex (along with the two edges meeting at the vertex) in such a manner that the knot never acquires a singularity during the process. It would, of course, be easy to express this third operation in purely combinatorial terms.

Two knots will be said to be the same *type* if, and only if, one of them is transformable into the other by a finite succession of operations of the three kinds just described. A knot will be said to be *unknotted* if, and only if, it is of the same type as a sensed triangle.

To make our descriptions a trifle more vivid we shall often allow ourselves considerable freedom of expression, with the tacit understanding that, at bottom, we are really looking at the problem from the combinatorial point of view. Thus, we shall sometimes talk of a knot as though it were a smooth elastic thread subject to actual physical deformations. There will, however, never be any real difficulty about translating any statement that we make into the less expressive language of pure, combinatorial analysis situs. In the figures, we shall picture a knot by a smooth curve rather than by a polygon. A purist may think of the curve as a polygon consisting of so many tiny sides that it gives an impression of smoothness to the eye.

A knot will be represented schematically by a 2-dimensional figure, or *diagram.* In the *plane of the diagram* a curve, called the *curve of the diagram,* will be traced picturing the knot as viewed from a point of space sufficiently removed so that the entire knot comes, at one time, within the field of vision. The curve of the diagram will ordinarily have singularities, but we shall assume that the point of observation is in a general position so that the singularities are all of the simplest possible sort: that is to say, double points with distinct tangents. The singularities of the curve of the diagram will be called *crossing points*, and the regions into which it subdivides the plane *regions of the diagram.* At each crossing point, two of the four corners will be dotted to indicate which of the two branches through the crossing point is to be

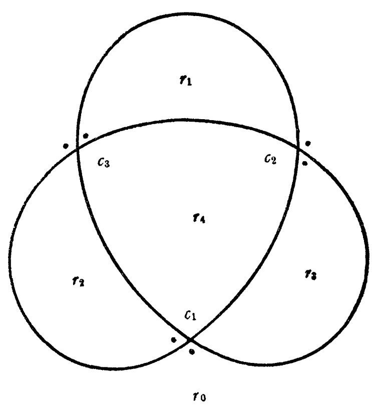

FIG. 1

thought of as the one passing under, or behind the other. The convention will be to place the dots in such a manner that an insect crawling in the positive sense along the "lower" branch through a crossing point would always have the two dotted corners on its left. Two corners will be said to be of *like signatures* if they are either both dotted or both undotted; they will be said to be of *unlike signatures* if one is dotted, the other not. Figure 1 represents a diagram of one of the two so-called trefoil knots.

To each region of a diagram a certain integer, called the *index* of the region, will be assigned. We shall allow ourselves to choose the index of any one region at random, but shall then fix the indices of all the remaining regions by imposing the requirement that whenever we cross the curve from right to left (with reference to our imaginary insect crawling along the curve in the positive sense) we must pass from a region of index p, let us say, to a region of next higher index $p+1$. Evidently, this condition determines the indices of all the remaining regions fully and without contradiction. To save words, we shall say that a corner of a region of index p is itself of index p.

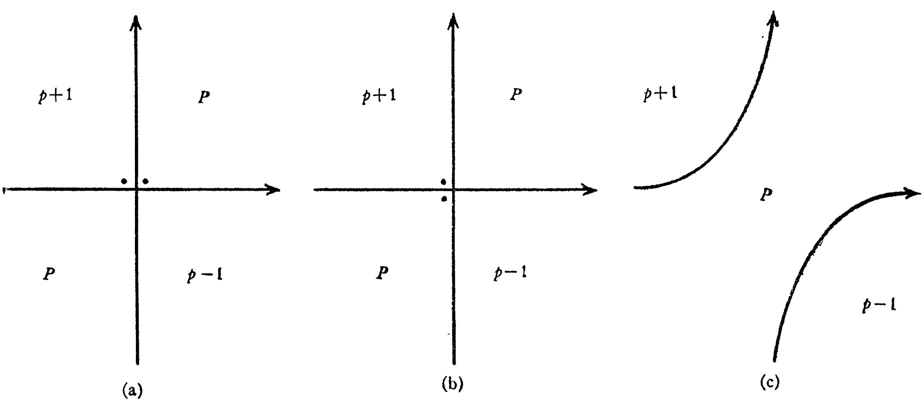

FIG. 2

It is easy to verify that at any crossing point c there are always two opposite corners of the same index p and two opposite corners of indices $p-1$ and $p+1$ respectively. The index p associated with the first pair of corners will be referred to as the *index* of the crossing point c. Two kinds of crossing points are to be distinguished according to which branch through the point passes under, or behind, the other. A crossing point of the first kind, Fig. 2a, will be said to be *right handed*, one of the second kind, Fig. 2b, *left handed*. At either kind of point the two undotted corners are of indices $p-1$ and p respectively, the two dotted ones of indices p and $p+1$. However, at a right handed point the dotted corner of index p precedes the dotted corner of index $p+1$ as we circle around the point in the counter clockwise sense, whereas at a left handed point it follows the other. At a crossing point c, the two corners of like index p may belong to the same region of the diagram. We observe for future reference that on the boundary of a region of index p only crossing points of indices $p-1$, p, and $p+1$ may appear. Finally, we recall again that the entire system of indices is determined to within an additive constant only, since the index of some one region or crossing point has to be assigned before the indexing of the figure as a whole becomes determinate.

3. **The equations of a diagram.** In reality, the same diagram represents an infinite number of different knots, but this indetermination is, if anything, an advantage, as the knots so represented are all of the same type. The knot problem is the problem of recognizing when two different diagrams represent knots of the same type. Now, to tell the type of knot determined by a diagram it is evidently not necessary to know the exact shapes of the various elements of the diagram, but only the relations of incidence between the elements and the signatures at the corners of the regions. Because of this fact, the essential features of a diagram may all be displayed schematically by a properly chosen system of linear equations, as we shall now prove.

If a diagram has ν crossing points

$$c_i \qquad (i = 1, 2, \cdots, \nu), \tag{3.1}$$

we find, by a simple application of Euler's theorem on polyhedra, that it must have $\nu+2$ regions

$$r_j \qquad (j = 0, 1, \cdots, \nu + 1). \tag{3.2}$$

Now, suppose the four corners at a crossing point c_i belong respectively to the regions r_j, r_k, r_l, and r_m, that we pass through these regions in the cyclical order just named as we go around the point c_i in the counterclockwise sense, and that the two dotted corners are the ones belonging to the regions r_j and r_k respectively. Then, corresponding to the crossing point c_i we shall write the following linear equation:

$$c_i(r) = xr_j - xr_k + r_l - r_m = 0. \tag{3.3}$$

The ν equations (3.3) determined by the ν crossing points c_i will be called the *equations of the diagram.* The cyclical order of the terms in the left hand members of these equations plays an essential rôle and is not to be disturbed. The distribution of the coefficients x determines in which corners of the diagram the dots are located.

By way of illustration we shall write out the equations of the diagram of the trefoil knot (Fig. 1). They are as follows:

$$
\begin{aligned}
c_1(r) &= xr_2 - xr_0 + r_3 - r_4 = 0,\\
c_2(r) &= xr_3 - xr_0 + r_1 - r_4 = 0,\\
c_3(r) &= xr_1 - xr_0 + r_2 - r_4 = 0.
\end{aligned} \tag{3.4}
$$

The equations of a diagram determine the structure of the diagram completely unless there happen to be two or more edges incident to the same pair of regions. For, barring this exceptional case, two cyclically consecutive terms in any equation correspond to a pair of regions that are incident along one edge only, and, therefore, determine the edge itself. In other words, the equations of the diagram tell us the incidence relations between the edges and crossing points. But they also tell us the relative position of the four edges at a crossing point; therefore, we have all the information needed to reconstruct the curve of the diagram. Moreover, the distribution of the coefficients x tells us how the corners must be dotted.

In the exceptional case, where the boundaries of two regions have more than one edge in common we are either dealing with the diagram of a *composite knot* K or with a diagram that admits of obvious simplification. Suppose the edges e_1 and e_2 are on the boundary of each of two regions r_1 and r_2. Then, if we join a point P_1 of the edge e_1 to a point P_2 of the edge e_2 by means of an arc α lying wholly within the region r_1, the extremities of the arc α will subdivide the curve of the diagram into two non-intersecting arcs γ_1 and γ_2 which may be combined respectively with the arc α to form the two closed curves

$$\alpha + \gamma_1, \qquad \alpha + \gamma_2.$$

Moreover, these last two curves may be regarded as the diagram curves of a pair of non-interlinking knots K_1 and K_2 in space. If neither of the knots K_1 nor K_2 is unknotted we may regard K_1 and K_2 as *factors* of the composite knot K. If one of them, K_1, is unknotted, the knot K must evidently be of the same type as the other one, K_2. Hence, in this case, the diagram of the knot K may be replaced by the simpler diagram of the knot K_2.

