Notes on Groups and Matrices

LK

Let $e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \ldots, e_n = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \in \mathbb{R}^n.$

$\beta = \{e_1, e_2, \ldots, e_n\}$ is a basis for $\mathbb{R}^n$.
Every $v \in \mathbb{R}^n$ can be uniquely written in the form $v = a_1 e_1 + \cdots + a_n e_n$ where $a_1, \ldots, a_n$ are real numbers. Note that

$$a_1 e_1 + \cdots + a_n e_n = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} = a.$$

If M is an $n \times n$ matrix, then

$$\boxed{Me_i = \text{the } i^{th} \text{ column of } M}.$$

$$M = (m_{ij}) \Longrightarrow Me_K = \begin{pmatrix} m_{1K} \\ m_{2K} \\ \vdots \\ m_{nK} \end{pmatrix} = K^{th} \text{ column of } M.$$

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e.g. $\begin{pmatrix} a & b & c \\ d & e & f \\ h & K & l \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} b \\ e \\ K \end{pmatrix}$

$$Me_2 = 2^{nd} \text{ column of } M.$$

An $n \times n$ matrix P is said to be a <u>permutation matrix</u> if

1) Every column is of the form e_i for some i, $1 \le i \le n$.
2) If $c_1, \ldots, c_n$ are the columns of M, then $(c_1, \ldots, c_n)$ is a permutation of $(e_1, \ldots, e_n)$.

$$P = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$ is a permutation matrix.

Note that
$$\left.\begin{aligned} Pe_1 &= e_2 \\ Pe_2 &= e_1 \\ Pe_3 &= e_3 \end{aligned}\right\}$$

So we say that
$$\sigma(P) = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$$ is the <u>permutation associated with</u> P.

More generally, if
$$P = \left(e_{\sigma_1}\ e_{\sigma_2} \cdots e_{\sigma_n} \right)$$ is P given in terms of its columns e_{σ_i}, then $$\sigma(P) = \begin{pmatrix} 1 & 2 & \cdots & n \\ \sigma_1 & \sigma_2 & \cdots & \sigma_n \end{pmatrix}$$ is the permutation associated with P.

Suppose P and Q are two $n \times n$ permutation matrices. Then we have the

<u>Proposition</u>. $\sigma(PQ) = \sigma(P)\sigma(Q)$.

That is, the product of two permutation matrices is a permutation matrix, and the permutation associated with a product is the product of the corresponding permutations.

<u>Example</u>. $P = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $\sigma(P) = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$

$$Q = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad \sigma(Q) = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$$

$$PQ = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \sigma(PQ) = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$$

$$\parallel$$

and $\sigma(P)\sigma(Q) = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$

<u>Proof</u>. We write $P = (e_{\sigma(P)_1}\ e_{\sigma(P)_2}\ \cdots\ e_{\sigma(P)_n})$,

$$Q = (e_{\sigma(Q)_1}\ e_{\sigma(Q)_2}\ \cdots\ e_{\sigma(Q)_n}).$$

Then $PQ = P\,(e_{\sigma(Q)_1}\ e_{\sigma(Q)_2}\ \cdots\ e_{\sigma(Q)_n})$

$$= (Pe_{\sigma(Q)_1}\ \ Pe_{\sigma(Q)_2}\ \ \cdots\ \ Pe_{\sigma(Q)_n})$$

$$= (e_{\sigma(P)(\sigma(Q)(1))}\ \cdots\ e_{\sigma(P)(\sigma(Q)(n))})$$

$$= (e_{[\sigma(P)\sigma(Q)](1)}\ \cdots\ e_{[\sigma(P)\sigma(Q)](n)})$$

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But by definition,

$$PQ = \left(e_{\sigma(PQ)(1)} \quad \cdots \quad e_{\sigma(PQ)(n)} \right).$$

$$\therefore \quad \sigma(PQ)(k) = (\sigma(P)\sigma(Q))(k), \quad k = 1, \ldots, n.$$

$$\Longleftrightarrow \quad \sigma(PQ) = \sigma(P)\sigma(Q) \quad //$$

We have proved (check the remaining details) that

$$\sigma : \begin{Bmatrix} n \times n \text{ Permutation} \\ \text{Matrices} \end{Bmatrix} \longrightarrow S_n$$

$$P \longmapsto \sigma(P)$$

is an isomorphism of groups.

Since we know that every finite group G is isomorphic to a subgroup of S_n for some n, this result implies that every finite group is isomorphic with a subgroup of $n \times n$ permutation matrices for some n. <u>Every finite group can be represented as a group of matrices.</u>

Here is a way to directly construct permutation matrices corresponding to the multiplication table of a finite group. I will give the instructions and illustrate with some examples.

