Math 435 — Algebra Notes — Week 4

LK

(Problems are at the end of the notes.)

1° Linear Transformations and Matrices

$\mathbb{R}^n = \{(a_1, a_2, \ldots, a_n) \mid a_i \in \mathbb{R} \text{ for } i = 1, \ldots, n\}$.

$\mathbb{R}$ = the real numbers

$\mathbb{C} = \{a + bi \mid a, b \in \mathbb{R},\ i^2 = -1\}$ = the complex numbers.

$e^{i\theta} = \cos(\theta) + i\sin(\theta)$

(See the notes on complex numbers for more about $e^{i\theta}$ as a series.)

We know from previous notes that $e^{i\theta}e^{i\varphi} = e^{i(\theta+\varphi)}$, hence have geom interpretation of complex nos:

$$a+bi = \sqrt{a^2+b^2}\left(\frac{a}{\sqrt{a^2+b^2}} + \frac{b}{\sqrt{a^2+b^2}}\,i\right)$$
$$= r\left(\cos(\theta) + i\sin(\theta)\right)$$
$$= re^{i\theta} \qquad ((a,b) \neq (0,0)).$$

A mapping (function) $T : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is said to be <u>linear</u> if

(a) $T(kv) = kT(v)$

& (b) $T(v+w) = T(v) + T(w)$

$\forall k \in \mathbb{R};\ v, w \in \mathbb{R}^n$

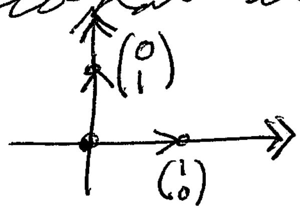
$$\boxed{\begin{array}{l} k(a_1, \ldots, a_n) = (ka_1, \ldots, ka_n) \\ (a_1, \ldots, a_n) + (b_1, \ldots, b_n) = (a_1+b_1, \ldots, a_n+b_n) \end{array}}$$

Here we work with $\mathbb{R}^2$. $\forall \begin{pmatrix} a \\ b \end{pmatrix} \underset{\text{def}}{=} (a, b)$.

$$T\begin{pmatrix} a \\ b \end{pmatrix} = T\left(a\begin{pmatrix} 1 \\ 0 \end{pmatrix} + b\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right)$$

$$= aT\begin{pmatrix} 1 \\ 0 \end{pmatrix} + bT\begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Thus a linear transformation is completely determined by what it does on $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ & $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$.



Suppose $T\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} r \\ s \end{pmatrix}$

$T\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} t \\ u \end{pmatrix}$.

Then $T\begin{pmatrix} a \\ b \end{pmatrix} = a\begin{pmatrix} r \\ s \end{pmatrix} + b\begin{pmatrix} t \\ u \end{pmatrix} = \begin{pmatrix} ar+bt \\ as+bu \end{pmatrix}$.

Defn. $$\underbrace{\begin{pmatrix} r & t \\ s & u \end{pmatrix}}_{\substack{2\times 2 \\ \text{matrix}}} \underbrace{\begin{pmatrix} x \\ y \end{pmatrix}}_{\text{vector}} \underset{\text{def}}{=} \underbrace{\begin{pmatrix} rx+ty \\ sx+uy \end{pmatrix}}_{\substack{\text{result of} \\ \text{applying} \\ \text{matrix to} \\ \text{vector.}}}$$

$\boxed{\text{Matrices represent linear transformations.}}$

We write $[T] = \begin{pmatrix} r & t \\ s & u \end{pmatrix}$.

Suppose $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} r & t \\ s & u \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} rx + ty \\ sx + uy \end{pmatrix}$

$$S\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & c \\ b & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + cy \\ bx + dy \end{pmatrix}$$

are two given linear transformations.

