Combinatorics and Topology - Francois Jaeger's Work in Knot Theory

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Abstract

This paper is dedicated to the memory of Francois Jaeger.

1 Introduction

Francois Jaeger found a number of beautiful connections between combinatorics and the topology of knots and links, culminating in an intricate relationship between link invariants and the Bose-Mesner algebra of association schemes. We give an elementary introduction to this connection.

We begin by first recalling the construction of the bracket polynomial (a state summation model for the Jones polynomial). With this example of a combinatorial state model in hand, we then devote a section to Jaeger's discovery of a combinatorial state model for the Homfly polynomial. The next section shows how the bracket polynomial can be translated into a (so-called) spin model in terms of the checkerboard graph of a knot or link. This gives an introduction to spin models and the opportunity to explain Jaeger's discovery of the connection of spin models with association schemes.

Acknowledgement. It gives the author pleasure to thank the National Science foundation for support of this research under NSF Grant DMS-9802859 and the National Security Agency for support of the research under grant MDA904-97-1-0015.

2 Recalling the Bracket Polynomial

We must first recall that the theory of knots and links is the study of smooth (infinitely differentiable) or piecewise linear embeddings of circles and collections of circles into Euclidean three dimensional space R^3. Knots are embeddings of single circles and links are embeddings of multiplicities of circles. The number of circles is called the number of *components* of the link. Two knots or links are said to be *ambient isotopic* if there is a smooth (or piecewise linear) family of embeddings parametrized on the unit interval that starts with one link and ends up with the other. The problem in knot theory is to determine when two embeddings are ambient isotopic.

A knot is said to be *knotted* if it is not ambient isotopic to a planar embedding of a circle. A link is said to be *linked* if it is not ambient isotopic to a disjoint embedding of circles in the plane. Thus all knots are links, but not conversely.

The plane is a key figure in this theory because it is always possible to project a knot or link from three-space to the plane (or to the surface of a two dimensional sphere about the origin) so that the curve(s) in space become curve(s) on the plane with finitely many transverse self-intersections as shown in Figure 1.

A projection to the plane can be arranged for any link so that the only multiple points are double points corresponding to transverse interesections of the curves in the projection. The curve(s) formed by the projection of the knot or link can be then be construed as a 4-regular (four local edges per vertex) plane multigraph (a multigraph can have a multiplicity of edges between two of its vertices). The vertices of this multigraph correspond to the places where the projection of the link has double points in the plane. This combinatorial structure can be used to encode all the information one needs about the knot or link for topological purposes. In particular it is customary as in Figure 1(a) to use a convention of labelling the 4-regular graph at its vertices to indicate how the corresponding curves in space cross over one

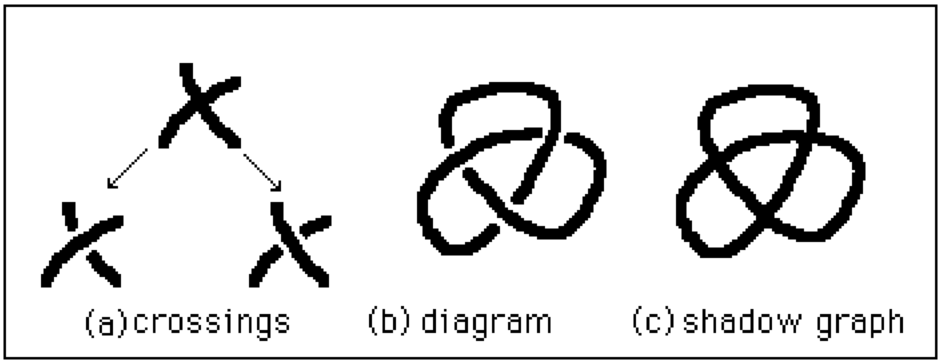

Figure 1 - Crossings and Diagrams

another. This convention for a crossing is indicated in Figure 1(b). Thus we see that the overcrossing line is shown as continuous while the undercrossing line is shown with deletion just before it would meet the overcrossing line and continuing just afterwards. A knot or link diagram of this type can be used as weaving instructions to make a corresponding embedding into three-space. See Figure 1(b) for an illustration of the trefoil knot T in this mode and for its shadow graph.

An edge in a graph G either joins two distinct vertices in G or, if the edge is a loop, it joins a vertex to itself. Let the *boundary* of an edge e be the multi-set of those vertices, with the case of a loop giving a set of multiplicity two. Let ∂e denote the boundary of e. If C is a set of edges in G let ∂C denote the union (with multiplicities) of the boundaries of the edges in C. A set C is said to be a *(mod-2) cycle* if every member of ∂C has even multiplicity.