1. Write down the multiplication table. e.g. $G = (1, r, r^2 \mid r^3 = 1)$.

	1	r	r^2
1	1	r	r^2
r	r	r^2	1
r^2	r^2	1	r

We will stop writing these labels, ok?

2. To ~~get~~ a set of permutations in S_n (for $\#G = n$) that represent G label the elements of G $\{1, 2, \ldots, n\}$ and write the table again.

1	1	2	3
r	2	3	1
r^2	3	1	2

Write a permutation for each row:

$$\sigma(1) = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$$

$$\sigma(r) = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$$

$$\sigma(r^2) = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$$

You will find that $\sigma(gg') = \sigma(g)\sigma(g')$ where $\sigma(g)$ = perm assoc with row g.

e.g. $\sigma(r)\sigma(r) = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} = \sigma(r^2)$.

Lets discuss why this works. Suppose $g_1, g_2, \dots, g_n$ is the list of group elements and suppose g_K is fixed and we look at the K^{th} row in the mult table. Then we have

$$g_K \quad \underbrace{\boxed{g_Kg_1 \mid g_Kg_2 \mid g_Kg_3 \mid g_Kg_4 \mid \quad \cdots \quad \mid g_Kg_{n-1} \mid g_Kg_n}}_{\text{This is a permuted list of the elements of } G.}$$

$$\sigma(g_K) = \begin{pmatrix} g_1 & g_2 & g_3 & \cdots & g_n \\ g_Kg_1 & g_Kg_2 & g_Kg_3 & \cdots & g_Kg_n \end{pmatrix}$$

Here the permutation is written symbolically using the names of the group elements.

$$\text{So } \sigma(g_\ell g_K) = \begin{pmatrix} g_1 & g_2 & \cdots & g_n \\ (g_\ell g_K)g_1 & (g_\ell g_K)g_2 & \cdots & (g_\ell g_K)g_n \\ \| & \| & & \\ g_\ell(g_Kg_1) & g_\ell(g_Kg_2) & & g_\ell(g_Kg_n) \end{pmatrix}$$

$$= \begin{pmatrix} g_1 & g_2 & \cdots & g_n \\ g_\ell g_1 & g_\ell g_2 & \cdots & g_\ell g_n \end{pmatrix} \begin{pmatrix} g_1 & g_2 & \cdots & g_n \\ g_Kg_1 & g_Kg_2 & \cdots & g_Kg_n \end{pmatrix}$$

$$\sigma(g_\ell g_K) = \sigma(g_\ell)\sigma(g_K).$$

This gives the isomorphism of G with a subgroup of specific permutations in S_n.

Before going to matrices, let's apply this to the group of symmetries of a triangle: $\Gamma = \left(I, R, R^2, F, FR, FR^2 \;\middle|\; \begin{array}{l} R^3 = I \\ F^2 = I \\ RF = FR^2 \end{array} \right)$

I	I	R	R^2	F	FR	FR^2
R	R	R^2	R^3	FR^2	F	FR
R^2	R^2	I	R	FR	FR^2	F
F	F	FR	FR^2	I	R	R^2
FR	FR	FR^2	F	R^2	I	R
FR^2	FR^2	F	FR	R	R^2	I

1	2	3	4	5	6
2	3	1	6	4	5
3	1	2	5	6	4
4	5	6	1	2	3
5	6	4	3	1	2
6	4	5	2	3	1

$$\sigma(I) = (\;)$$

$$\sigma(R) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 1 & 6 & 4 & 5 \end{pmatrix} = (123)(465)$$

$$\sigma(R^2) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 1 & 2 & 5 & 6 & 4 \end{pmatrix} = (132)(456)$$

$$\sigma(F) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 1 & 2 & 3 \end{pmatrix} = (14)(25)(36)$$

$$\sigma(FR) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 6 & 4 & 3 & 1 & 2 \end{pmatrix} = (15)(26)(34)$$

$$\sigma(FR^2) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 4 & 5 & 2 & 3 & 1 \end{pmatrix} = (16)(24)(53)$$

3. Now take the multiplication table and rearrange (permute) the columns so that the squares labeled with the identity element are on the diagonal.