<u>Theorem</u>. The matrix for $T \circ S$ is given by the formula

$$[T \circ S] = \begin{pmatrix} ra+tb & rc+td \\ sa+ub & sc+ud \end{pmatrix}.$$

Thus we define the <u>product of matrices</u> by the formula:

$$\boxed{\begin{pmatrix} r & t \\ s & u \end{pmatrix}\begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{pmatrix} ra+tb & rc+td \\ sa+ub & sc+ud \end{pmatrix}.}$$

<u>Proof</u>. $(T \circ S)\begin{pmatrix} x \\ y \end{pmatrix} = T\left(S\begin{pmatrix} x \\ y \end{pmatrix}\right) = T\begin{pmatrix} ax + cy \\ bx + dy \end{pmatrix}$

$$= \begin{pmatrix} r(ax+cy) + t(bx+dy) \\ s(ax+cy) + u(bx+dy) \end{pmatrix}$$

$$= \begin{pmatrix} (ra+tb)x + (rc+td)y \\ (sa+ub)x + (sc+ud)y \end{pmatrix} = \begin{pmatrix} ra+tb & rc+td \\ sa+ub & sc+ud \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}.$$

//

Examples. (i) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ x \end{pmatrix}$.

The matrix $\eta = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ permutes the coordinates x, y. $\eta^2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$.
We let $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ denote the identity matrix.
Note $I\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$.

(ii) $J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, then

$$J^2 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = -I.$$

Thus $J^2 = -I$.
(Note. $k\begin{pmatrix} a & c \\ b & d \end{pmatrix} \underset{def}{=} \begin{pmatrix} ka & kc \\ kb & kd \end{pmatrix}$)
$J =$ "$\sqrt{-I}$", a matrix analog of $\sqrt{-1} = i$.

(iii) $aI + bJ = a\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + b\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

$$= \begin{bmatrix} a & -b \\ b & a \end{bmatrix}.$$

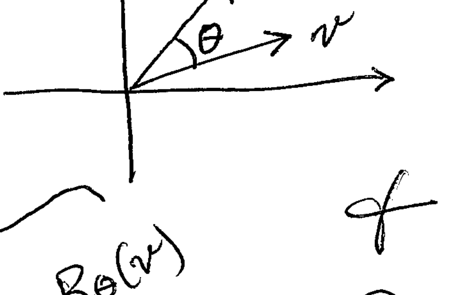
$\hat{\mathbb{C}} = \{ aI + bJ \mid a, b \in \mathbb{R} \}$ behaves just like the complex numbers $\mathbb{C}$.
In fact, (next page)

$$\cos(\theta) I + \sin(\theta) J = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} = R_\theta$$

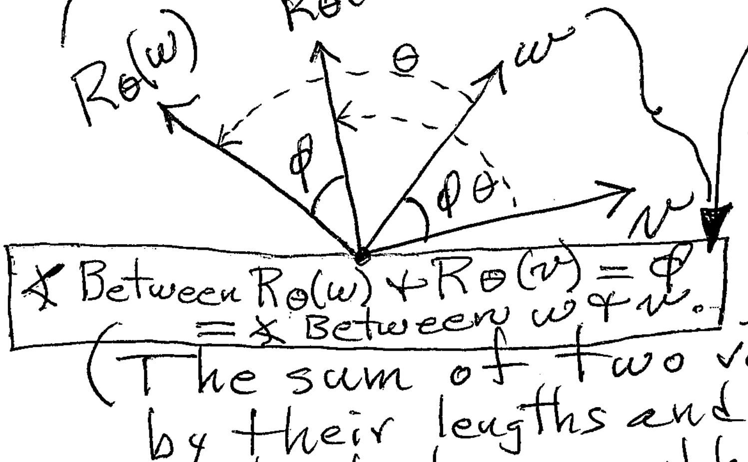
our familiar rotation matrix corresponding to multiplication by $e^{i\theta}$.

2° Rotation Revisited

Let R_θ denote rotation of $\mathbb{R}^2$ counterclockwise by angle θ. Suppose this is <u>all</u> we know about R_θ.