4. **The invariant polynomial $\Delta(x)$.** Let us now treat the equations of the diagram as a set of ordinary linear equations E in which the ordering of the terms in the various left hand members is immaterial. Then, the matrix of the coefficients of equations E will be a certain rectangular array M of ν rows and $\nu+2$ columns, one row corresponding to each crossing point and one column to each region of the diagram. We shall presently show that the

matrix M has a genuine invariantive significance; for the moment, let us merely observe that it has the following property:

If the matrix M is reduced to a square matrix M_0 by striking out two of its columns corresponding to regions with consecutive indices p and $p+1$, the determinant of the residual matrix M_0 will be independent of the two columns struck out, to within a factor of the form $\pm x^n$.

To prove the theorem, let us introduce the symbol R_p to denote the sum of all the columns corresponding to the regions of index p and the symbol 0 to denote a column made up exclusively of zero elements. Then, we obviously have the relation

$$\sum_p R_p = 0\,; \tag{4.1}$$

for in each row of the matrix there are only four non-vanishing elements, namely x, $-x$, 1, and -1, and the sum of these four elements is zero. We also have the relation

$$\sum_p x^{-p} R_p = 0\,; \tag{4.2}$$

for if we multiply the elements of each column by a factor x^{-p}, where p is the index of the (region corresponding to) the column, the four non-vanishing elements in a row of index q become x^{1-q}, $-x^{1-q}$, x^{-q} and $-x^{-q}$ respectively, so that their sum is again zero. By properly combining relations (4.1) and (4.2) we obtain the relation

$$\sum_p (x^{-p} - 1) R_p = 0 \tag{4.3}$$

in which the term in R_0 disappears.

Now, let

$$\pm\, \Delta_{pq}(x) = \pm\, \Delta_{qp}(x)$$

be the determinant of any one of the matrices M_{pq} obtained by striking out from the matrix M a pair of columns of indices p and q respectively. Then, by (4.3), we clearly have

$$(x^{-q} - 1)\Delta_{0p}(x) = \pm\, (x^{-p} - 1)\Delta_{0q}. \tag{4.4}$$

For relation (4.3) tells us that a column of index p multiplied by the factor $x^{-p}-1$ is expressible as a linear combination of the other columns of the matrix M of indices different from zero (that is to say, of columns of the matrix M_{0p}), and that in this linear combination the coefficients of the columns of index q are $-(x^{-q}-1)$. Moreover, since indices are determined to within an additive constant only, relation (4.4) gives us

$$(x^{r-q} - 1)\Delta_{rp} = \pm (x^{r-p} - 1)\Delta_{rq},$$

$$(x^{q-s} - 1)\Delta_{qr} = \pm (x^{q-r} - 1)\Delta_{qs};$$

whence,

$$\Delta_{rp} = \pm \frac{x^{q-r}(x^{r-p} - 1)}{x^{q-s} - 1}\Delta_{qs}. \tag{4.5}$$

But, as a special case of (4.5), we have the relation

$$\Delta_{r(r+1)} = \pm x^{q-r}\Delta_{q(q+1)}, \tag{4.6}$$

which proves the theorem.

Let us now divide the determinant $\Delta_{r(r+1)}$ by a factor of the form $\pm x^n$ chosen in such a manner as to make the term of lowest degree in the resulting expression $\Delta(x)$ a positive constant. Then,

The polynomial $\Delta(x)$ is a knot invariant.

The theorem will be proved in §6 and again in §10, as a corollary to a more general theorem.

Let us actually evaluate the invariant $\Delta(x)$ in a simple, concrete case. From the equation of the diagram of the trefoil knot, (3.4), we obtain the matrix

$$\begin{matrix} -x & 0 & x & 1 & -1 \\ -x & 1 & 0 & x & -1 \\ -x & x & 1 & 0 & -1. \end{matrix} \tag{4.7}$$

Now, if we assign indices in such a way that the first row of the matrix is of index 2, the next three rows will be of index 1 and the last row of index 0. The determinant Δ_{01} obtained after striking out the last two rows of the matrix (4.7) will be

$$\Delta_{01} = -x(1 - x + x^2);$$

the determinant obtained after striking out the first two rows,

$$\Delta_{12}(x) = -(1 - x + x^2).$$

The difference between these two expressions is of the sort predicted by relation (4.6). The invariant $\Delta(x)$ is, of course,

$$\Delta(x) = 1 - x + x^2.$$

5. **Further new invariants.** It will now be necessary to obtain a somewhat more precise theorem about the matrix M than the one proved in §4.

Any two columns of the matrix M of consecutive indices p and $p+1$ may be expressed as linear combinations of the remaining ν columns, where the coefficients of the two linear combinations are polynomials in x with integer coefficients.

Here, and elsewhere throughout the discussion, we shall use the term "polynomial" in the broad sense, so as to allow terms in negative as well as positive powers of the mark x to be present.

Since indices are determined to within an additive constant only, we may assume that p is zero in proving the theorem. Now, in relation (4.3) there is no term in R_0, and the coefficient of the term in R_1 is $x^{-1}-1$. Let us divide the coefficients of all the terms in (4.3) by this last expression so as to make the coefficient of R_1 equal to unity. The coefficients of the remaining terms will then be expressible as polynomials in the broad sense; for if p is positive, we have

$$x^{-p} - 1/x^{-1} - 1 = x^{-p+1} + x^{-p+2} + \cdots + 1,$$

while if p is negative, we have

$$x^{-p} - 1/x^{-1} - 1 = - x^{-p} - x^{-p-1} - \cdots - x.$$

Therefore the simplified relation (4.3) tells us that any column of index 1 is expressible as a linear combination with polynomial coefficients of columns *of indices different from zero*. But if we start from the relation

$$\sum (x^{-p} - 1) R_p = 0$$

which also follows, at once, from (4.1) and (4.2), we conclude, by a similar argument, that any column of index 0 is expressible as a linear combination with polynomial coefficients of columns of *indices different from one*. The theorem follows at once.

Two matrices M_1 and M_2 will be said to be *equivalent* if it is possible to transform one of them into the other by means of the ordinary elementary operations allowed in the theory of matrices with integer coefficients:

(α) Multiplication of a row (column) by -1.

(β) Interchange of two rows (columns).

(γ) Addition of one row (column) to another.

(δ) Bordering the matrix with one new row and one new column, where the element common to the new row and column is 1 and the remaining elements of the new row and column are 0's; or the inverse operation of striking out a row and a column of the type just described.

Two matrices M_1 and M_2 will be said to be *ϵ-equivalent* if it is possible to transform one of them into the other by means of the operations (α), (β), (γ), (δ), along with the further operation

(ϵ) Multiplication or division of a row (column) by x. Two polynomials will be said to be ϵ-equivalent if they differ, at most, by a factor of the form $\pm x^n$. We now state the following theorem, which will be proved in the next section and again in §10.

If two diagrams represent knots of the same type their matrices M are ϵ-equivalent.

As a corollary to this theorem it follows that

If two diagrams represent knots of the same type the elementary factors of their matrices M are ϵ-equivalent, barring factors of the form $\pm x^p$ (those ϵ-equivalent to unity).

For operations (α), (β), and (γ) leave the elementary factors invariant; operation (δ) merely introduces or suppresses a unit factor; operation (ϵ) merely changes one of the factors by a factor x. By an *elementary factor* of a matrix M we here mean the highest common factor of all the minors of the matrix M of any given order p divided (for $p>1$) by the highest common factor of all minors of order $p-1$. If all the minors of order p vanish, the corresponding factor is zero.

The theorem about the invariance of the polynomial $\Delta(x)$ announced in §4 is an obvious consequence of the corollary, for $\Delta(x)$ is ϵ-equivalent to the product of the elementary factors of the matrix M.

The theorem suggests the problem of finding normal forms for the matrices M under operations (α), (β), (γ), (δ), and (ϵ). Under this particular group of operations, the elementary factors of a matrix M are not a complete set of invariants. They would be if we replaced operation (α) by the more general operation

(α') Multiplication of a row (column) by an arbitrary rational number.

The matrix M_0 obtained by deleting from the matrix M two columns of consecutive indices p and $p+1$ is ϵ-equivalent to the matrix M.

