APPENDIX. The Classical Alexander Polynomial

In this appendix we shall sketch one approach to the Alexander polynomial. This material is standard, and is based upon Alexander's original paper [1].

Let G be a finitely presented, finitely related group that is equipped with a surjective homomorphism $f : G \longrightarrow Z$ where Z denotes the group of additive integers. Assume that $\mathrm{Kernel}(f) = G'$, the commutator subgroup of G. Let $H = G'/G''$ be the abelianization of this commutator subgroup. If s is an element of G such that $f(s) = 1$ and x denotes the (multiplicative) generator of the group ring $\Gamma = Z[Z] = Z[x,x^{-1}]$, then H becomes a Γ-module via the action on G': $x(g) = sgs^{-1}$.

We say that the pair (G,f) is an indexed group. Two indexed groups (G_1,f_1), (G_2,f_2) are isomorphic if there is a group isomorphism $h : G_1 \longrightarrow G_2$ such that $f_2 \circ h = f_1$. If $H(G,f)$ denotes G'/G'' with module structure as above, then $H(G,f)$ is an isomorphism invariant of the indexed group (G,f).

As we shall see, $H(G,f)$ is finitely presented and related over Γ. Suppose that there exists such a presentation with an equal number n of generators and relations. (This will be the case for the knot group.) Then there is an exact sequence

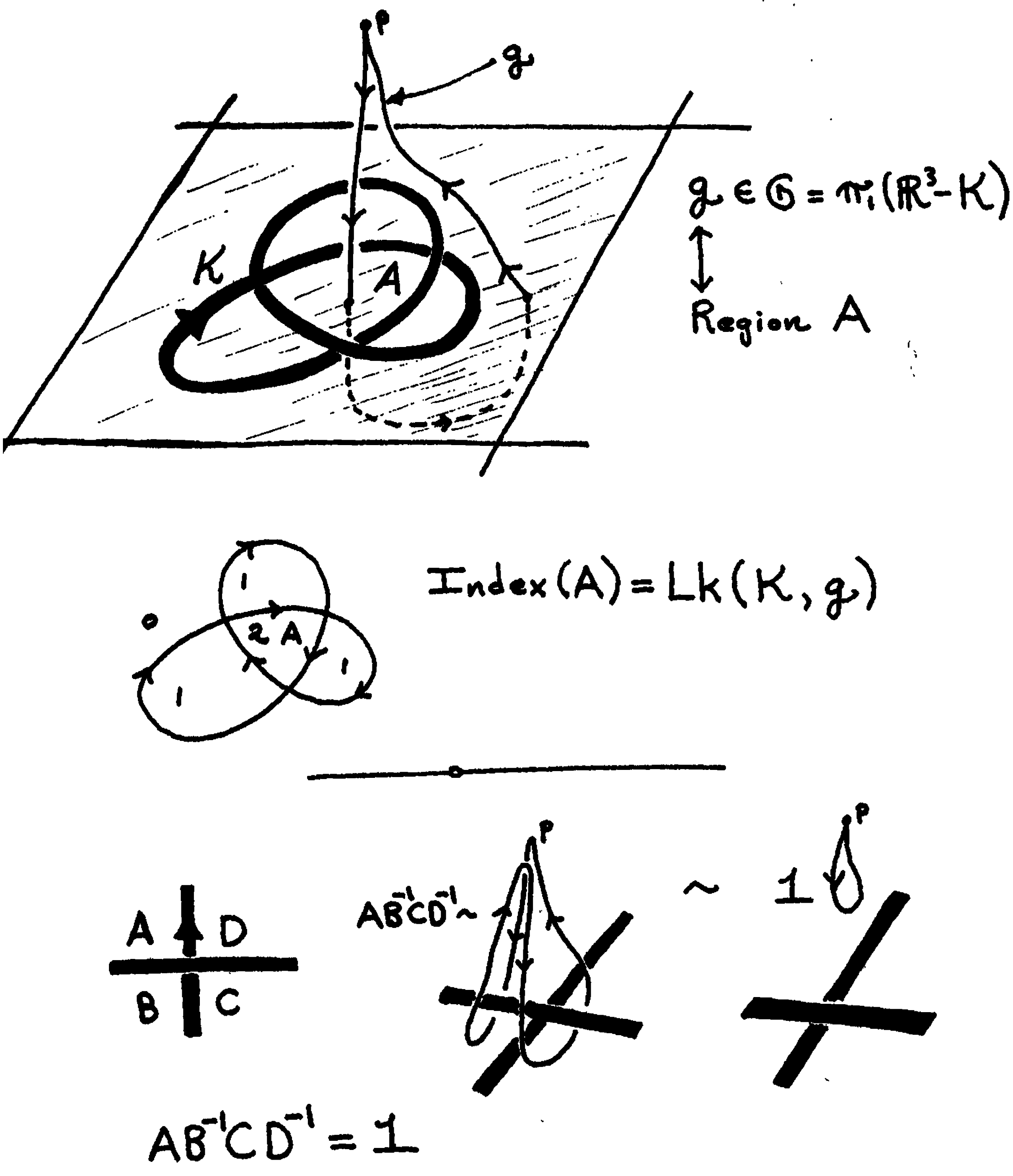
$$\Gamma^n \xrightarrow{A} \Gamma^n \longrightarrow H(G,f) \longrightarrow 0$$

where A is an $n \times n$ matrix with entries in Γ.