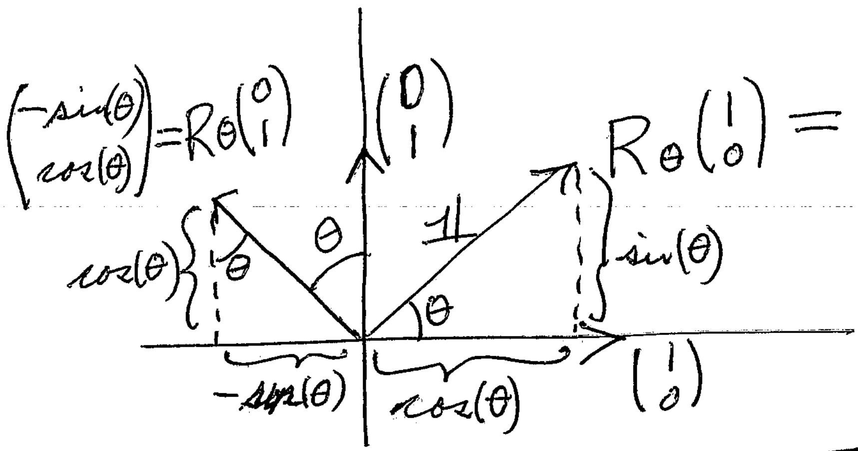
But this means that we <u>do know</u> that R_θ <u>preserves lengths</u>, & R_θ <u>preserves angles between vectors</u>.



<u>This means that R_θ is a linear transformation.</u>

(The sum of two vectors is determined by their lengths and directions & the angle between them. This implies that $R_\theta(v+w) = R_\theta(v) + R_\theta(w)$. Since R_θ preserves lengths, it follows ~~that~~ $R_\theta(kv) = kR_\theta(v)$.)

Since R_θ is linear, it is represented by a matrix, and the matrix is determined by $R_\theta\begin{pmatrix}1\\0\end{pmatrix}$ and $R_\theta\begin{pmatrix}0\\1\end{pmatrix}$.



$$\Longrightarrow \quad [R_\theta] = \left[R_\theta\begin{pmatrix}1\\0\end{pmatrix} \;\; R_\theta\begin{pmatrix}0\\1\end{pmatrix} \right] = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

(matrix of)

columns

The columns of our familiar rotation matrix are determined & so we can write

$$R_\theta = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$$

<u>without using any trig identities!</u>

But we know, from the geometric definition of R_θ as a rotation, that $R_{(\theta+\varphi)} = R_\theta R_\varphi$ where $R_\theta R_\varphi$ is the matrix product.

$$\therefore \begin{pmatrix} \cos(\theta+\varphi) & -\sin(\theta+\varphi) \\ \sin(\theta+\varphi) & \cos(\theta+\varphi) \end{pmatrix} = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} \cos(\varphi) & -\sin(\varphi) \\ \sin(\varphi) & \cos(\varphi) \end{pmatrix}$$

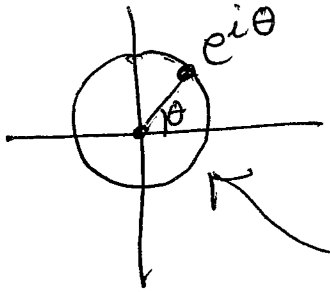
This implies (by multiplying the matrices on the right) ~~that~~

$$\cos(\theta+\varphi) = \cos(\theta)\cos(\varphi) - \sin(\theta)\sin(\varphi)$$
$$\sin(\theta+\varphi) = \sin(\theta)\cos(\varphi) + \cos(\theta)\sin(\varphi).$$

We have deduced basic trigonometry from the linearity of rotations.

Note that $\left\{ \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = R_\theta \;\middle|\; 0 \leq \theta \leq 2\pi \right\}$

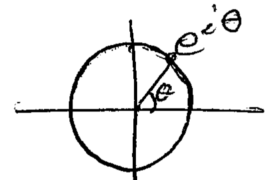
$\updownarrow$ 1-1 correspondence



$\longleftrightarrow \left\{ e^{i\theta} \;\middle|\; 0 \leq \theta \leq 2\pi \right\} = S^1$

the set of points on the unit circle in the complex plane.

$$S^1 = \{ e^{i\theta} \mid 0 \leq \theta \leq 2\pi \} \longleftrightarrow$$



is a group under multiplication of complex numbers.