This follows, immediately, from the first theorem proved in this section.

It should be remarked that the matrix N obtained by changing the signs of all negative elements of the matrix M is equivalent to the matrix M. For if we change the signs of all the elements of M belonging to the columns of odd indices we obviously obtain a matrix of such a form that the elements in any given row are of like sign. Therefore, by further changing the signs of the elements in the rows containing no positive elements we obtain the matrix N. Thus, the matrix M is transformable into the matrix N by elementary transformations. For theoretical purposes, the matrix M offers certain advantages over the matrix N; when actual computations are to be made, the matrix N is generally to be preferred, as mistakes in sign are less likely to be made when it is used.

6. **Diagram transformations.** When a knot is deformed, the equations of its diagram remain invariant so long as the topological structure of the diagram does not change. Now, a change in the structure of the diagram may come about in one or another of the following ways:

(A) The curve of the diagram may acquire a loop and crossing point (Fig. 3) or it may lose a loop and crossing point by a deformation of the inverse sort.

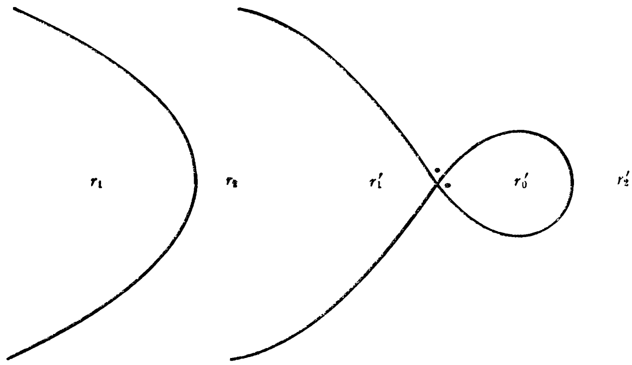

FIG. 3

(B) One branch of the curve may pass under another with the creation of two new crossing points (Fig. 4); or by a deformation of the inverse sort, one branch may slide out from under another with the loss of two cross-

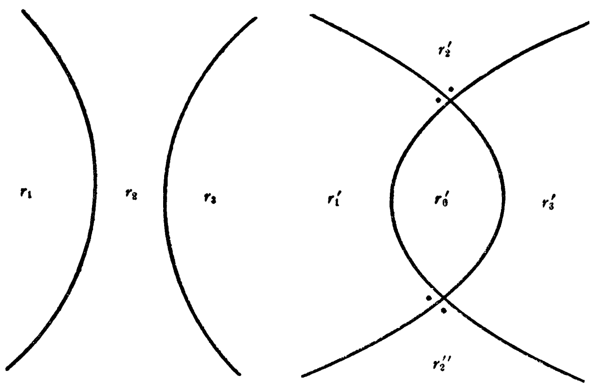

FIG. 4

ing points. In this case it must be borne in mind that the corners at the two crossing points must be so dotted as to imply that the lower branch at one is also the lower branch at the other.

(*C*) If there is a three-cornered region in the diagram, bounded by three arcs and three crossing points, and if the branch corresponding to one of the three arcs passes beneath the branches corresponding to the other two, then any one of the three branches may be deformed past the crossing point formed by the intersection of the other two (Fig. 5). The effect is the same, topologically speaking, whichever of the three branches undergoes the deformation.

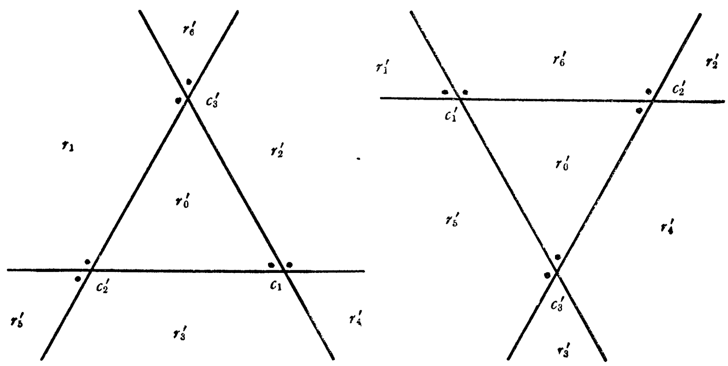

FIG. 5

It is a simple matter to verify that any allowable variation in the structure of the diagram may be compounded out of variations of the three simple types indicated above.*

With these facts before us, it is now easy to prove the theorem about the ϵ-equivalence of the matrices of two diagrams which determine knots of the same type. For it is sufficient to show that under each of the transformations (*A*), (*B*), and (*C*) the matrix of the diagram is carried into an ϵ-equivalent one.

First, consider case (*A*), where a branch of the curve acquires a new loop and crossing point. Let M_0 be the matrix of the original diagram after the two redundant columns corresponding to the region r_1 and r_2 (Fig. 3), have been struck out. Then, the effect of the transformation is merely to border the matrix M_0 with a new row and column in which all the elements are zero except the one which the row and column have in common. This last element will be ± 1 or $\pm x$ according to how the corners at the new crossing points are dotted. Evidently, the new matrix is ϵ-equivalent to the original one.

* Cf., for example, Alexander and Briggs, *On types of knotted curves*, Annals of Mathematics, (2), vol. 28 (1927), pp. 563–586.

Under case (B), let M_0 be the matrix of the original diagram after the two redundant columns corresponding to the regions r_1 and r_2 (Fig. 4) have been struck out, and let M_0' be the corresponding transformed matrix. Then, if we add column r_0' of M_0' to column r_3' we obtain the original matrix M_0 bordered by two new rows and columns in the manner indicated schematically by the following figure:

$$\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 1 & \Phi \\ \hline 0 & 0 & M_0 \end{array}.$$

Thus, clearly, the new matrix is again ϵ-equivalent to the old. It may happen that the corners are not dotted in the manner indicated in the figure, but that the two corners of the region r_0' are dotted ones. The method of proof is, however, essentially the same in this case as in the case just considered.

In disposing of case (C), we shall replace the matrix M of the original diagram by the equivalent matrix N, as defined at the end of §6, so as not to be troubled about the correct evaluation of the signs of the various elements. Moreover, we shall change the rôles of the rows and columns of the matrix N and think of this last as the matrix of a set of equations in the symbols c_i associated with the crossing points, rather than in the symbols r_j associated with the regions. Then, with the aid of Fig. 5, we may verify, at once, that the matrix N undergoes the transformation induced by the following change of symbols:

$$\begin{aligned} c_1' &= -xc_1, \\ c_2' &= xc_1 + c_3, \qquad (6.1) \\ c_3' &= c_1 + c_2. \end{aligned}$$

In making the verification we must not overlook the relations

$$xc_1 + c_2 + c_3 = 0$$

and

$$c_1' + xc_2' + xc_3' = 0$$

corresponding to the regions r_0 and r_0' (Fig. 5) respectively. Now, the substitution (6.1) is, clearly, the product of a substitution

$$c_1' = xc_1$$

which induces an ϵ-operation on the matrix N, and a substitution of determinant unity which induces a set of elementary operations, by a well known theorem. Therefore the matrix N is carried over into an ϵ-equivalent one. The cases where the corners are dotted in a different manner from that shown in Fig. 5 are treated in a similar manner.

This completes the proof of the invariantive character of the matrix M, whence, also, it follows that the polynomial $\Delta(x)$ is an invariant.

In the next sections we shall establish the connection between the matrix M and the so-called *group* of the knot, as defined by Dehn. This will necessitate the interpolation of a few preliminary remarks bearing, very largely, on questions of terminology and notation.

7. **Abstract groups.** In expressing the resultant of two or more operations of a group A we shall use the summation, rather than the product notation. Thus, if the symbols $a_i, a_j, \cdots$ represent operations of the group, the symbol $-a_i$ will represent the inverse of the operation a_i, the symbol a_i+a_j the resultant of the operation a_i followed by the operation a_j, the symbol 0 the identical operation. Furthermore, the symbol λa_i, where λ is any positive integer, will denote the resultant of the λ-fold repetitions of the operation a_i, and the symbol $-\lambda a_i$ the resultant of the λ-fold repetitions of the operation $-a_i$. It goes without saying that we must distinguish, in general, between the operations a_i+a_j and a_j+a_i.

