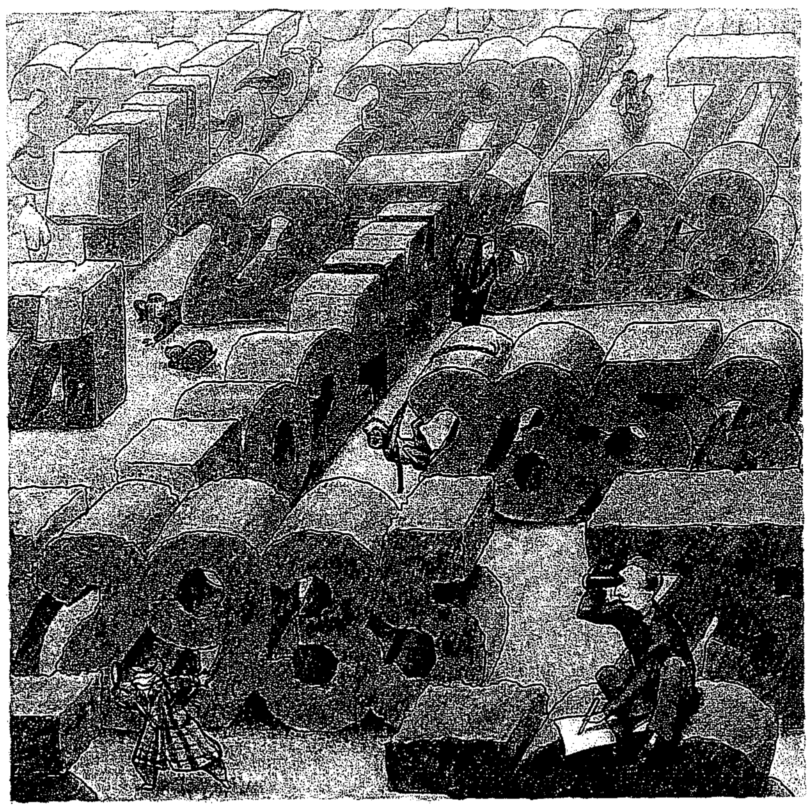
THE BOOK OF *Numbers*



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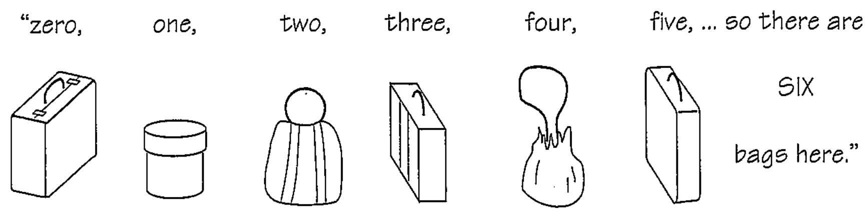


FIGURE 10.1 *What Sierpiński should have said.*

THE EMPTY SET

One of the advantages of the new system is that it works even when you are counting no objects at all. If Sierpiński's luggage all gets lost en route, then, at the other end of his journey he should say:

" ", so there are ZERO bags here!

The usual system of counting doesn't work for counting zero objects, since there isn't a last number that you used.

CANTOR'S ORDINAL NUMBERS

The great German mathematician Georg Cantor was the earliest person to construct a coherent theory of counting collections that may be infinite. For this he extended the ordinary series of numbers used for counting, as follows:

$0, 1, 2, \ldots$ as usual,
then $\omega, \omega+1, \omega+2, \ldots$ then $\omega+\omega, \omega+\omega+1, \ldots$

and so on.

The important point about these numbers (and, in essence, their definition) is that, no matter how many of them you've used, there's always a (uniquely determined) earliest one that you haven't. Cantor's opening infinite number,

$$\omega = \{0, 1, 2, \ldots | \}$$

is defined to be the earliest number greater than all the finite counting numbers. We'll use

$$\{a, b, c, \ldots | \}$$

for the earliest ordinal number after $a, b, c, \ldots$. The vertical bar signals the place where we've cut off the number sequence $a, b, c, \ldots$, for example,

$$\{0, 1, 2|\} = 3, \{0, 1, 2, \ldots|\} = \omega, \{0|\} = 1, \{|\} = 0, \{0, 1, 2, \ldots \omega|\} = \omega + 1.$$

To avoid inventing lots of new words, the symbols $\omega + 1, \omega + 2, \ldots$ are used as proper names for the ordinary numbers following ω, just as "hundred and one" is the proper name of the number you get by adding "a hundred" and "one."