We can extend this to a larger group $O(1) = \underbrace{\langle S^1 \cup \{F\} \rangle}_{\text{group generated by}}$

where $F(z) = \bar{z}$, the complex conjugate of z.

$$R_\theta \in O(1), \quad R_\theta(z) = e^{i\theta} z$$

Thus
$$(R_\theta F) z = e^{i\theta} \bar{z}$$
$$= \overline{e^{-i\theta}}\, \bar{z}$$
$$= \overline{e^{-i\theta} z}$$

$$(R_\theta F) z = (F R_{-\theta}) z.$$

Thus $R_\theta F = F R_\theta^{-1}$ (since $R_\theta^{-1} = R_{-\theta}$).

Here we view $O(1)$ as the group of all linear transformations of the form $\{R_\theta\} \cup \{R_\theta F\}$.

(next page)

Note $(R_\theta F)(R_\phi F) = (R_\theta F)(F R_{-\phi})$

$$= R_\theta (F^2) R_{-\phi}$$

$$= R_\theta R_{-\phi} = R_{\theta-\phi}.$$

So it is clear that

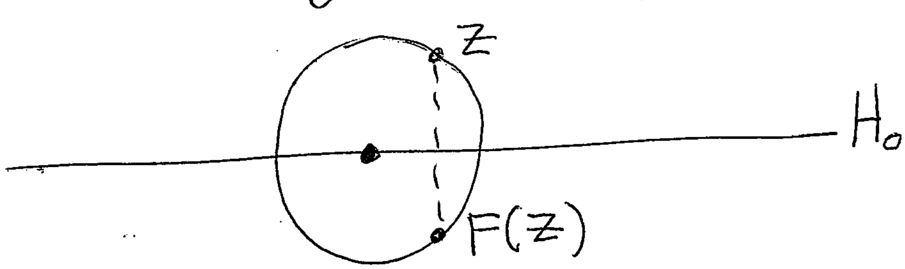
$$O(1) = \{R_\theta \mid 0 \le \theta \le 2\pi\} \cup \{R_\phi F \mid 0 \le \phi \le 2\pi\}$$

is closed under composition and so forms a group. You can show that

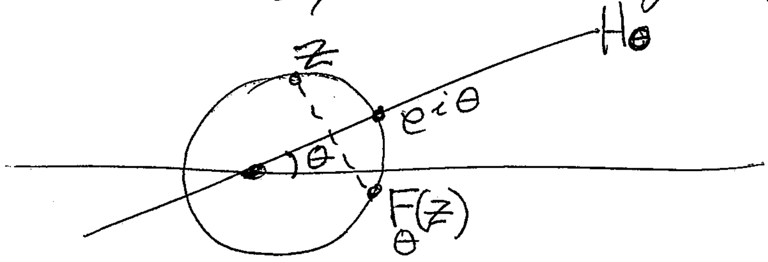
$$O(1) = \left\{ \begin{pmatrix} a & c \\ b & d \end{pmatrix} \,\middle\|\, |ad-bc| = 1 \right\}.$$

Note that $F = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $ad-bc = -1$ for F.

Here is an exercise for you:
$O(1) \ni F$ and F is a flip about the horizontal axis:



Show that there are flips F_θ for each axis at angle θ with $F_\theta \in O(1)$.



Among other things, you want $F_\theta(e^{i\theta}) = e^{i\theta}$.