If two consecutive terms of a sum of operations

$$c(a_i) = \lambda_1 a_{i_1} + \lambda_2 a_{i_2} + \cdots + \lambda_n a_{i_n}$$

involve the same letter a_{i_p} they may be contracted into a single term in a_{i_p}. After all possible contractions of the sort have been made the sum $c(a_i)$ will be said to be in its *reduced form*. We shall use the identity sign between two sums,

$$c(a_i) \equiv d(a_i),$$

to indicate that the sums, when reduced, are formally identical. An equality sign between two symbols

$$c = d$$

will merely indicate that the two symbols represent the same operation of the group, without implying their formal identity.

Let

$$(7.1) \qquad a_i \qquad (i = 1, 2, \cdots, m)$$

be a set of *generators* of a group A: that is to say, a set of operations of the group A in terms of which all the operations of A may be expressed. Then, in most cases, there will exist certain identical combinations of the generators of the form

$$(7.2) \qquad c_j(a_i) = 0 \qquad (j = 1, 2, \cdots, m).$$

Now, if we know any set of identities (7.2) among the generators there are three standard processes whereby we may enlarge the set (7.2) by the formation of new identities:

(i) The process of inversion, giving identities of the form

$$-c_j(a_i) = 0\,; \tag{7.3}$$

(ii) The process of summation, giving identities of the form

$$c_j(a_i) + c_k(a_i) = 0\,; \tag{7.4}$$

(iii) The process of transformation, giving identities of the form

$$e(a_i) + c_j(a_i) - e(a_i) = 0, \tag{7.5}$$

where $e(a_i)$ is any operation of the group.

The set (7.2) will be said to be *complete* if there is no identical combination of the generators which cannot ultimately be brought into the set by repeated application of the three processes just indicated. Thus, if the set (7.2) is complete, the most general identical combination of the generators must be of the form

$$c \equiv \sum_i (e_i \pm c_{j_i} - e_i) = 0. \tag{7.6}$$

A group A is fully determined by a set of generators (7.1) together with a complete set of identities (7.2) among them. In all of the discussion we shall confine our attention to the case when the number of generators (7.1) and defining identities (7.2) is finite.

With every group A there is associated a commutative group A_c determined by adjoining to the defining identities (7.2) of A all possible relations among the generators of the form

$$a_i + a_j = a_j + a_i.$$

The group A_c may also be thought of as the one determined by the generators (7.1) and identities (7.2) alone, where, however, we must now assign a new meaning to our symbolism and regard addition as commutative. If we do this, equations (7.2) simplify, by collecting terms in like symbols, to the form

$$c_j = \sum_i \epsilon_{ij} a_i = 0 \tag{7.7}$$

while relation (7.6) which displays the form of the most general identical combination among the generators becomes

$$c \equiv \sum c_j = 0. \tag{7.8}$$

The group A_c is, obviously, an invariant of the group A, whence, also, its own invariants are invariants of A. For future reference, we quote without proof a classical theorem about the commutative group A_c. Let $\|\epsilon_{ij}\|$ denote the matrix of the coefficients in equation (7.7). Then

The elementary factors of the matrix

$$\|\epsilon_{ij}\|$$

which differ from unity form a complete set of invariants of the group A_c. *Therefore, also, they are invariants of the group* A.

8. **The knot group.** The group R of a knot, as defined by Dehn,* is merely the ordinary topological group of the space S exterior to the knot. Let us fix upon some point of the space S, such as the observation point P from which we are supposed to be viewing the knot when we look at its diagram. Then, in the space S, each closed, sensed curve beginning and ending at P determines an operation of the group R. Moreover, two different sensed curves determine the same operation if, and only if, one may be deformed continuously into the other within the space S, while its ends remain fixed at the point P. During the deformation the curve may cut through itself at will, but it must never come into contact with the knot, as that would involve its leaving the space S. The condition that a curve determine the identical operation is that it be continuously deformable into the point P itself. If two sensed curves C_1 and C_2 correspond respectively to the operations r_1 and r_2 of the group R, the sensed curve C_1+C_2 obtained by joining the initial point of C_2 to the terminal point of C_1 determines the operation r_1+r_2.

The group R of a knot may be obtained, at once, from a diagram of the knot.† Let us flatten out the knot until it coincides, sensibly, with the curve of its diagram. Then, if we pick out, at random, a region r_0 of the diagram, there will be one generator of the group corresponding to each of the other $\nu+1$ regions r_j, where the generator in question is the operation determined by a curve which starts from the point P, crosses the region r_j, passes behind the plane of the diagram, and returns to the point P by way of the region r_0. It is easy to see, by inspection, that to each crossing point of the diagram there corresponds an identical relation of the form

$$r_j - r_k + r_l - r_m = 0, \tag{8.1}$$

* M. Dehn, *Topologie des dreidimensionalen Raumes*, Mathematische Annalen, vol. 69 (1910), pp. 137–168.

† Dehn, loc. cit.

where if the symbol r_0 appears in this relation it must be set equal to zero. Moreover, it is not difficult to verify† that the set of relations (8.1) is complete. We, therefore, have the following theorem:

The group R of a knot is the one determined by the equations of the diagram, §3, *when we set x equal to unity, together with one more equation of the form*

$$r_0 = 0.$$

9. **Indexed groups.** We shall now make another short digression leading to a generalization of the theorem quoted at the end of §7. A group A will be said to be *indexed* if with each operation of the group there is associated an integer, called the *index* of the operation, such that

(i) The index of the identical operation is zero;

(ii) There exists an operation of index unity;

(iii) The index of the resultant of two operations is the sum of the indices of the two operations.

Two indexed groups will be said to be *directly equivalent* if they are related by a simple isomorphism pairing elements of like indices, and *inversely equivalent* if they are related by a simple isomorphism pairing elements of index p ($p=0, \pm 1, \pm 2, \cdots$) with elements of index $-p$.

Let A be an indexed group determined by a finite number of generators connected by a finite number of identical relations. Then, clearly, the generators may always be chosen in the *canonical form*

$$s,\ a_1,\ a_2,\ \cdots,\ a_n, \tag{9.1}$$

where the first generator s is of index 1 and the others a_i are of index 0. For any arbitrary finite set of generators may be reduced to the above form by a process analogous to the one used in finding the highest common factor of a set of integers. The defining identities of the group, expressed in terms of the generators (9.1), will be certain linear expressions which we shall denote by

$$c_j(s,\ a_1,\ a_2,\ \cdots,\ a_n) = 0. \tag{9.2}$$

Now, it will be observed that the operations of the group A which are of index 0 determine a self-conjugate subgroup A^* of A. Let a be any operation of this subgroup. Then if the operation a is expressed in terms of the generators (9.1) of the group A,

$$a = a(s,\ a_1,\ a_2,\ \cdots,\ a_n), \tag{9.3}$$

the sum of the coefficients of the terms in s must evidently vanish. Con-

† Dehn, loc. cit.

sequently, if we interpolate between every two terms of (9.3) a pair of redundant terms of the form $-\lambda s+\lambda s$, where the various coefficients λ are suitably determined, we shall obtain a representation of the operation a in the form

$$(9.4) \qquad a = \sum_i (\lambda_i s \pm a_{p_i} - \lambda_i s).$$

It will be convenient to introduce the *abridged notation*

$$\pm x^\lambda a_i = \lambda s \pm a_i - \lambda s$$

to denote a succession of three terms like the ones appearing in the sum (9.4). We shall then be able to express the sum a in the form

$$(9.5) \qquad a = \sum_i \pm x^{\lambda_i} a_{p_i}.$$

Conversely, every sum of terms $\pm x^{\lambda_i} a_{p_i}$ represents an operation of index 0: that is to say, an operation of the subgroup A^* of A. In particular, the defining relations (9.2) of the group A may be written

$$(9.6) \qquad c_j(x, a) = \sum_i \pm x^{\lambda_{ij}} a_{p_{ij}} = 0.$$

Now, let us reexamine the three processes (7.3), (7.4), and (7.5) of §7, whereby new identities are to be formed from the identities of a given set (9.6). The first two processes require no particular comment; in the new notation they may be expressed by

$$(9.7) \qquad -c_j(x, a) = 0$$

and

$$(9.8) \qquad c_j(x, a) + c_k(x, a) = 0$$

respectively. The third process, however, is expressed by

$$(9.9) \qquad e(x, a) + x^\lambda c_j(x, a) - e(x, a) = 0,$$

where the presence of the coefficient x^λ before the middle term is to be accounted for by the fact that in reducing expression (7.5) to the form (9.9) we must, in general, interpolate a pair of redundant terms $-\lambda s+\lambda s$ after the term $e(a_i)$ in order to obtain a group of terms

$$e(x, a) = e(a_i) - \lambda s$$

of index zero. Relation (7.6) which exhibits the most general identity among the generators (9.6) becomes, in the abridged notation,

$$(9.10) \qquad c(x, a) = \sum [e_i(x, a) \pm x^{\lambda_i} c_{j_i}(x, a) - e_i(x, a)] = 0.$$

We may now regard the subgroup A^* of A as determined by the n generators

$$(9.11)\qquad a_1, a_2, \cdots, a_n$$

of A of indices zero together with the identities (9.6), which, for convenience, we shall now rewrite:

$$(9.12)\qquad c_j(x, a) = 0.$$

Relation (9.10) shows us how to form the most general identity in the generators a_i.