In a 4-regular plane multigraph M, we say that a subset C of the edges of M is a *through cycle* (denoted t-cycle) if every vertex in the set ∂C is shared by exactly two edges in C, and these edges are *not* adjacent in the cyclic order of edges around the vertex defined by the planar embedding of M. In other words, a through cycle corresponds to the projection of one of the components of a link that projects to M. Thus a graph M with only one t-cycle is the projection of a knot, while a graph M with μ t-cycles is the projection of a link with μ components.

In the 1920s Reidemeister [1] discovered a set of moves on knot and link diagrams that capture ambient isotopy in combinatorial form. His moves are illustrated in Figure 2.

These moves indicate local changes that can be performed on the link diagram. Thus in the first move one locates a region with one edge and one vertex and eliminates it or creates it by performing the move in reverse. In the second move one creates or destroys a two-sided region. In the the third move one changes the configuration of a three-sided region. In each case the crossings at the boundary of the region are as indicated in Figure 2.

Two knots or links in three-space are *ambient isotopic* if and only if diagrams for them can be obtained one from the other by a sequence of Reidemeister moves. Consequently, it is a good strategy to find functions of the diagrams and to attempt to adjust these functions so that they are invariant under the moves. A function F defined on diagrams so that $F(K) = F(K')$ whenever K and K' differ by a Reidemeister move is called an *invariant* of knots and links. Certainly, if $F(K)$ is not equal to $F(L)$ then it follows (by Reidemeister's theorem) that $K$and L are not related by a sequence of the Reidemeister moves, and hence that K and L are not ambient isotopic.

We now give an example of the construction of such a function. We shall first define a function $[K]$ on knot and link diagrams such that $[K]$ is well-defined on diagrams and it is a polynomial in the commuting variables A,B and d. Then we shall see how to specialize A,B and d so that $[K]$ is invariant under the Reidemeister moves. See [22], [23].

To define $[K]$ we need the concept of a *state* of the link diagram K. A state S is obtained by *smoothing* each crossing of the diagram K. In smoothing, a crossing is replaced by two parallel arcs and the new diagram loses this vertex, but gains a *label* of either A or B depending on the sense of the smoothing relative to the crossing. Each crossing has possible two smoothings, as shown in Figure 3. The conventions for smoothing are labelling are illustrated in this figure. Since there are two choices for smoothing each crossing, there will be 2^n states for a diagram with n crossings. Each state is a labelled configuration of disjoint loops in the plane. Let $||S||$ denote the number of loops in the state and $[K|S]$ denote the product of the labels. Note that $[K|S]$ is a product of A's and B's, commuting variables. Now define $[K]$ to

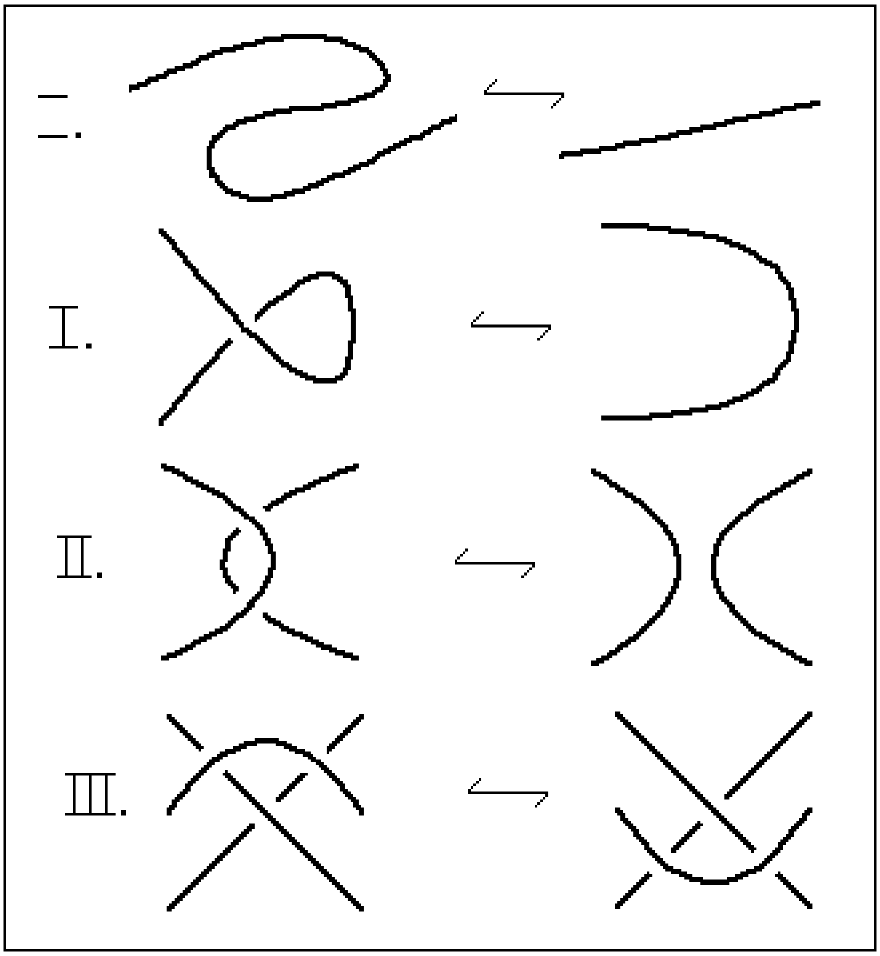

Figure 2 - Reidemeister Moves

be the sum over all the states of the products $[K|S]d^{||S||}$:

$$[K] = \sum_S [K|S]d^{||S||}.$$

This defines the *three-variable bracket polynomial.*

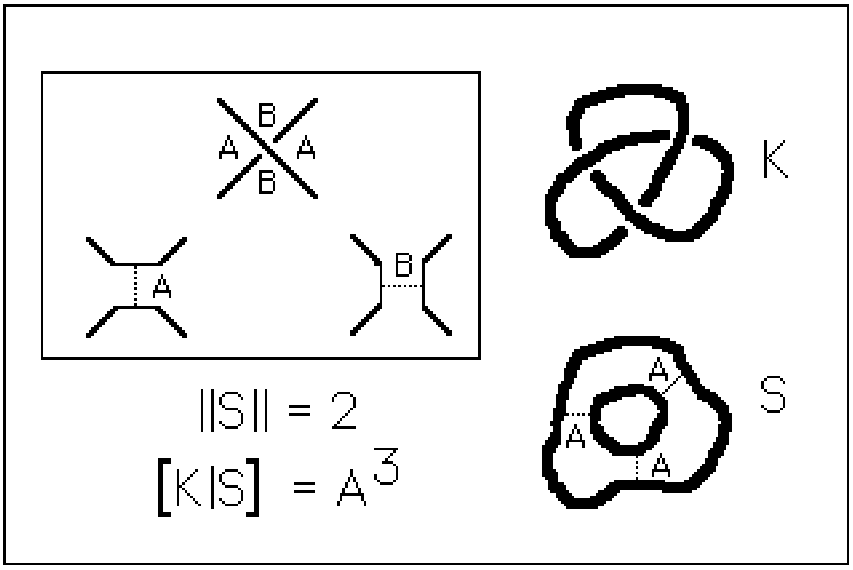

Figure 3- Smoothings and States

The following lemma shows how to recursively compute the bracket polynomial.

Lemma 1. Let K be a given link diagram and let K' and K'' denote two smoothings of K at a given crossing of K. Let K' be the smoothing with label A and K'' be the smoothing with label B. Interpret $[K']$ and $[K'']$ to be the bracket polynomials of the links obtained from these smoothings by removing the labels A and B. Then

$$[K] = A[K'] + B[K''].$$

(Figure 4 for an illustrates this relation.) Let O denote the unknotted circle and let OK denote any link diagram obtained from K by taking a disjoint union with a circle in the complement of the diagram. Then

$$[OK] = d[K].$$

Proof. These properties follow directly from the state summation definition of the bracket polynomial. //

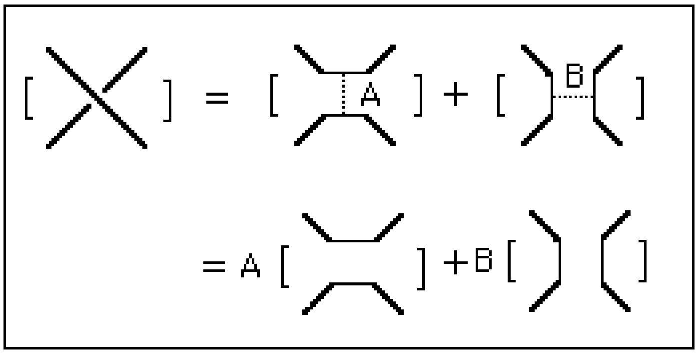

Figure 4 - Bracket Identity

A diagram K is said to be *oriented* if there is a chosen direction for each of its components. We assign a *sign* to each oriented crossing by the convention that a crossing is positive when a counterclockwise rotation of the overcrossing line makes it coincide with the undercrossing line. See Figure 5. In Figure 5 we also illustrate the two types of "curls". A *curl* is a local appearance of a single-edge region in the diagram. Either assignment of orientation to the component supporting a curl yields the same sign at the curl's crossing. Thus we can designate curls as positive or negative. Let $K(+)$ denote a diagram with a positive curl, and let K denote the corresponding diagram with the curl removed (by a Reidemeister move of type I). Similarly let $K(-)$ denote a diagram with a negative curl.