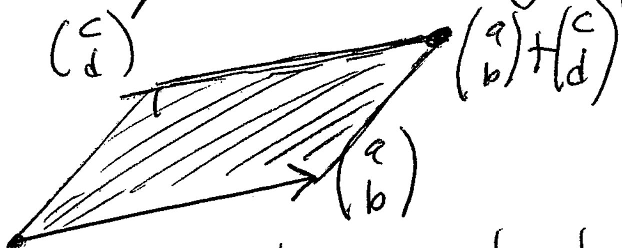
3° Determinants

Definition.

$$\mathrm{Det}\begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{vmatrix} a & c \\ b & d \end{vmatrix} = ad - bc.$$

Fact 1. $\left|\mathrm{Det}\begin{pmatrix} a & c \\ b & d \end{pmatrix}\right| = |ad-bc|$

= area of parallelogram spanned by $\begin{pmatrix} a \\ b \end{pmatrix}$ and $\begin{pmatrix} c \\ d \end{pmatrix}$.



See our linear algebra notes for the proof.

Fact 2. $\begin{pmatrix} d & -c \\ -b & a \end{pmatrix}\begin{pmatrix} a & c \\ b & d \end{pmatrix} = \Delta I$

where $\Delta = \begin{vmatrix} a & c \\ b & d \end{vmatrix} = ad - bc.$

Thus when $\mathrm{Det}(M) \neq 0$, then M has an inverse matrix and $M^{-1} = \frac{1}{\Delta}\,\mathrm{adj}(M)$ where

$\mathrm{adj}(M) = \begin{pmatrix} d & -c \\ -b & a \end{pmatrix}.$

Among other things, this means

$$\textcircled{G} = \left\{ M \,\middle|\, \begin{array}{l} M \text{ 2x2 matrix with real entries} \\ \text{and } Det(M) \neq 0 \end{array} \right\}$$

forms a group.

<u>Theorem.</u> If M and N are 2×2 matrices then $Det(MN) = Det(M)Det(N)$.

<u>Proof.</u> This will be part of your homework. //

e.g. $Det\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = 4 - 6 = -2$

$Det\begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} = 2$

$Det\left(\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}\right) = Det\begin{pmatrix} 1 & 5 \\ 3 & 11 \end{pmatrix} = 11 - 15 = -4 = (-2)(2)$ ✓

4° <u>The Cayley Hamilton Theorem</u>

Define the <u>characteristic polynomial</u> $C_M(x)$ of a 2×2 matrix by the formula:

$$C_M(x) = Det(M - xI).$$

Thus if $M = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$, then

$$C_M(x) = \left| \begin{pmatrix} a & c \\ b & d \end{pmatrix} - x \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right|$$

$$= \left| \begin{pmatrix} a-x & c \\ b & d-x \end{pmatrix} \right| = (a-x)(d-x) - bc$$

$$= ad - (a+d)x + x^2 - bc$$

$$= x^2 - (a+d)x + (ad-bc).$$

$$\boxed{C_M(x) = x^2 - tr(M)x + Det(M)}$$

(where $tr\begin{pmatrix} a & c \\ b & d \end{pmatrix} = a+d =$ sum of diagonal entries.)

<u>Cayley-Hamilton Theorem</u>. Any matrix M is a root of its own characteristic polynomial. That is, if $C_M(x) = x^2 - tr(M)x + Det(M)$, then

$$M^2 - tr(M)M + Det(M)I = 0.$$

note the I.

<u>Proof</u>. This will be an exercise. //

example. $M = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$

$$C_M(x) = x^2 - x - 1.$$

Cayley Hamilton $\Rightarrow M^2 - M - I = 0$

check: $M^2 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$

$$M^2 - M - I = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2-1-1 & 1-1 \\ 1-1 & 1-0-1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

So Cayley Hamilton says ~~that~~ every 2×2 matrix M satisfies a quadratic equation:

$$\boxed{M^2 = tr(M)M - Det(M)I}.$$

example. $M = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

$$C_M(x) = x^2 - (-1)1 = x^2 + 1.$$

Cayley Hamilton $\Longrightarrow M^2 + I = 0.$

We already knew this.

