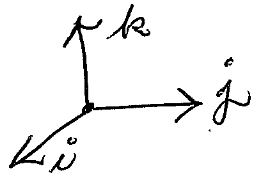
Math 435 — Problems Due Tuesday April 15, 2014

Do Problems 1 → 5

First recall the quaternions and some facts about them:

$$\mathbb{H} = \{a + bi + cj + dk \mid a, b, c, d \in \mathbb{R}\}$$



$$\begin{cases} i^2 = j^2 = k^2 = -1 \\ ij = k,\ jk = i,\ ki = j \\ ji = -k,\ kj = -i,\ ik = -j \end{cases}$$

We regard $\mathbb{R}^3 = \{ri + sj + tk \mid r, s, t \in \mathbb{R}\}$

$$S^3 = \{a + bi + cj + dk \mid a^2 + b^2 + c^2 + d^2 = 1\}$$

These are the <u>unit quaternions</u>.

An element of $\mathbb{R}^3$ is called a <u>pure quaternion</u>.

If $w = a + ib + jc + kd$, the conjugate define $\overline{w} = a - ib - jc - kd$, real numbers quaternion. (Note that commute with i, j, k so that $ai = ia$.)

1°. (a) Prove that if w and z are two quaternions $\in \mathbb{H}$, then
$$\overline{wz} = \overline{z}\,\overline{w}.$$

(b) Prove that if $w = a + bi + cj + dk$, then $w\overline{w} = a^2 + b^2 + c^2 + d^2$.

(c) Prove that if $u, v \in \mathbb{R}^3$ are pure quaternions, then
$$uv = -u \cdot v + u \times v$$
where (next page)

If $u = u_1 i + u_2 j + u_3 k$
$v = v_1 i + v_2 j + v_3 k$

then $u \times v \underset{\text{def}}{=} \mathrm{Det}\begin{bmatrix} i & j & k \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{bmatrix}$

$= \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} i - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} j + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} k$

is the vector cross-product in $\mathbb{R}^3$.

Note that you have shown that if $u \in \mathbb{R}^3$ then $uu = -u \cdot u + 0$.
Thus when $u \cdot u = 1$ (u has unit length in $\mathbb{R}^3$), then $u^2 = -1$ in the quaternions.

2° Let $u, v, w \in \mathbb{R}^3$ be pure quaternions.
Show that $(uv)w = u(vw)$.
You can use the following formulas about the vector cross product in your proof.

$$(u \times v) \times w = -(v \cdot w)u + (u \cdot w)v$$
$$u \times (v \times w) = -(u \cdot v)w + (u \cdot w)v$$

(You do not have to prove these identities. We'll discuss them in class.)

3° Assume that quaternion multiplication is associative and that you only know that $i^2 = j^2 = k^2 = ijk = -1$.
Prove the rest of the identities about i, j and k just from these assumptions.

4° Let
$$SU(2) = \left\{ \begin{pmatrix} z & w \\ -\bar{w} & \bar{z} \end{pmatrix} \middle| \begin{array}{l} z = a+ib \\ w = c+id \\ \text{and } z\bar{z} + w\bar{w} = 1 \end{array} \text{complex numbers} \right\}$$

This is a set of 2×2 matrices with entries in the complex numbers as indicated above.

(a) Prove that $SU(2)$ is a ~~group~~ group under matrix multiplication. (Note that you need to show closure under multiplication. For inverses note that if $U = \begin{pmatrix} z & w \\ -\bar{w} & \bar{z} \end{pmatrix}$ and $U^* = \begin{pmatrix} \bar{z} & -w \\ \bar{w} & z \end{pmatrix}$ (conjugate transpose) then $UU^* = U^*U = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.)

(b) Show
$$\begin{pmatrix} a+bi & c+di \\ -c+di & a-bi \end{pmatrix} = a\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + b\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} + c\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + d\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

Let $E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $I = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$, $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $K = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$.

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$E=\begin{pmatrix}1&0\\0&1\end{pmatrix}$, $I=\begin{pmatrix}i&0\\0&-i\end{pmatrix}$, $J=\begin{pmatrix}0&1\\-1&0\end{pmatrix}$, $K=\begin{pmatrix}0&i\\i&0\end{pmatrix}$.

Show that $I^2=J^2=K^2=IJK=-E$.

Since you can think of E as $\mathbb{1}$, this shows that $\{E, I, J, K\}$ generate a matrix model for the quaternions and that $su(2)$ is a matrix model for the unit quaternions.

5. We have seen that if

$g = e^{u\theta} = \cos(\theta) + u\sin(\theta)$

for u a unit vector in $\mathbb{R}^3$,

then $v \overset{R}{\longmapsto} gv\bar{g}$

gives us a rotation by 2θ around the axis v in $\mathbb{R}^3$.

Work out the result of $S \circ R$ where

R = 180° rotation about k.

S = 90° rotation about i.

(a) algebraically using quaternions.

(b) geometrically using a cube.