With the group A^* there is associated a commutative group A_c^* determined by adding to the identities (9.12) all relations of the form

$$x^\lambda a_i \pm x^\mu a_j = \pm x^\mu a_j + x^\lambda a_i.$$

The group A_c^* bears the same relation to the group A^* as the group A_c of §7 to the group A. It may be thought of as the one determined by the generators (9.11) and identities (9.12) alone, where, as in §7, we change the meaning of our notation and regard addition as commutative. Here, however, we must bear in mind that it is only when we express the operations of the group A_c^* in abridged notation that addition is commutative. The operations a and

$$x^\lambda a = \lambda s + a - \lambda s$$

are still to be regarded as distinct, otherwise we would get only trivial results. The defining identities (9.12) of the commutative group A_c^* may evidently be simplified, by collecting the terms in the various symbols a_i, to the form

$$(9.13)\qquad c_j(x, a) = \sum_i X_{ij} a_i = 0,$$

where the coefficients X_{ij} are polynomials in x with integer coefficients. (Here, again, we are using the term "polynomial" in the broad sense so as to allow negative as well as positive powers of x to be present.) Relation (9.10) exhibiting the form of the most general identity in the generators simplifies, in this case, to

$$(9.14)\qquad c(x, a) = \sum_i X_i c_{j_i}(x, a) = 0,$$

where the coefficients X_i are also polynomials in the broad sense.

Now, let $\|X_{ij}\|$ be the matrix of the coefficients of equations (9.13). Then, as a generalization of the theorem quoted at the end of §7, we shall have the following proposition:

If two indexed groups A and B are directly equivalent, their associated matrices $\|X_{ij}\|$ and $\|Y_{ij}\|$ are ϵ-equivalent.

The proof will be made in a series of easy steps. Let us choose a canonical set of generators

$$s,\ a_1,\ a_2,\ \cdots,\ a_m \tag{9.15}$$

of the group A, and a set of defining identities

$$c_j(x,\ a) = 0 \qquad (xa = s + a - s). \tag{9.16}$$

Then, corresponding to these last, we shall have the identities

$$\sum X_{ij}a_i = 0 \tag{9.17}$$

of the commutative group A_c^* associated with the group A.

Now, let us observe the following simple facts:

(i) If we enlarge the set (9.16) by the adjunction of one new relation which is dependent on the ones already in the set, the matrix $\|X_{ij}\|$ is, thereby, transformed into an ϵ-equivalent one. For, by relation (9.4), the matrix $\|X_{ij}\|$ merely acquires a new row, expressible as a linear combination of the old ones with polynomial coefficients.

(ii) If we adjoin a new generator a_{m+1} of index zero to the set (9.15) and, at the same time, add to relations (9.16) an identity of the form

$$c_{m+1}(x, a) + a_{m+1} = 0$$

expressing the new generator in terms of the old ones, the matrix $\|X_{ij}\|$ is again transformed into an ϵ-equivalent one. For the matrix $\|X_{ij}\|$ is merely bordered by a new row and column, where the elements of the new column are all zero except the one in the new row which is unity.

(iii) If we replace the generator s by another operation t of index unity such that the operations

$$t,\ a_1,\ a_2,\ \cdots,\ a_m \tag{9.18}$$

also generate the group A we leave the matrix $\|X_{ij}\|$ invariant. For suppose we write

$$y^\lambda a = \lambda t + a - \lambda t$$

to denote transformation through this new operation t. Then, since s and t are both of weight unity, it must be possible to write

$$s = t + \phi(y,\ a) \tag{9.19}$$

where $\phi(y,\ a)$ is of index zero and, therefore, expressible in the abridged notation. Therefore, we have

$$(9.20)\qquad xa = s + a - s = t + \phi + a - \phi - t = y(\phi + a - \phi).$$

But suppose we make the substitution (9.20) in equation (9.17). Since, in these last equations, addition is to be regarded as commutative, the substitution (9.20) produces the same effect as the substitution $x=y$. Therefore, the matrix $\|X_{ij}\|$ is left invariant, except for a change of notation.

With the above facts established, the proof of the theorem is immediate. Let the generators of the group B, written in canonical form, be

$$(9.21)\qquad t,\ b_1,\ b_2,\ \cdots,\ b_n,$$

and the defining relations,

$$(9.22)\qquad d_j(y, b) = 0.$$

Then, since the groups A and B are directly isomorphic, we may express the isomorphism either by the identities

$$(9.23)\qquad t = s + \phi(x, a) \qquad [yb = x(\phi + b - \phi)],$$

$$(9.24)\qquad b_i = \phi_i(x, a)$$

or by the inverted identities

$$(9.25)\qquad s = t + \psi(y, b) \qquad [xa,\ y(\psi + a - \psi)],$$

$$(9.26)\qquad a_i = \psi_i(y, b).$$

Now, starting with the generators (9.15) and identities (9.16), let us adjoin successively the generators b_i of the group B along with the corresponding relations (9.24) expressing these last in terms of the generators of the group A. Moreover, let us next adjoin successively relations (9.22) and (9.26), in which we think of y as expressed in terms of

$$x,\ a_1,\ \cdots,\ a_m,\ b_1,\ \cdots,\ b_n$$

by means of relation (9.23). We finally obtain the group A determined by the generators

$$s,\ a_1,\ \cdots,\ a_m,\ b_1,\ \cdots,\ b_n$$

with the defining identities (9.16), (9.22), (9.24) and (9.26). Moreover, by (ii) and (i), the matrix $\|X'_{ij}\|$ corresponding to this new mode of definition is ϵ-equivalent to the matrix $\|X_{ij}\|$.

By a similar argument, we may define the group B by means of the generators

$$t,\ a_1,\ \cdots,\ a_m,\ b_1,\ \cdots,\ b_n$$

along with the same defining identities (9.16), (9.24), and (9.26), where the matrix $\|Y'_{ij}\|$ corresponding to the new mode of definition is ϵ-equivalent to the matrix $\|Y_{ij}\|$. Finally, by (iii) the matrix $\|X'_{ij}\|$ must be identical with the matrix $\|Y'_{ij}\|$ except for a change of notation. Therefore, the matrices $\|X_{ij}\|$ and $\|Y_{ij}\|$ must be ϵ-equivalent.

If two indexed groups A and B are inversely equivalent, the matrix $\|X_{ij}\|$ *associated with the group A goes over into a matrix which is* ϵ*-equivalent to the matrix* $\|Y_{ij}\|$ *associated with the group B if we make the change of marks* $x' = x^{-1}$.

The proof is similar to that of the previous theorem. The one essential difference is that in place of relation (9.19) we must write

$$s = -t + \phi(y, a) ;$$

whence, in place of (9.20), we have

$$xa = y^{-1}(\phi + a - \phi).$$

The matrices $\|X_{ij}\|$ *and* $\|X'_{ij}\|$ *corresponding to two different ways of defining an indexed group A are* ϵ*-equivalent. Moreover, the effect on the matrix of changing the signs of the indices of all the operations of an indexed group A is that produced by the substitution* $x' = x^{-1}$.

This is, of course, a consequence of the two previous theorems, when A and B are regarded as symbols for the same group.

10. Application to knots. The group R of a knot may evidently be thought of as an indexed group, for with each curve C determining an operation r of the group there is associated a certain integer measuring the number of times (in the algebraical sense) that the curve C winds around or loops the knot. This integer will be defined as the *index* of the operation r. With proper conventions as to what shall be the positive sense of winding around the knot, the index of an operation r_i of the group will evidently be equal to the index of the region r_i diminished by the index of the region r_0 or, if we choose the additive constant at our disposal so as to make the index of the last named region equal to zero, the index of the operation r_i will simply be the index of the region r_i.