Lemma 2.

$$[K(+)] = (Ad + B)[K]$$

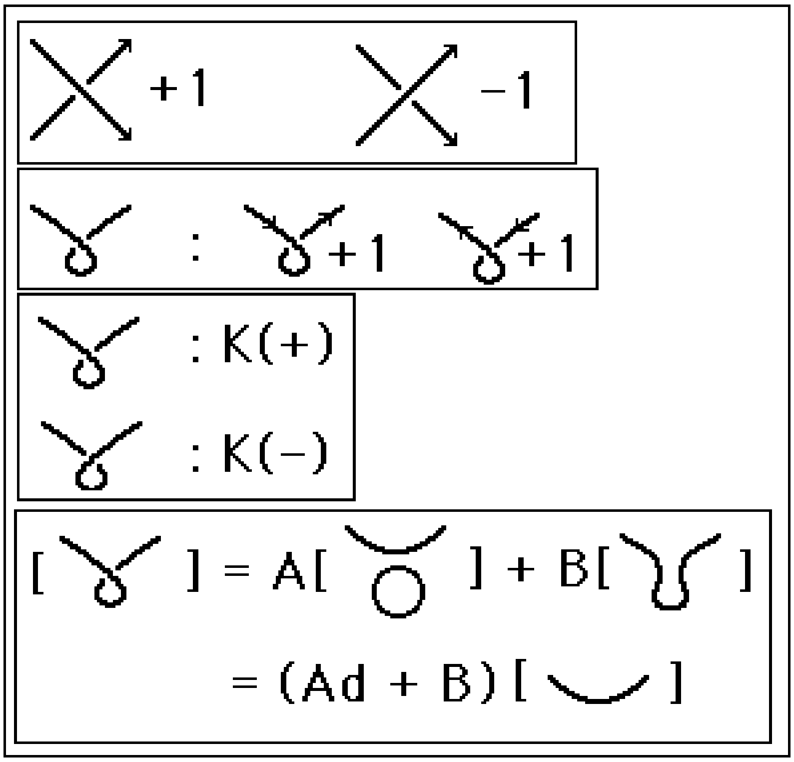

Figure 5 - Crossing Signs and Curls

and

$$[K(-)] = (A + Bd)[K].$$

Proof. See Figure 5.//

Lemma 3. Let $K(II)$ denote a diagram containing a two sided region that is capable of simplification under the type II move. Let K denote the result of doing the type II simplification and let $K*$ denote the opposite smoothing to this type II site, as shown in Figure 6. Then

$$[K(II)] = AB[K] + (ABd + A^2 + B^2)[K*].$$

Proof. See Figure 6.//

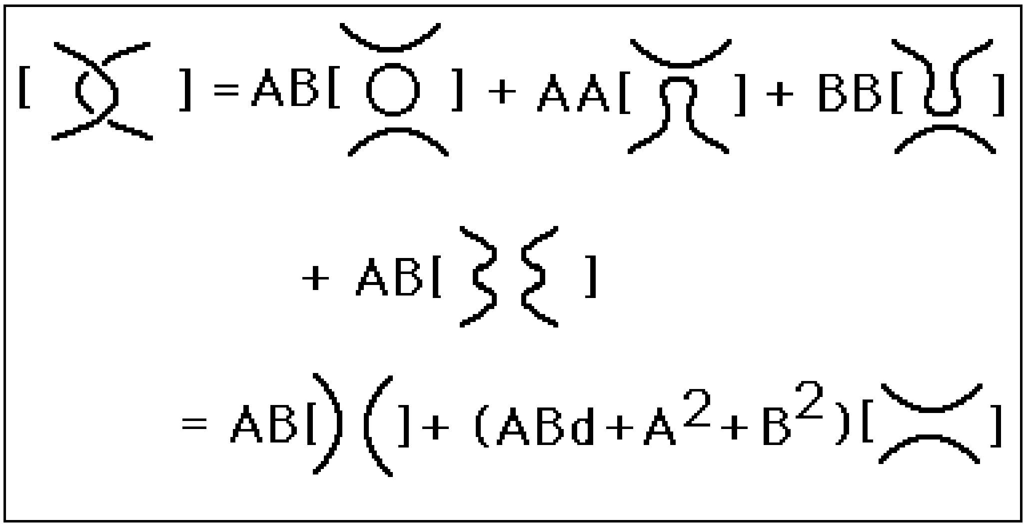

Figure 6 - Bracket Identity Relative to Move II

It follows from this Lemma that the bracket polynomial will be invariant under the second Reidemeister move if $B = A^{-1}$ and $d = -(A^2 + A^{-2})$. In fact, with these conditions it then follows easily that $[K]$ is invariant under the third Reidemeister move as well. See Figure 7 for an illustration of this. Note that with these assumptions

$$Ad + B = A(-A^2 - A^{-2}) + A^{-1} = -A^3$$

and

$$A + Bd = -A^{-3}.$$

Thus

$$[K(+)] = (Ad + B)[K] = -A^3[K]$$

and

$$[K(-)] = (A + Bd)[K] = -A^{-3}[K].$$

Let *regular isotopy* denote the equivalence relation on diagrams generated by the second and third Reidemeister moves, plus planar isotopy of diagrams. Then $[K]$ is an invariant of regular isotopy.