When you count things, you are really ordering them in a special way:

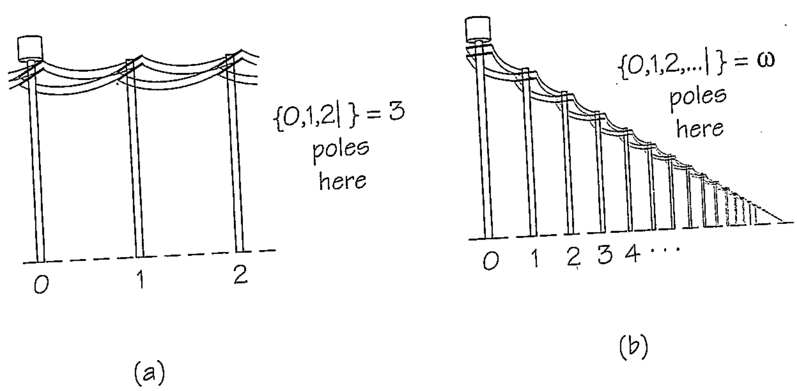
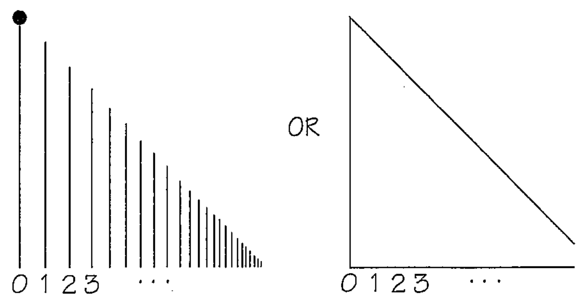


FIGURE 10.2 *Various numbers of poles.*

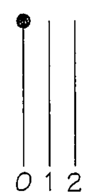
To count the poles in Figure 10.2(a), you'd say, "0, 1, 2, so there are $\{0, 1, 2|\} = 3$ poles here." But now look at Figure 10.2(b), where we imagine that the road is infinite, with a pole for each of the ordinary integers $0, 1, 2, \ldots$. Obviously, we should now say:

"$0, 1, 2, \ldots$, so there are ω poles here."

In the future, we'll represent such an infinite sequence of objects by

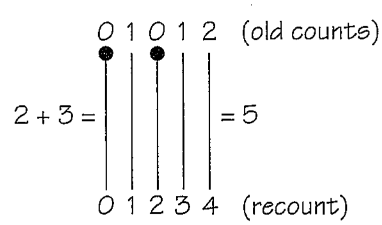


(recalling Figure 10.2(b)), but we'll represent a finite sequence by poles of equal height:

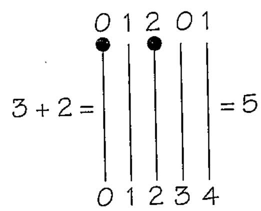


(recalling Figure 10.2(a)).

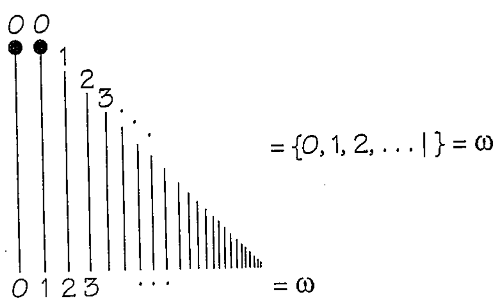
To add two of Cantor's ordinal numbers, you just put their pictures side by side and do a recount: For instance,



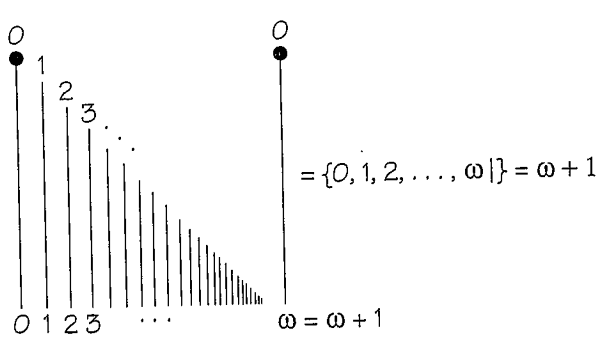
Of course, we get the same answer from 3 + 2, although the recounting's in a different order:



But infinite numbers give some surprises! We find that $1 + \omega$ is the same as ω:

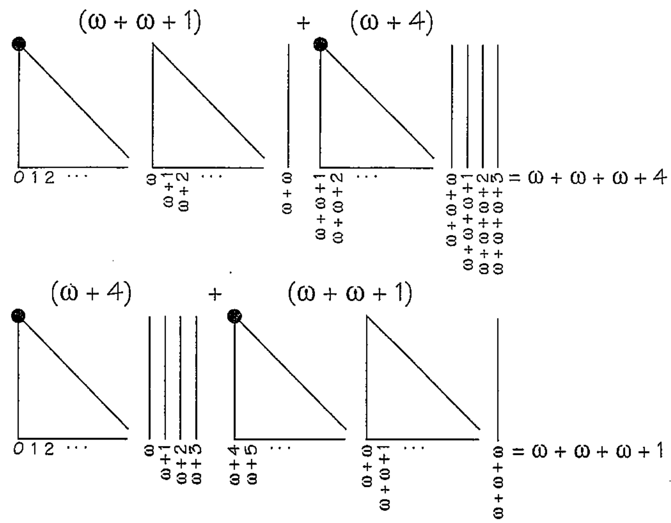


but $\omega + 1$ is bigger:



In other words, this kind of addition usually *fails* to satisfy the commutative law; $\beta + \alpha$ may be larger or smaller than $\alpha + \beta$.

As a bigger example, we'll add $\alpha = \omega+\omega+1$ to $\beta = \omega+4$, both ways around,



Since two numbers, α, β, in their two orders, can give two distinct sums, you might expect that three ordinal numbers, α, β, γ, could give six different sums,

$$\alpha+\beta+\gamma,\ \alpha+\gamma+\beta,\ \beta+\gamma+\alpha,\ \beta+\alpha+\gamma,\ \gamma+\alpha+\beta,\ \gamma+\beta+\alpha,$$

but it turns out that at least two of these six are equal, so that no three ordinal numbers can have more than five different sums.

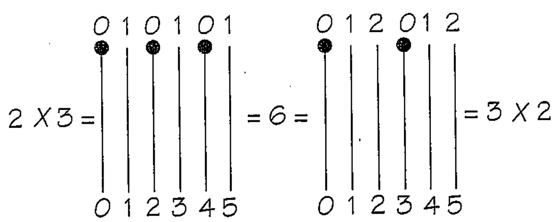
By taking the largest possible number of different sums of n ordinal numbers for $n = 1, 2, 3, \ldots$, we get the sequence

1	2	5	13	33
81	193	449	33^2	33×81
81^2	81×193	193^2	$33^2 \times 81$	33×81^2
81^3	$81^2 \times 193$	81×193^2	193^3	33×81^3
and	from here on	you multiply	the previous	row by 81:
81^4	$81^3 \times 193$	$81^2 \times 193^2$	81×193^3	$33 \times 81^4 \ldots$

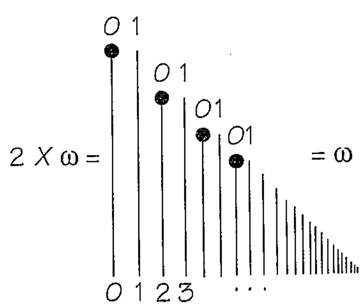
So the largest number of different sums that n ordinals can have behaves rather strangely. For 15 or more numbers, it will be either a power of 193 times a power of 81, or 33 times a power of 81.