Let $D(A)$ denote the determinant of A. The Γ-module structure on $H(G,f)$ is induced by scalar (Γ) multiplication on Γ^n. Let $\hat{A}$ denote the adjoint matrix to A so that $A\hat{A} = D(A)I$ where I is the $n \times n$ identity matrix. This shows that for all a, $D(A)a = A(\hat{A}a)$, and hence $D(A)a$ is in the image of A. Therefore $D(A)[a] = 0$ for all $[a] \in H(G,f)$ Thus $D(A)$ is an annihilating element for $H(G,f)$ as a Γ-module. We shall see that the Alexander polynomial takes the form of $D(A) = \Delta_K(x)$ for an approrpriate matrix A.

Before doing more algebra, let's turn to the geometry. Let $G = \pi_1(S^3 - K)$ be the fundamental group of the knot complement. This group is finitely presented and related, with a particularly useful presentation known as the Dehn presentation. In the Dehn presentation each region of the knot diagram corresponds to an element of $\pi_1(S^3-K)$ via the following conventions: Replace S^3-K by R^3-K and assume that the knot lies in the $(x,y,0)$ plane, except for over and under-crossings. These crossings deviate in the z-direction (third variable) by $0 < |z| \ll 1$. Let the base-point $p = (0,0,1)$ be a point above the knot diagram. Associate to each region R a loop that starts at p, descends to pierce R once, and then returns by piercing the unbounded region once. See Figure 49.

In this form each region of the knot diagram corresponds to a generator of the fundamental group, except for the unbounded region, which corresponds to the identity element.



Crossing relation in Dehn presentation

Figure 49

Each crossing in the knot diagram corresponds to a relation in the fundamental group (as illustrated in Figure 49). This gives a complete set of relations for the group. The mapping $f : \pi_1(S^3-K) \longrightarrow Z$ exists since a fundamental group of a knot complement abelianizes to Z. The map can be specifically described by linking numbers: $f([g]) = Lk(K,g)$ where Lk denotes linking numbers of curves in R^3. With this interpretation we see that, when we let elements of the group correspond to regions R in the knot diagram as described above, then $f(R) = Index(R)$ where Index denotes the Alexander index of the region R (See Lemma 3.4.). (The unbounded region is assigned index zero. See Figure 49.)

Remark. The Γ-module $H(G,f)$ has the following interpretation. Let $q : X \longrightarrow S^3-K$ be the covering space corresponding to the representation $f : G \longrightarrow Z$. The space X is the infinite cyclic cover of the knot complement (see [26]). The first homology group, $H_1(X;Z)$, is a Γ-module via the action of the group of covering translations of X. With this structure, $H_1(X;Z)$ and $H(G,f)$ are isomorphic Γ-modules. This interpretation is very important, but will not be pursued here.

Returning to algebra, we wish to describe a presentation for G', and thence compute G'/G''. Suppose that G has a presentation of the form: $G = (s,g_1,g_2,\ldots,g_n/R_1,\ldots,R_m)$ with

1. $n = m$ (True for the Dehn presentation since there are two more regions than crossings, and one region corresponds to the identity element.) or $m \geq n$.
2. $f(s) = 1$, $f(g_1) = f(g_2) = \ldots = f(g_m) = 0$.

The second condition is accomplished from an arbitrary presentation by choosing an s $(f(s) = 1)$, and re-defining the other generators via multiplication by approrpriate powers of s, to insure that they all hit zero under f.

Recall that $\Gamma = Z[x,x^{-1}]$ acts on G via $xg = sgs^{-1}$. It is easy to see that G' is generated by the set $\{x^k g_i / k \in Z,\ i = 1,\ldots,n\}$. In particular, each relation R_k can be rewritten in terms of these generators. Let $\rho(R_k)$ denote this rewriting of R_k. Then $\{x^k \rho(R_k) / k \in Z,$ $i = 1,\ldots,m\}$ is a set of relations for G' (proof via covering spaces or combinatorial group theory). Thus $G' = (\{x^k g_i\} / \{x^k \rho(R_j)\})$. By abelianizing these generators and relations, and writing them additively, we obtain the structure of $H(G,f)$.