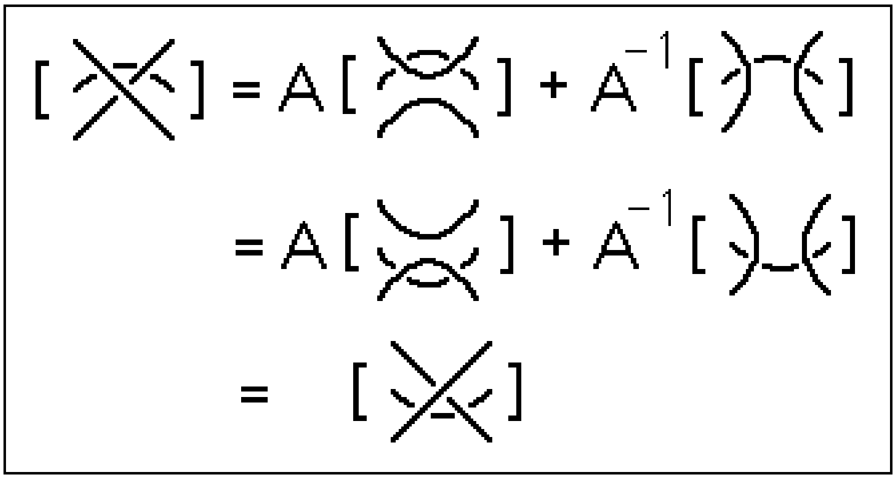

Figure 7 - Move III Invariance

With $B = A^{-1}$ and $d = -(A^2 + A^{-2})$ it is convenient to define

$$< K >= [K]/d$$

so that $< O >= 1$. This is the usual topological bracket polynomial.

Let the *writhe* $w(K)$ of an oriented link K be the sum of the signs of its crossings. We normalise the bracket to the polynomial

$$f_K(A) = (-A^3)^{-w(K)} < K >$$

where $< K >$ is computed by forgetting the orientation assigned to K. The polynomial f_K is an invariant of ambient isotopy. It gives a model for the original Jones polynomial $V_K(t)$ by the substitution

$$V_K(t) = f_K(t^{-1/4}).$$

By now there are many other invariants of knots and links beyond the classical Alexander-Conway polynomial, but the Jones polynomial and the bracket polynomial are important for the initial relationship with purely combinatorial state sums, graph polynomials and statistical mechanics. Furthermore the following conjecture remains outstanding:

Conjecture 4. For a link K of one component, $V_K(t) = 1$ implies that K is unknotted.

This conjecture is closely related to the fact that the spanning tree expansion of the bracket polynomial has no cancellation among its terms for K an alternating diagram. See [26], [27].

3 Jaeger's State Model for the Homfly Polynomial

The Homfly polynomial [20] is a common generalization of the Jones polynomial and the classical Alexander-Conway polynomial. For the purposes of this section we take the following axiomatic definition of the Homfly polynomial $P_K(z, a)$ in terms of the framing polynomial $H_K(z, a)$:

0. $H_K(z, a)$ assigns to each oriented link diagram a Laurent polynomial in the commuting variables z and a. If two diagrams are regularly isotopic (see the previous section for the definition of regular isotopy) then they receive the same H-polynomial.

1. $H_O = 1$.

2. $H_{K_+} - H_{K_-} = zH_{K_0}$ whenever K_+, K_-, K_0 form a triple of diagrams that differ only at one site, with K_+ having a positive crossing at that site, K_- a negative crossing and K_0 the oriented smoothing of that crossing.

3. $H_{K(+)} = aH_K$ and $H_{K(-)} = a^{-1}H_K$. Recall from the previous section that $K(+)$ and $K(-)$ denote diagrams with a positive or negative curl, with K denoting the result of removing this curl by a type I Reidemeister move.

4. $P_K(z,a) = a^{-w(K)}H_K(z,a)$ where $w(K)$ denotes the writhe of K as defined at the end of the last section, and $H_K(z,a)$ is described above.

The H-polynomial is a framed version of the Alexander-Conway polynomial. It has the same "skein identity"

$$H_{K_+} - H_{K_-} = zH_{K_0}$$

as Alexander-Conway, but the extra variable a keeps track of the framings of the diagrams via the curl identities in 3. above. The Homfly polynomial P_K is an invariant of ambient isotopy. It is not hard to see that our axioms imply that

$$H_{OK} = \delta H_K$$

where

$$\delta = (a - a^{-1})z^{-1}.$$

This is the "loop value" for the Homfly polynomial.