MULTIPLYING ORDINAL NUMBERS

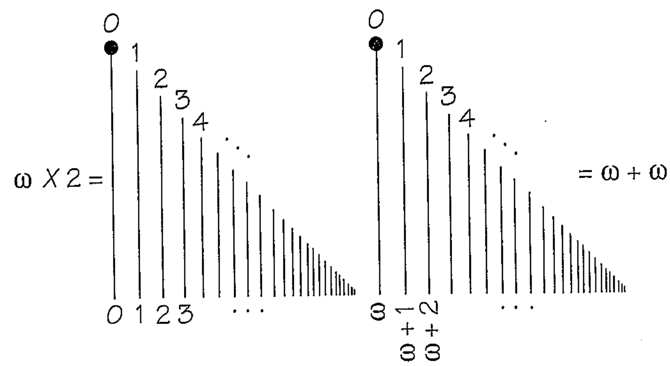
Now let's see how to multiply Cantor's numbers. The product $\alpha \times \beta$ is what you get by placing β copies of α in sequence: for instance,



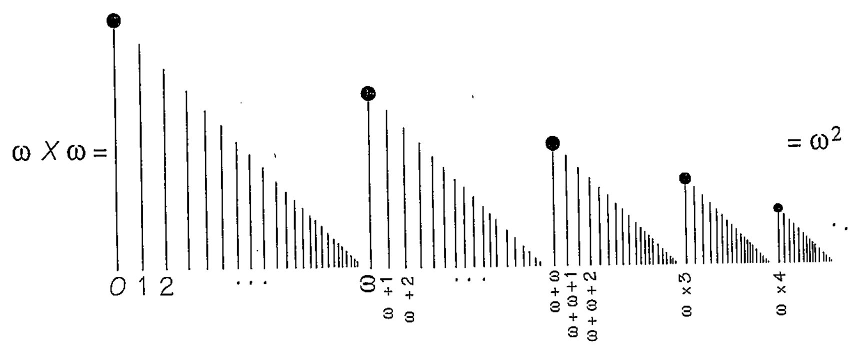
as you might expect, but infinite numbers continue to surprise us. When we take ω copies of 2, we see that $2 \times \omega$ is just ω:



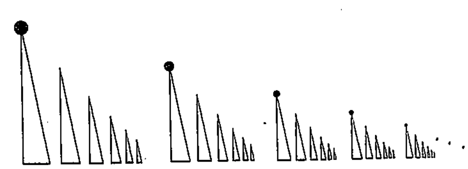
but $\omega \times 2$ (2 copies of ω) is the same as $\omega + \omega$:



What is $\omega \times \omega$ (which we can write as ω^2)? It's a much larger number than the ones we've seen before. It consists of ω copies of ω, placed in sequence:



What about ω^3, ω^4, . . . ? Well, of course, $\omega^3 = \omega^2 \times \omega$. We can get it by having ω copies of a pattern of ω^2:



Then you get ω^4 from ω copies of this; then ω^5 from ω copies of that, and so on—we won't draw the pictures for $\omega^4, \omega^5, \ldots$—and there are lots of other numbers. For instance,

$$\omega^6 \times 49 + \omega^3 \times 8 + \omega^2 \times 3 + \omega \times 57 + 1001$$

lies between ω^6 and ω^7. Figure 10.3 shows a pattern for the number $\omega^2 \times 2 + \omega \times 3 + 7$.

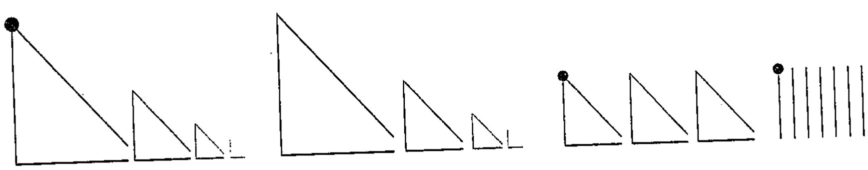


FIGURE 10.3 $(\omega^2 \times 2) + (\omega \times 3) + 7$.