Now, let us choose our notation so that r_0 and $r_{\nu+1}$ are two regions with consecutive indices 0 and 1. Moreover, let us denote by p_i the index of a general region r_i. Then, if we make the substitution

$$r_i = p_i s + r_i' \qquad (i = 1, 2, \cdots, \nu), \tag{10.1}$$

$$r_{\nu+1} = s$$

the new set of generators

$$s, r_1', r_2', \cdots, r_\nu'$$

will evidently be in canonical form, for the index of the first one will be unity and the indices of the others zero. Let us examine the form that the defining relations

$$r_i - r_j + r_k - r_l = 0 \tag{10.2}$$

of the group R take when written in the abridged notation. If an equation (10.2) corresponds to a right handed crossing point of index p it may be expressed as

$$ps + r_i' - r_j' - (p+1)s + ps + r_k' - r_l' - (p-1)s = 0$$

in terms of the canonical generators. Therefore, if we put

$$xr' = s + r' - s$$

it reduces, after we leave off the primes, to

$$xr_i - xr_j + r_k - r_l = 0.$$

A similar reduction leading to the same final result may be made if equations (10.2) correspond to a left-handed crossing point. Therefore,

The equations of the diagram taken in conjunction with two more equations of the form

$$r_0 = 0, \quad r_{\nu+1} = 0,$$

corresponding to regions with consecutive indices are the equations of the group of the knot written in abridged notation.

In other words, the matrix M_0, §5, obtained by striking out two columns of consecutive indices from the matrix M is simply the matrix $\|X_{ij}\|$, §9, of the group equations written in abridged notation. This gives us a second proof of the ϵ-invariantive character of the matrix M from which most of the other theorems in §§4 and 5 are immediately deducible.

11. **Links.** A *link* will be defined as a figure composed of the vertices and sensed edges of a finite number of non-intersecting knots. The most obvious link invariant is the number of knots into which the link may be resolved. We shall call this number the *multiplicity* μ of the link. A knot will thus be a link of multiplicity one. Evidently, the entire discussion up to this point applies not only to knots but to links of arbitrary multiplicities. That is to say, with every link there will be associated a matrix M having the same ϵ-invariantive significance as for the case of a knot, an invariant polynomial $\Delta(x)$, and so on. In the case of a link of higher multiplicity a broader generalization is, however, possible, as we shall now indicate.

Let L be a link of multiplicity μ made up of the elements of μ different knots

$$K_1,\ K_2,\ \cdots,\ K_\mu. \tag{11.1}$$

Then, at each crossing point c_i of the diagram of the link L the lower branch will belong to some knot K_a of system (11.1), the upper branch to some knot K_b which may, or may not be the same as the knot K_a. To the crossing point c_i we shall attach the number a associated with the knot K_a determined by the lower branch through the point. Moreover, we shall replace the equation (3.3) of the diagram associated with the crossing point c_i by a similar equation

$$x_a r_j - x_a r_k + r_l - r_m = 0, \tag{11.2}$$

where the coefficient x of the original equation has been replaced everywhere in the equation by the coefficient x_a. The matrix M_μ of the system of equations (11.2) determined by the various crossing points c_i will thus be an array in the marks 0, ± 1, and $\pm x_a$ $(a=1, 2, \cdots, \mu)$. It will reduce to the matrix M, as defined in §4, if we replace all of the marks x_a by one single mark x.

Two matrices M_μ will be said to be ϵ-equivalent if it is possible to transform one of them into the other by means of a finite number of elementary operations (α), (β), (γ), (δ), §5, in combination with a finite number of operations of the following type:

(ϵ) Multiplication or division of a row (column) by x_a.

This last operation is, of course, the natural generalization of the operation of the same name defined in §4 for the case where we have a simple mark x.

We now have the following broad generalization of one of the theorems of §5:

If two diagrams represent links of the same type their matrices M_μ are ϵ-equivalent.

The theorem may be verified directly by the elementary method of §6. It may also be derived by group theoretical considerations analogous to those developed in §§9 and 10. We shall indicate, briefly, the second method of proof.

The group A of a link of multiplicity μ is a *μ-tuply indexed group*; for with each operation a of the group there may be associated a composite index

$$(p_1,\ p_2,\ \cdots,\ p_\mu) \tag{11.3}$$

such that the number p_i is the linkage number of a curve determining the operation with the ith component knot K_i of the group. By a process analogous to the one used in finding the highest common factor of a set of integers, it is easy to reduce any set of generators of the group A to the *canonical form*

$$s_1, s_2, \cdots, s_\mu, a_1, \cdots, a_m, \tag{11.4}$$

where the index of each generator s_i of the first type is composed of zeros except for the number p_i which is one, and where the index of each generator a_i of the second type is composed exclusively of zeros. The identical relations in the generators will be certain linear expressions of the form

$$c_j(s_1, \cdots, s_\mu, a_1, \cdots, a_m) = 0. \tag{11.5}$$

We shall denote by A' the group determined by the relations (11.5) together with all additional relations of the form

$$s_i + s_j = s_j + s_i \tag{11.6}$$

expressing that any two generators (11.4) of the first type are commutative. Moreover, we shall denote by A^* the self conjugate subgroup of the group A' consisting of all operations of the group A' of index $(0, 0, \cdots, 0)$. Let us now use the *abridged notation*

$$x_i a = s_i + a - s_i. \tag{11.7}$$

Then, by an obvious extension of the argument used in §9, we may show that the operations of the group A^* are precisely the ones which may be represented by sums of the form

$$\sum \pm x_1^{\lambda_{1ij}} x_2^{\lambda_{2ij}} \cdots x_\mu^{\lambda_{\mu ij}} a_{ij}.$$

Finally, if we impose a further set of relations making any two operations of the group A^* commutative, we obtain a group A_c^* consisting of all operations which may be represented in the form

$$\sum_i X_i a_i, \tag{11.8}$$

where X_i is a polynomial in the marks $x_1, x_2, \cdots, x_\mu$. In the last expression, addition is, of course, to be regarded as commutative. The group A_c^* will be the one determined by the generators

$$a_1, a_2, \cdots, a_m$$

together with the identities

$$\sum X_{ij} a_j = 0 \tag{11.9}$$

to which the identities (11.5) reduce when expressed in the abridged notation and when addition is regarded as commutative.

By the methods of §10 it may be verified without difficulty that the matrix of the group A of a link is the matrix $\|X_{ij}\|$ of the coefficients in (11.8).

To obtain the direct generalization of the theory developed for knots, we must set

$$x_1 = x_2 = \cdots = x_\mu = x.$$

A certain number of easily calculable invariants are obtainable by setting all but one of the marks x_i equal to unity.

12. **Miscellaneous theorems.** Let the number of regions of a link diagram be $\nu+2$. Then, if the curve of the diagram is a connected point set the number of crossing points must be ν, just as the special case of a knot diagram. If the curve of the diagram is not a connected point set but is made up of $\kappa+1$ connected pieces, the number of crossing points is only $\nu-\kappa$. We must then adjoin to the equations of the diagram a set of κ equations of the form $0=0$ so that the matrices M and N, §4, shall have two less rows than columns. For we want the matrix M_0 from which we compute the invariant $\Delta(x)$ to be a square array of order ν. If the number κ is greater than unity the invariant $\Delta(x)$ evidently vanishes.

Several theorems are to be obtained by observing that when we set x equal to 1 in the matrix $M(x)=M$ the form of the resulting matrix $M(1)$ will be independent of how the corners at the crossing points are dotted and, therefore, independent of which branch through a crossing point is regarded as the one passing under the other. Suppose, for example, we start with the theorem that the invariant $\Delta(x)$ of an unknotted knot is unity, as may be verified, at once, by direct calculation. Then, as an immediate consequence, we may obtain to the following theorem about knots in general:

The sum of the coefficients of the invariant $\Delta(x)$ of a knot is always numerically equal to unity.

For, in the notation of §4, we have

$$\Delta(x) = \pm\, x^p \Delta_{r(r+1)}(x), \tag{12.1}$$

whence, the sum of the coefficients of the invariant $\Delta(x)$ must be given by

$$\Delta(1) = \pm\, \Delta_{r(r+1)}(1).$$

Now, by changing upper into lower branches at a suitably chosen set of crossing points we may always "unknot" the knot. For one obvious way of doing this is to reverse crossings in such a manner that if we start at a specified point P of the curve of the diagram and describe the curve in the positive sense we never pass through a crossing point along an upper branch without previously having passed through it along a lower one. But, as we have already observed, such a reversal of crossings leaves invariant the value of the determinant $\Delta_{n(n+1)}(1)$. Therefore, since $\Delta(x)$ is unity for an unknotted knot, we have

$$1 = \pm \Delta_{r(r+1)}(1),$$

which proves the theorem.