After the discovery of the bracket state model there came a large number of models for new link invariants that were based on this idea of using the diagram of the link as a basis for forming a state summation. The key to this prolixity of models is the use of ideas from statistical mechanics where more general vertex weights prevail and one designates states by assigning "colors" to the edges or to the faces of the link diagram in such a way that a local assignment of colors at vertex gives rise to a specific vertex weight. Nevertheless, even at this level of generality there did not seem to be a state summation for the full two variable Homfly polynomial. It was in this context that Jaeger found a state summation model for the whole Homfly polynomial [5]. We will describe that model.

In Jaeger's state sum for the Homfly polynomial the states of a diagram K are defined by first replacing each crossing of the diagram by either a flat crossing or a smoothing (See Figure 8) to form the *state diagram* S_0.

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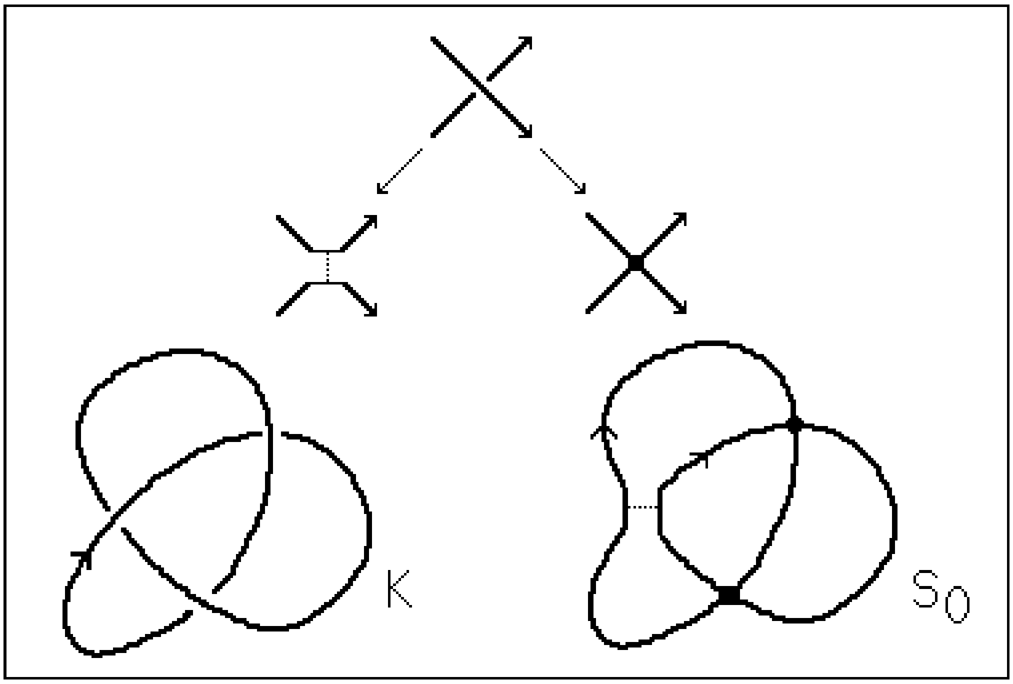

Figure 8 - State Diagrams for Jaeger's State Sum

We then take a series of walks on the t-cycles (See section 2 for the definition of a t-cycle) of this 4-regular plane graph S_0. The initial location for the walk is chosen by a *template* where the template is a labelling of the edges of K from any subset of the positive integers. The edge with the least template label is the starting point for the first walk. One then takes the walk on the t-cycle determined by this starting point. (Note that in an oriented diagram, the t-cycle and direction of the walk is completely determined by the choice of initial edge.) The next walk begins on the edge with least unused label. The process continues in this way until walks have been taken on all the t-cycles in the state diagram S. Each walk creates labels on the sites of the state diagram S_0. If a walk passes through a smoothing for the first time, we mark the leg of the smoothing where the first passage occurs with a dot as shown in Figure 9. If the walk passes through a flat crossing for the first time, we draw this passage as an overpass on the state diagram S_0.

Once the walks have all been performed, the state diagram has a labelling at each crossing that indicates the first passage in the walks made on it. At this point the state diagram S_0 has become a state S for the state summation. The vertex weights for the state summation are then obtained from the labels on S by the following prescription:

1. At a smoothing in S, if the crossing in K is positive and the first passage is along the bottom of the smoothing (See Figure 9) then the weight is z.

2. At a smoothing in S, if the crossing in K is negative and the first passage is along the top of the smoothing (See Figure 9) then the weight is $-z$.

3. At a crossing in S, if the crossing is positive then the weight is a.

4. At a crossing in S, if the crossing is negative then the weight is a^{-1}.

The product of these weights is denoted by $< K|S >$.
Let $||S||$ denote the number of t-cycles in S.
Then Jaeger's state sum formula for the H-polynomial is simply

$$H_K(z,a) = \sum_S < K|S > \delta^{||S||-1}$$

where $\delta = (a - a^{-1})z^{-1}$ is the loop value as discussed above. After the normalization this gives a model for the Homfly polynomial. The same method applies to the Kauffman polynomial [27].