For example, re-arranging ① on last page:

I	1	3	2	4	5	6
R	2	1	3	6	4	5
R^2	3	2	1	5	6	4
F	4	6	5	1	2	3
FR	5	4	6	3	1	2
FR^2	6	5	4	2	3	1

Let $P_1, \ldots, P_n$ be the permutation matrices with

$$P_i : \begin{cases} 1\text{'s at locations labeled } i \\ 0 \text{ elsewhere} \end{cases}$$

$$P_1 = \begin{pmatrix} 1 & & & & & \\ & 1 & & & 0 & \\ & & 1 & & & \\ & & & 1 & & \\ & 0 & & & 1 & \\ & & & & & 1 \end{pmatrix}, \quad P_2 = \begin{array}{|c|c|c|c|c|c|} \hline & & 1 & & & \\ \hline 1 & & & & & \\ \hline & 1 & & & & \\ \hline & & & & 1 & \\ \hline & & & & & 1 \\ \hline & & & 1 & & \\ \hline \end{array},$$

$$P_3 = \begin{array}{|c|c|c|c|c|c|} \hline & 1 & & & & \\ \hline & & 1 & & & \\ \hline 1 & & & & & \\ \hline & & & & & 1 \\ \hline & & & 1 & & \\ \hline & & & & 1 & \\ \hline \end{array}, \quad P_4 = \begin{array}{|c|c|c|c|c|c|} \hline & & & 1 & & \\ \hline & & & & 1 & \\ \hline & & & & & 1 \\ \hline 1 & & & & & \\ \hline & 1 & & & & \\ \hline & & 1 & & & \\ \hline \end{array}$$

(Note: In terms of the original group elements, $P_1 = P_I$, $P_2 = P_R$, $P_3 = P_{R^2}$, $P_4 = P_F$, $P_5 = P_{FR}$, $P_6 = P_{FR^2}$.)

$$P_5 = \begin{bmatrix} & & & & 1 & \\ & & & & & 1 \\ & & & 1 & & \\ & & 1 & & & \\ 1 & & & & & \\ & 1 & & & & \end{bmatrix}, \quad P_6 = \begin{bmatrix} & & & & & 1 \\ & & & 1 & & \\ & & & & 1 & \\ & 1 & & & & \\ & & 1 & & & \\ 1 & & & & & \end{bmatrix}$$

Note: $\sigma(P_1) = ()$

$$\sigma(P_2) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 1 & 6 & 4 & 5 \end{pmatrix} = \sigma(R)$$

$$\sigma(P_3) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 1 & 2 & 5 & 6 & 4 \end{pmatrix} = \sigma(R^2)$$

$$\sigma(P_4) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 1 & 2 & 3 \end{pmatrix} = \sigma(F)$$

$$\sigma(P_5) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 6 & 4 & 3 & 1 & 2 \end{pmatrix} = \sigma(FR)$$

$$\sigma(P_6) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 4 & 5 & 2 & 3 & 1 \end{pmatrix} = \sigma(FR^2)$$

The permutateions associated with these permutation matrices derived from the multiplication table are exactly the permutations we took directly from the table. The table produces its own matrices!

Exercise: Prove this this method always works.

Read On.

Since we already know that the permutations associated with group elements multiply correctly and that $\sigma(PQ) = \sigma(P)\sigma(Q)$ for permutation matrices, it follows that all these representations fit correctly together. So what needs to be proved is that

$$\sigma(P_g) = \sigma(g)$$

where $\sigma(P_g)$ is the permutation associated with the permutation matrix associated with matrix entries labeled g, and $\sigma(g)$ is the permutation associated with the g-row in the multiplication table.

Now our result is a tricky tautology:

$\lambda = \sigma(P_K)i$ = the standard position of g_K in the new ith column.

The new ith column $\leftrightarrow g_\ell$ s.t. $g_i g_\ell = 1$ (so there is a 1 in the ith place).

Then for λ we need $g_\lambda g_\ell = g_K$.

Thus $g_\lambda g_i^{-1} = g_K$.

Thus $g_\lambda = g_K g_i$

Thus $\sigma(g_K)i = \lambda$. $\Rightarrow \boxed{\sigma(g_K) = \sigma(P_K)}$.

Problem. Here is the 8 element quaternion group: $\{\pm 1, \pm i, \pm j, \pm k\} = \mathcal{H}$

$$i^2 = j^2 = k^2 = -1$$
$$ij = k \,,\quad ji = -k$$
$$jk = i \,,\quad kj = -i$$
$$ki = j \,,\quad ik = -j$$

Note: $(-1)x = -x$, $-(-x) = x$ for each $x \in \mathcal{H}$.

Write out the multiplication table for $\mathcal{H}$ and determine the eight permutation matrices of size 8×8 that represent $\mathcal{H}$.

Remark. The quaternions were discovered by Sir William Rowan Hamilton in 1843. He found them as a non-commutative generalization of the complex numbers and used them to study rotations and to formulate physics.

His quaternions are linear combinations of $1, i, j, k$.

$$\text{Quaternions} = \{a + bi + cj + dk \mid a, b, c, d \in \mathbb{R}\}$$

This was the first example of a non-commutative algebra (prior to matrix algebra).