Problems for Week 4

1. Consider a cubic equation of the form
$$x^3 = px + q.$$

(a) Suppose that
$$x^3 - px - q = (x-\alpha)(x-\beta)(x-\gamma).$$
(you do not have to know the values of α, β, γ.)

Prove that

(i) $\alpha + \beta + \gamma = 0.$

(ii) $p = -(\alpha\beta + \alpha\gamma + \beta\gamma).$

(iii) $q = +\alpha\beta\gamma.$

(b) In solving the cubic we do the following: $x = a + b \Longrightarrow$
$$\left.\begin{aligned} a^3b^3 &= p^3/27 \\ a^3 + b^3 &= q \end{aligned}\right\} \begin{aligned} RS &= p^3/27 \\ R + S &= q \end{aligned}$$
$$\Longrightarrow R(q-R) = p^3/27$$
$$\Longrightarrow R^2 - qR + p^3/27 = 0$$
$$\Longrightarrow R = \frac{q \pm \sqrt{q^2 - 4p^3/27}}{2}$$
$$R = \frac{\dfrac{q}{2} \pm \dfrac{\sqrt{27q^2 - 4p^3}}{2\sqrt{27}}}{}$$

Show that ~~if~~
$$D = (\alpha-\beta)(\alpha-\gamma)(\beta-\gamma),$$
then $\boxed{-D^2 = 27q^2 - 4p^3}$.

Either illustrate some examples, or prove in general.

2. See pages 8 + 9 of these notes.
We have $O(1)$ = group generated by $R_\theta : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, $0 \leq \theta \leq 2\pi$ & $F : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, $F(z) = \overline{z}$ $\left(F\binom{a}{b} = \binom{a}{-b}\right)$.

$$\left(R_\theta\binom{a}{b} = \begin{pmatrix} \cos\theta & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}\binom{a}{b}\right)$$

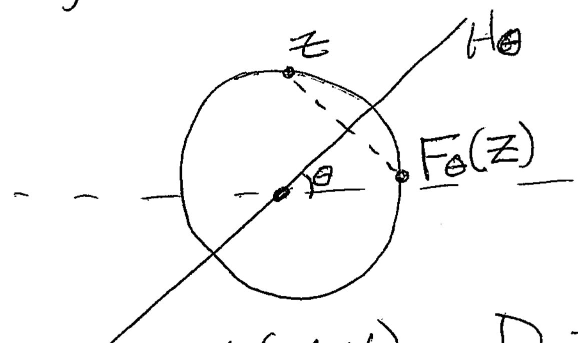
$$R_\theta z = e^{i\theta} z.$$

You can think of $R_\theta, F : \mathbb{C} \longrightarrow \mathbb{C}$.

Define $F_\theta = R_\theta F R_{-\theta}$.

(i) Show $F_\theta(e^{i\varphi}) = e^{i(2\theta - \varphi)}$.

(ii) Show that, geometrically, F_θ is a flip about the axis H_θ whose angle is θ.



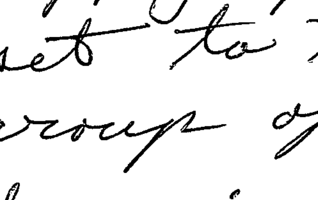
3. Prove that $Det(MN) = Det(M)Det(N)$ for M and N any 2×2 matrices.

4. Let $M = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$. See pages 12, 13 of these notes. Prove that
$$M^2 = (a+d)M - (ad-bc)I.$$
This is the Cayley-Hamilton Theorem for 2×2 matrices.

5. Recall Problem #4 from the Week3 homework.
You had $R = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}$.

Find $C_R(x)$, the characteristic polynomial for R.
Verify that $R^2 + R + I = 0$ is the Cayley-Hamilton Theorem for R.

6. Apply problem 2. of this set to the symmetry group of a <u>square</u> by using $\{1, -1, i, -i\}$ as the vertices of the square and $R(z) = iz$
$$F(z) = \bar{z}$$
as the group generators.



Tell everything you can about this group and its subgroups.