Jaeger's proof of the validity of this state sum can be simplified by recognising its relationship with the recursive process inherent in the skein identity for the H-polynomial. In [25] I show that the states in this model are in one to one correspondence with the diagrams that occur at the bottom of the tree generated in a skein calculation. Thus the model follows directly from

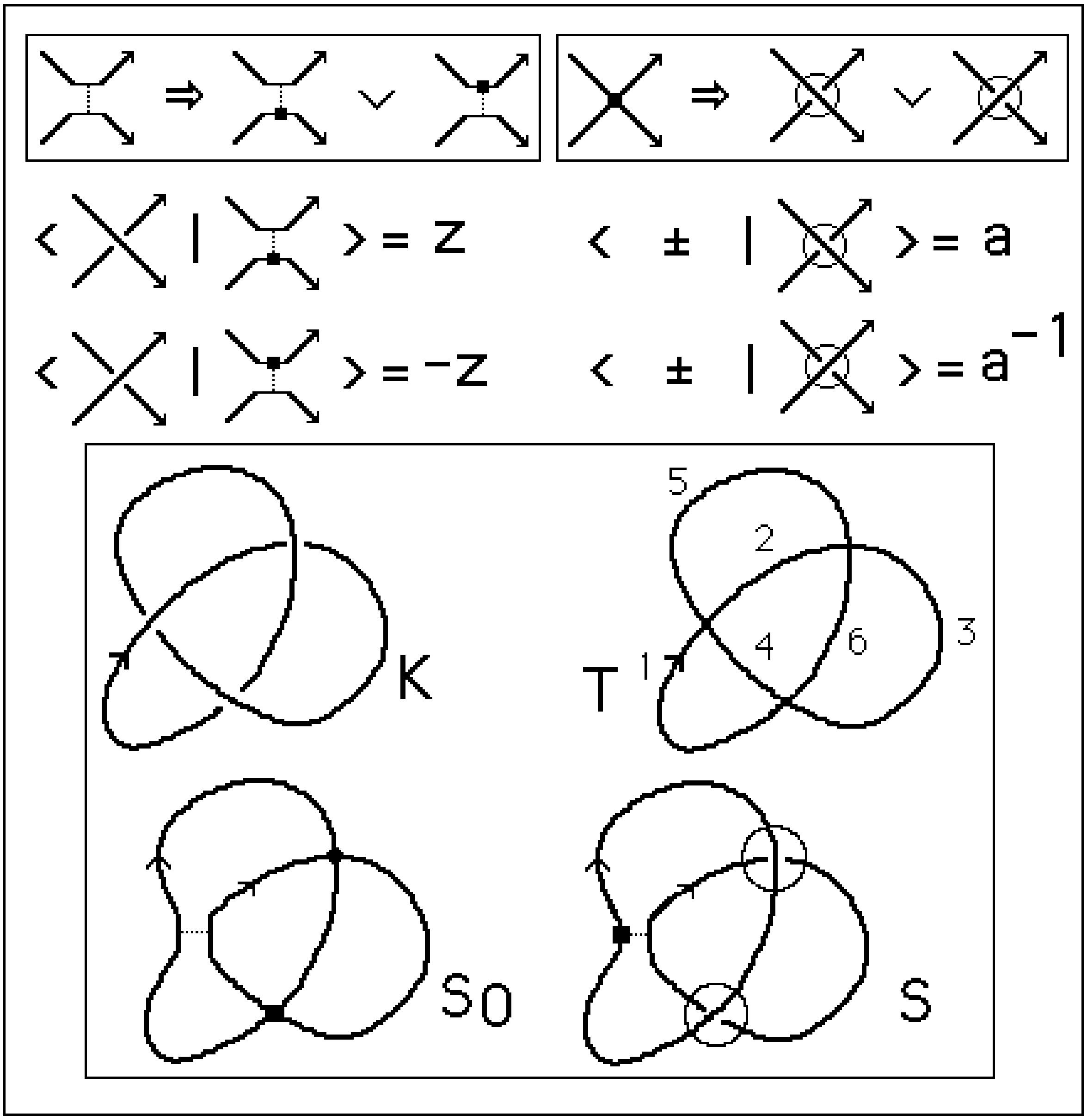

Figure 9 - States Arise From Walks on State Diagrams

this form of calculation. Nevertheless it is quite interesting to have a direct formula of this kind for the Homfly polynomial. I believe that there is much more that can be done with Jaeger's state model and that the corresponding polynomials defined for 4-regular plane graphs will be of interest to combinatorialists.

4 Medial Graphs and Graphical Reidemeister Moves

Recall that a *signed graph* is a graph with labels of $+1$ or -1 on each edge. To each knot or link diagram K there is an associated signed graph $G(K)$. We recall the construction of $G(K)$ below and also how the Reidemeister moves on the knot and link diagrams translate to moves on the graphs.