Consider the Dehn presentation. Let s correspond to a region of index 1 (or -1 if necessary). Suppose A, B, C, D are the regions around a crossing with indices $f(A) = p$, $f(B) = f(D) = p+1$, $f(C) = p+2$ as in Figure 50. Then we have new generators a, b, c, d with $A = s^p a$, $B = s^{p+1} a$, $C = s^{p+2} c$, $D = s^{p+1} d$. The relation $R = AB^{-1}CD^{-1}$ becomes

$$
\begin{aligned}
R &= AB^{-1}CD^{-1} \\
&= (s^p a)(s^{p+1} b)^{-1}(s^{p+2} c)(s^{p+1} d)^{-1} \\
&= s^p ab^{-1} scd^{-1} s^{-p-1}.
\end{aligned}
$$

The next calculation rewrites R in terms of the generators $x^k a$, $x^k b$, $x^h c$, $x^k d$ of G'.

$$\begin{aligned} R &= s^p a s^{-p} s^p b^{-1} s^{-p} s^{p+1} c s^{-p-1} s^{p+1} d^{-1} s^{-p-1} \\ &= (x^p a)(x^p b)^{-1}(x^{p+1}c)(x^{p+1}d)^{-1} = \rho(R). \end{aligned}$$

Upon abelianizing G' to form $H(G,f)$, this relation becomes the additive relation $x^p(a - b + xc - xd)$. Thus, as a module over the group ring Γ, $H(G,f)$ is generated by symbols $a,b,c,d,\ldots$ corresponding to the regions of the knot diagram, with generating relations, one per crossing, of the form $a - b + xc - xd$. These relations can be remembered by placing ± 1, $\pm x$ around the crossing in the regions that they correspond with (as shown in Figure 50).

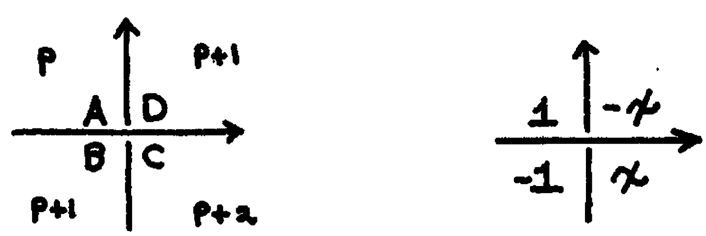


Figure 50

Earlier in the notes we have referred to this labelling as the Alexander code. The symbol corresponding to the unbounded region is set equal to zero, and an adjacent region corresponds to s $(f(s) = 1)$ and is also eliminated. The resulting square relation matrix is exactly what we have described in section 3 as the Alexander Matrix for this code. Its determinant is the Alexander polynomial, $\Delta_K(x)$.

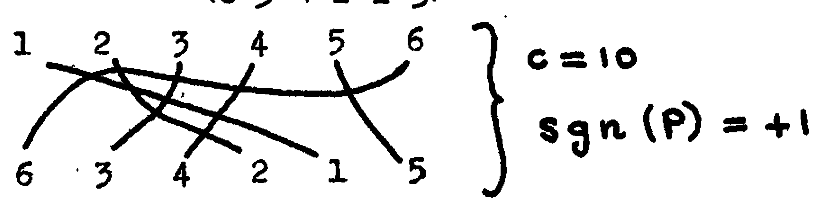
That the Alexander polynomial is well-defined up to sign and powers of x and that it is a topological invariant of the knot K can be verified by examining the combinatorial group theory that we have sketched.

Alexander apparently felt that the algorithm should stand on its own right, and he wrote his first paper on the polynomial from a combinatorial standpoint with slight mention of the background group theory and topology.

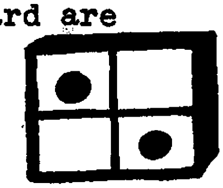
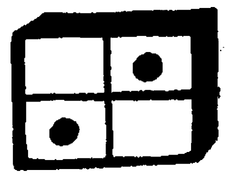
Remark on Determinants

In these notes we have used a formulation of certain determinants (of Alexander matrices) as state summations over the states of a universe (knot or link graph). The signs in the determinant expansion come from the geometry of the universe. I wish to point out here that the formula for any determinant follows a similar pattern.

The sign of a permutation is determined geometrically by the following prescription. List the numbers $1,\ldots,n$ in order, and the permutation of them on a line below. Connect corresponding numbers by arcs so that all arcs intersect transversely at double points. Then the sign of the permutation is $(-1)^c$ where c is the number of intersections of the arcs. For example let $P = \begin{pmatrix}1&2&3&4&5&6\\6&3&4&2&1&5\end{pmatrix}$:



The combinatorics underlying the determinant expansion of an $n \times n$ matrix consists of the $n!$ grid states of an $n \times n$ grid. A grid state is a pattern of n rooks on the $n \times n$ chessboard so that no two rooks attack each other. (Rooks move only on horizontal and vertical files. Thus no file contains more than one rook.) For example, the grid states for a 2×2 board are

 and  .

Given a grid state S and matrix A, define $\langle A|S\rangle$ to be the product of the entries of A that are in boxes corresponding to rooks of S (when the matrix and grid state are superimposed).

The sign of a grid state S, denoted $\sigma(S)$, is the sign of the permutation of the rows that produces this state from the diagonal state (all rooks on the main diagonal). To obtain this sign directly, draw arcs outside the grid from top row positions to left column positions so that each arc marks the row and column position of a corresponding rook. Then $\sigma(S) = (-1)^c$ where c is the number of crossings of the arcs. Then $\mathrm{Det}(A) = \sum_{\mathcal{S}} \sigma(S)\langle A|S\rangle$ where $\mathcal{S}$ is the collection of all grid states for the $n \times n$ grid.