Can we go further? Yes! In Cantor's system you can *always* go further! The number

$$\omega^\omega = 1 + \omega + \omega^2 + \omega^3 + \omega^4 + \cdots$$

is obtained by juxtaposing all the patterns for $1, \omega, \omega^2, \omega^3, \omega^4, \ldots$, in that order. Then you have

$$\omega^\omega + 1, \omega^\omega + 2, \ldots \omega^\omega + \omega, \ldots \omega^\omega + \omega \times 2, \ldots \omega^\omega + \omega \times 3, \ldots$$
$$\omega^\omega + \omega^2, \omega^\omega + \omega^2 + 1, \ldots \omega^\omega + \omega^2 + \omega, \ldots, \omega^\omega + \omega^3, \ldots$$
$$\omega^\omega + \omega^\omega = \omega^\omega \times 2, \omega^\omega \times 2 + 1, \ldots \omega^\omega \times 3, \ldots, \omega^\omega \times 4, \ldots$$
$$\omega^\omega \times \omega = \omega^{\omega+1}, \ldots \omega^{\omega+1} + \omega, \ldots \omega^{\omega+1} + \omega^2, \ldots$$
$$\omega^{\omega+1} + \omega^\omega, \ldots \omega^{\omega+2} \ldots, \omega^{\omega+3}, \ldots \omega^{\omega\times 2}, \ldots \omega^{\omega\times 3}, \ldots$$
$$\omega^{\omega^2}, \ldots \omega^{\omega^3}, \ldots \omega^{\omega^4}, \ldots \omega^{\omega^\omega}, \ldots \omega^{\omega^{\omega+1}}, \ldots \omega^{\omega^{\omega^\omega}}, \ldots \omega^{\omega^{\omega^{\omega^\omega}}}, \ldots$$

The "limit" of all these is a number that it is natural to write as

$$\omega^{\omega^{\omega^{\omega^{\cdot^{\cdot^{\cdot}}}}}}$$

where there are ω omegas. This famous number was called ϵ_0 by Cantor. It's the first ordinal number that you can't get from smaller ones by a finite number of additions $\alpha+\beta$, multiplications $\alpha\times\beta$, and exponentiations α^β. Another formula for it is

$$\epsilon_0 = 1 + \omega + \omega^\omega + \omega^{\omega^\omega} + \omega^{\omega^{\omega^\omega}} + \cdots.$$

It is also the first number that satisfies Cantor's famous equation $\omega^\epsilon = \epsilon$. You'd think that this couldn't happen, because

ω^1 is much bigger than 1,
ω^2 even more so than 2,
ω^3 still more so than ω,

but Cantor showed that his equation has lots of solutions. The next is

$$\epsilon_1 = (\epsilon_0 + 1) + \omega^{\epsilon_0+1} + \omega^{\omega^{\epsilon_0+1}} + \omega^{\omega^{\omega^{\epsilon_0+1}}} + \cdots.$$

Then come

$$\epsilon_2,\ \epsilon_3,\ \ldots\ \epsilon_\omega,\ \epsilon_{\omega+1},\ \ldots\ \epsilon_{\omega\times 2},\ \ldots\ \epsilon_{\omega^2},\ \ldots\ \epsilon_{\omega^\omega},\ \ldots$$
$$\epsilon_{\epsilon_0},\ \epsilon_{\epsilon_0+1},\ \ldots\ \epsilon_{\epsilon_0+\omega},\ \ldots\ \epsilon_{\epsilon_0+\omega^\omega},\ \ldots\ \epsilon_{\epsilon_0\times 2},\ \ldots\ \epsilon_{\epsilon_1},\ \ldots$$
$$\epsilon_{\epsilon_2},\ \ldots\ \epsilon_{\epsilon_\omega},\ \ldots\ \epsilon_{\epsilon_{\epsilon_0}},\ \ldots\ \epsilon_{\epsilon_{\epsilon_1}},\ \ldots\ \epsilon_{\epsilon_{\epsilon_\omega}},\ \ldots\ \epsilon_{\epsilon_{\epsilon_{\epsilon_0}}},\ \ldots$$

and eventually

$$\epsilon_{\epsilon_{\epsilon_{\epsilon_{\ddots}}}},$$

which is the first solution of $\epsilon_\alpha = \alpha$.

How Far Can We Go?

The ordinal numbers go on for an awfully long time! No matter how big the set of them you've already got, there's always another one, and another, and another, and . . . . The precise situation was guessed by Cantor and proved a quarter of a century later by his student Zermelo in 1904: there are enough ordinals to count the members of any set of objects, no matter how big it is. Zermelo's proof showed that this depends on a hitherto unrecognized principle in mathematics: the so-called **axiom of choice**.