The multiplicity μ of a link L is equal to the number of zero elementary factors of the matrix $M(1)$ obtained by setting x equal to 1 in the matrix M.

For, by an elementary calculation, we verify that the theorem is true when the link L consists of μ unknotted and non-interlinking curves.

It should be remembered that if we assign a constant integral value c to x in the matrix M and then derive the elementary factors of the matrix $M(c)$, regarding the latter as a matrix in integer elements, we do not necessarily get the same result as if we derived the elementary factors of the matrix $M(x)$ regarded as a matrix in polynomial elements and then substituted the value c for x in these factors. For in calculating the factors of the matrix $M(x)$ we have to allow the operation of adding to one row (column) a rational multiple of another row (column), which operation is not allowed in calculating the factors of the matrix $M(c)$ unless the rational multiple is also integral. It may, therefore, be worth while noting the following theorem:

The elementary factors of the matrix $M(x)$ (c integral) that are not of the form $\pm c^p$ are all link invariants to within a multiple of c.

For the matrix operations (α), (β), and (γ) of §4 leave the elementary factors of $M(c)$ unaltered, the operation (δ) merely adds or takes away a unit factor, and the operation (ϵ) merely multiplies or divides one factor by c.

Let $\nu+2$ be the number of regions of a link diagram and p the number of distinct values taken on by the indices of the various crossing points (corresponding to any one way of assigning indices). Then the degree of the polynomial $\Delta(x)$ never exceeds $\nu-p$.

For, in the notation of §4, the degree of the polynomial $\Delta(x)$ never exceeds that of the polynomial $\Delta_{p(p+1)}(x)$. Moreover, by (4.6) we have

$$\Delta_{01}(x) = \pm x^p \Delta_{p(p+1)}(x).$$

But the degree of $\Delta_{01}(x)$ cannot exceed ν since this expression is a ν-rowed determinant with elements that are linear in x. Therefore, the degree of $\Delta_{p(p+1)}(x)$, and consequently also of $\Delta(x)$, never exceeds $\nu-p$.

If K is a composite knot, §3, made up of the factors K_1 and K_2, the invariant $\Delta(x)$ of the knot K is equal to the product of the corresponding invariants of the knots K_1 and K_2.

A composite knot is one which may be deformed in such a way that its diagram will be of the sort considered in the last paragraph of §3, where two edges e_1 and e_2 appear on the boundaries of two different regions r_1 and r_2 of the diagram. In the notation of §3, let $\alpha+\gamma_1$ and $\alpha+\gamma_2$ be the curves of the diagram of the two factors K_1 and K_2 of K. Moreover, let $\Delta_{r(r+1)}(x)$ be the determinant of the knot K after elimination from its matrix M of the two redundant columns corresponding to the regions r_1 and r_2 respectively. Thus, clearly the determinant $\Delta_{r(r+1)}(x)$ is equal to the product of the two similar determinants $\Delta'_{r(r+1)}(x)$ and $\Delta''_{r(r+1)}(x)$ corresponding to the two factors K_1 and K_2; for the determinant $\Delta_{r(r+1)}(x)$ is related to these other two in the manner illustrated schematically by

$$\Delta_{r(r+1)}(x) = \left|\begin{array}{c|c} \Delta'_{r(r+1)} & 0 \\ \hline 0 & \Delta''_{r(r+1)} \end{array}\right|.$$

The theorem therefore follows, at once.

We shall bring these miscellaneous remarks to a close by obtaining a relation between the polynomial invariants $\Delta(x)$ of three closely associated links. Consider a link diagram Z' with a right handed crossing point of index p (Fig. 2a). By changing this one crossing point into a left handed one (Fig. 2b) we obtain a new diagram Z''. Moreover, by cutting the two branches at the crossing point, separating them slightly, and rejoining them so as to unite into one the two regions of index p incident to the crossing point (Fig. 2c) we obtain a third diagram Z. Now, suppose we form the matrix N (§4) of the diagram Z' and arrange the rows and columns in such an order that the first row corresponds to the crossing point c and that the first four columns correspond to the four regions incident to the point c and represented in Fig. 2a by the first, second, third and fourth quadrants respectively. The first row of the matrix N' will, thus, start with the elements x, x, 1, 1 followed by zeros. To obtain the corresponding matrix N'' of the diagram Z we shall merely have to interchange the first and third elements in the first row of the matrix N'; to obtain the corresponding matrix N of the diagram Z we shall merely have to add the first column of the matrix N' to the third and then strike out the first row and first column. Now, let $\Delta'_{p(p+1)}$, $\Delta''_{p(p+1)}$ and $\Delta_{p(p+1)}$ be the determinants of the matrices obtained by striking out the first two columns of N', N'', and N, respectively. Then, clearly, the determinant $\Delta_{p(p+1)}$ will be the principal minor of both the determinants $\Delta'_{p(p+1)}$ and $\Delta''_{p(p+1)}$. Let Γ be the minor of the second element in the first row of $\Delta'_{p(p+1)}$ (and of $\Delta''_{p(p+1)}$). Then if we expand each of the determinants $\Delta'_{p(p+1)}$ and $\Delta''_{p(p+1)}$ in terms of the elements of its first row and their minors, we find

$$\Delta'_{p(p+1)} = \Delta_{p(p+1)} - \Gamma, \quad \Delta''_{p(p+1)} = x\Delta_{p(p+1)} - \Gamma,$$

whence,

$$\Delta'_{p(p+1)} - \Delta''_{p(p+1)} = (1 - x)\Delta_{p(p+1)}, \tag{12.2}$$

which is the relation we had in mind to establish.

The argument needs to be slightly modified if the two corners of index p at the crossing point c belong to the same region of the diagram. In this case, however, the curve of the diagram Z must be a disconnected point set, whence,

$$\Delta_{p(p+1)} = 0.$$

Moreover, we obviously have

$$\Delta'_{p(p+1)} = \Delta''_{p(p+1)},$$

so that relation (12.2) continues to hold.

13. n**-sheeted spreads.** Further invariants of the matrix M are to be obtained by regarding each element as the symbol for a certain square array of order n, where the connection between the symbols and the arrays which they represent is as follows:

(i) 0 is the symbol for an array composed entirely of zeros.

(ii) 1 is the symbol for an array with 1's along the main diagonal and 0's elsewhere.

(iii) x is the symbol for an array obtained from the array 1 either by permuting the columns cyclically so that the first column goes into the second or by permuting the rows cyclically so that the second row goes into the first; the effect of either permutation is the same. x^p is the symbol for the array obtained from the array 1 by making p successive permutations of the type just described.

Now, if we replace each element of the matrix M by the array which it symbolizes we obtain a new array M^n of νn rows and $(\nu+2)n$ columns, made up of integer elements.

The elementary factors of the array M^n that differ from unity are link invariants.

For to an elementary operation (α), (β), (γ), or (δ) on the matrix M there evidently corresponds an operation on the matrix M^n which may be resolved into elementary operations. Moreover, to an extended operation (ϵ) on the matrix M there merely corresponds a cyclical permutation of a block of n rows (columns) of the matrix M^n, which may again be resolved into elementary operations. This proves the theorem.

Corresponding to a pair of redundant columns of the matrix M there will be two sets of redundant columns of the matrix M^n composed of n columns each. We may, therefore, always replace the matrix M^n by a square matrix M_0^n of order νn.

The elementary factors of the array M_0^n have an interesting geometrical significance. If the link from which we start is a knot, the number of zero factors is equal to the connectivity numbers,

$$(P_1 - 1) = (P_2 - 1),$$

of an n-sheeted Riemann spread S^n (the 3-dimensional generalization of an n-sheeted Riemann surface) with the knot as branch curve (generalized branch point). The divisors that are different from zero and unity are the *coefficients of torsion* of the spread S^n.* This may all be proved very easily by making a suitable cellular subdivision of the spread S^n, as we shall now indicate.†

Let K be the knot under consideration and S the space containing it, where to simplify matters, we shall suppose that the space S closes up to a single point at infinity. Then, the first step will be to cut up the space S into cells, in the following manner. Wherever one branch of the knot appears to pass behind another, we shall join the upper branch to the lower one by a segment c_i'. The ends of the segment c_i' will be the vertices, or 0-cells, of the subdivision; the segments c_i' themselves, together with the arcs a_i' into which the ends of the segments c_i' subdivide the knot will be the 1-cells. Corresponding to each region r_i of the diagram we shall construct a 2-cell r_i' of which r_i will be the projection, bounded by the appropriate arcs a_i' and c_i'. The residual part of the space S will then consist of a pair of 3-cells. Thus, to sum up, the subdivision Σ that we have just described will consist of 2ν vertices, 3ν edges, $\nu+2$ 2-cells, and 2 3-cells. Corresponding to the subdivision Σ of the space S there will be a subdivision Σ^n of the spread S^n

* In a paper read before the National Academy in November, 1920, cf. Veblen, Cambridge Colloquium Lectures (1922), *Analysis Situs*, p. 150, I made the observation that the topological invariants of the n-sheeted spreads associated with a knot would be invariants of the knot itself, and showed by actual calculation that these invariants could be used to distinguish between a number of the more elementary knots. Later, these same invariants were discovered independently, by F. Reidemeister, *Knoten und Gruppen*, Abhandlungen aus dem Mathematischen Seminar der Hamburgischen Universität, 1926, pp. 7–23. See also a paper by Alexander and Briggs. *On types of knotted curves*, Annals of Mathematics, (2), vol. 28 (1927), pp. 563–586.

† Cf. Alexander and Briggs, loc. cit.

such that each cell of the subdivision Σ which is not composed of points of the knot will be covered by n cells of the subdivision Σ^n, one in each sheet of the spread S^n, and such that each cell of the subdivision Σ which is composed of points of the knot will be covered by just one cell of the subdivision Σ^n. Along the knot, of course, the n sheets of the spread S^n merge into one.

We may simplify the subdivision Σ^n somewhat by amalgamating all but one of the 2ν edges covering the edges a_i', along with their end points, into one single 0-cell; or, if we prefer, by treating this group of 0-cells and 1-cells as a single generalized vertex A. The remaining 1-cells of the subdivision will then represent closed 1-circuits beginning and ending at the vertex A, while the boundaries of the various 2-cells will give the relations of bounding among these circuits. Clearly the number of 1-circuits will be $\nu n+1$, as there will be n circuits

$$c_{ij}' \qquad (j = 1, 2, \cdots, n), \tag{13.1}$$

corresponding to each segment c_i', together with one additional circuit c corresponding to the residual arc of the branch system that has not been amalgamated into the generalized vertex A. Moreover, there will be $\nu n+2$ relations among these circuits corresponding to the $\nu n+2$ 2-cells

$$r_{ij}' \qquad (j = 1, 2, \cdots, n) \tag{13.2}$$

covering the 2-cells r_i' of the subdivision Σ.

Now, it is easy to verify that the matrix M^n is precisely the one exhibiting the incidence relations between the 1-circuits (13.1) and 2-cells (13.2). Therefore, to obtain a matrix exhibiting the incidence relations between all the 1-circuits and 2-cells we have only to adjoin to the matrix M^n a new row corresponding to the remaining 1-circuit c. As the 1-circuit c_i is on the boundary of two blocks of n 2-cells each, corresponding to a pair of contiguous regions r_0 and $r_{\nu+1}$ of the diagram, the added row will consist of a block of n 1's, a block of n -1's, and zeros. The elementary factors of the matrix with the added row will be the ones that determine the connectivity numbers and coefficients of torsion of the spread S^n. But this last matrix is evidently equivalent to the matrix M_0^n, for the $2n$ columns having the elements ± 1 in the last row correspond to the redundant columns of the matrix M^n; hence, by adding to these columns suitable linear combinations of the remaining νn ones we may make all their elements zero except the ones in the last row. Thus, the geometrical interpretation of the divisors of the matrix M_0^n is established.

Since the matrix with the added row may be transformed into one such that in certain columns all the elements will be zeros with the exception of an element 1 in the last column, it follows that the 1-circuit c must be a bounding curve,

$$c \sim 0\,;$$

for each column determines a relation of bounding among the 1-circuits determined by the rows. But the 1-circuit c, when we include the generalized point A which really forms a part of it, is simply the branch curve of the spread S^n itself. Hence we have the theorem†

The branch curve of the n-sheeted spread determined by a knot K is always a bounding curve of the spread.

The geometrical interpretation of the factors of the matrix M^n is not quite so satisfactory for the case of a link of multiplicity greater than unity. However, by a similar argument to the one made above, it is easy to show that these factors give the connectivity numbers and coefficients of torsion of the spread S^n when we treat the entire branch system of S^n as if it were a single generalized point.

14. **Tabulation of $\Delta(x)$.** At the end of the paper referred to in the last footnote a chart has been drawn up showing diagrams of the eighty-four knots of nine or less crossings listed as distinct by Tait and Kirkman; also a table giving the torsion numbers of the 2- and 3-sheeted Riemann spreads

3_{1a}	1−1	7_{7a}	1−5+9	9_{7a}	3−7+9	9_{11a}	1−5+7−7
4_{1a}	1−3	9_{45n}	1−6+9	8_{15a}	3−8+11	9_{17a}	1−5+9−9
5_{2a}	2−3	9_{47n}	1−7+11	9_{25a}	3−12+17	9_{20a}	1−5+9−11
6_{1a}	} 2−5*	8_{12a}	1−7+13	9_{41a}	3−12+19	9_{22a}	1−5+10−11
9_{46n}		7_{3a}	2−3+3	9_{39a}	3−14+21	8_{18a}	} 1−5+10−13*
7_{2a}	3−5	7_{5a}	2−4+5	9_{10a}	4−8+9	9_{24a}	
8_{1a}	3−7	8_{4a}	2−5+5	9_{13a}	4−9+11	9_{26a}	1−5+11−13
7_{4a}	} $4-7^{+}$	8_{6a}	2−6+7	9_{18a}	4−10+13	9_{27a}	1−5+11−15
9_{2a}		8_{8a}	2−6+9	9_{23a}	4−11+15	9_{28a}	} $1-5+12-15^{+}$
8_{3a}	4−9	8_{11a}	2−7+9	9_{38a}	5−14+19	9_{29a}	
9_{5a}	6−11	8_{13a}	2−7+11	8_{19n}	1−1+0+1	9_{30a}	1−5+12−17
9_{35a}	7−13	8_{14a}	} $2-8+11^{+}$	7_{1a}	1−1+1−1	9_{31a}	1−5+13−17
5_{1a}	1−1+1	9_{8a}		9_{43n}	1−3+2−1	9_{32a}	1−6+14−17
9_{42n}	1−2+1	9_{12a}	2−9+13	8_{2a}	1−3+3−3	9_{33a}	1−6+14−19
8_{20n}	1−2+3	9_{14a}	2−9+15	8_{5a}	1−3+4−5	9_{34a}	1−6+16−23
6_{2a}	1−3+3	9_{15a}	2−10+15	8_{7a}	1−3+5−5	9_{40a}	1−7+18−23
6_{3a}	1−3+5	9_{19a}	2−10+17	8_{9a}	1−3+5−7	9_{3a}	2−3+3−3
8_{21n}	} 1−4+5*	9_{21a}	2−11+17	8_{10a}	1−3+6−7	9_{6a}	2−4+5−5
9_{36a}		9_{37a}	2−11+19	9_{48n}	1−4+6−5	9_{9a}	2−4+6−7
9_{44n}	1−4+7	9_{4a}	3−5+5	8_{16a}	1−4+8−9	9_{16a}	2−5+8−9
7_{6a}	1−5+7	9_{49n}	3−6+7	8_{17a}	1−4+8−11	9_{1a}	1−1+1−1+1

† A and B, loc cit.

associated with the knots. We list below the values of the polynomial $\Delta(x)$ for these same knots. It appears that the polynomial $\Delta(x)$ of a knot is always of even degree and that its coefficients are arranged symmetrically with reference to the middle one. Therefore, to reduce the space occupied by the table we have merely indicated the values of the coefficients of $\Delta(x)$ up to, and including, the middle one. To illustrate how the table is to be used, let us turn, for example, to the entry

$$\|9_{33a} \mid 1 - 6 + 14 - 19\|.$$

This indicates that an alternating knot of nine crossings listed as the thirty-third in the chart referred to above has the polynomial invariant

$$\Delta(x) = 1 - 6x + 14x^2 - 19x^3 + 14x^4 - 6x^5 + x^6.$$

In the table, six repetitions of the same polynomial appear. In the three cases marked with (*) the two paired knots may be distinguished with the aid of other invariants of the matrix M, such, for example, as the coefficients of torsion of the associated Riemann spaces. In the three cases marked with ($^+$) the two paired knots have ϵ-equivalent matrices M and, therefore, cannot be distinguished (assuming that they actually are distinct, which Tait never really proves) by the methods of this paper.

PRINCETON UNIVERSITY,
PRINCETON, N. J.