First of all, it is always possible to color the regions of a knot or link diagram with two colors so that adjacent regions receive opposite colors. This two-color theorem for link diagrams follows directly from the Jordan curve theorem in the plane. (Smooth the crossings of the link diagram in such a way so that there is only one curve in the resulting state. Color the the inside of this curve black and the outside white. Take the corresponding coloring that is induced on the regions of the original link diagram. This is the desired coloring.) I shall refer to the two coloring of the regions of link diagram with the outer region colored white as the *checkerboard coloring* of the diagram.

The graph $G(K)$ is directly associated with the checkerboard coloring of K. The vertices of $G(K)$ are in one to one correspondence with the black regions of the coloring. There is an edge between two vertices whenever a crossing is shared by the corresponding regions. The sign of this edge is $+1$ when the overcrossing line can be rotated counterclockwise through the shaded region to coincide with the undercrossing line. See Figure 10.

The translation of the Reidemeister moves to graphical moves is shown in Figure 11. These graphical Reidemeister moves allow the generalization of knot theory to arbitrary graphs. (Planar graphs correspond to classical knots and links by inverting the construction of $G(K)$. For non-planar graphs

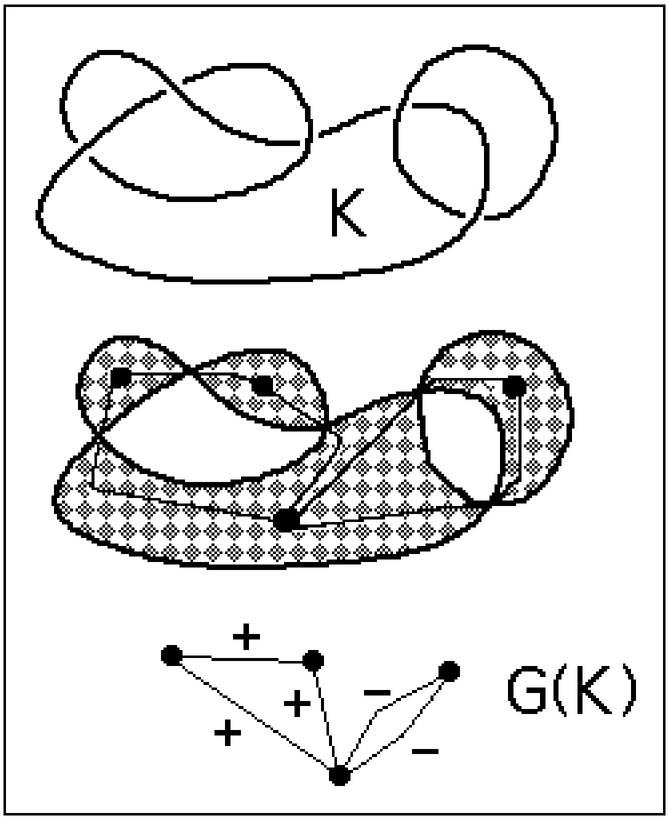

Figure 10 - The Checkerboard Graph G(K) of a Link Diagram K

this theory needs further exploration.)

Note the analogy of the graphical moves with series-parallel and star-triangle moves on electrical circuits. In fact [28] we can interpret the +1 or −1 on an edge of the graph G(K) as a conductance (negative conductance can be treated in an algebraically consistent way for circuits containing only conductors). Then the two versions of the second Reidemeister move and the third Reidemeister move are correct replacements for conductance in a circuit. We obtain invariants of knots and links by measuring the conductances of their corresponding graphs between two chosen points on the graph. The resulting conductance is an invariant of motions of the link that do not pass strands across the regions corresponding to the chosen points on the graph. Many knots and links (for example alternating links) exhibit non-zero conductance simply because all the conductances have the same sign.

5 Graphical Reformulation of the Bracket Polynomial as a Spin Model

We begin by considering states S of a link diagram K where these states are the states described in Section 2. Thus S is obtained from K by choosing a smoothing for each crossing of K. It is then clear that a state S of the link diagram K corresponds to labelling edges of the graph $G(K)$ *internal (i)* or *external (e)* as shown in Figure 12. An edge is internal if the local smoothing in S corresponding to that edge joins shaded regions in the checkerboard coloring of the diagram K. We let $I(S)$ denote the number of internal edges for the state S, and $E(S)$ the number of external edges for the state S.

As before, we let $||S||$ denote the number of components in the state S. We now define $|S|$ to be the number of shaded components in the checkerboard coloring of S. It is a direct consequence of the remarks above that $|S|$ is the number of components of $G(K)$ - { interiors of external edges }. We let $N = N(K)$denote the number of vertices of $G(K)$.

Proposition 5. Using the terminology established above, if S is a state of the link diagram K, then $||S|| = 2|S| + I(S) - N(K)$.

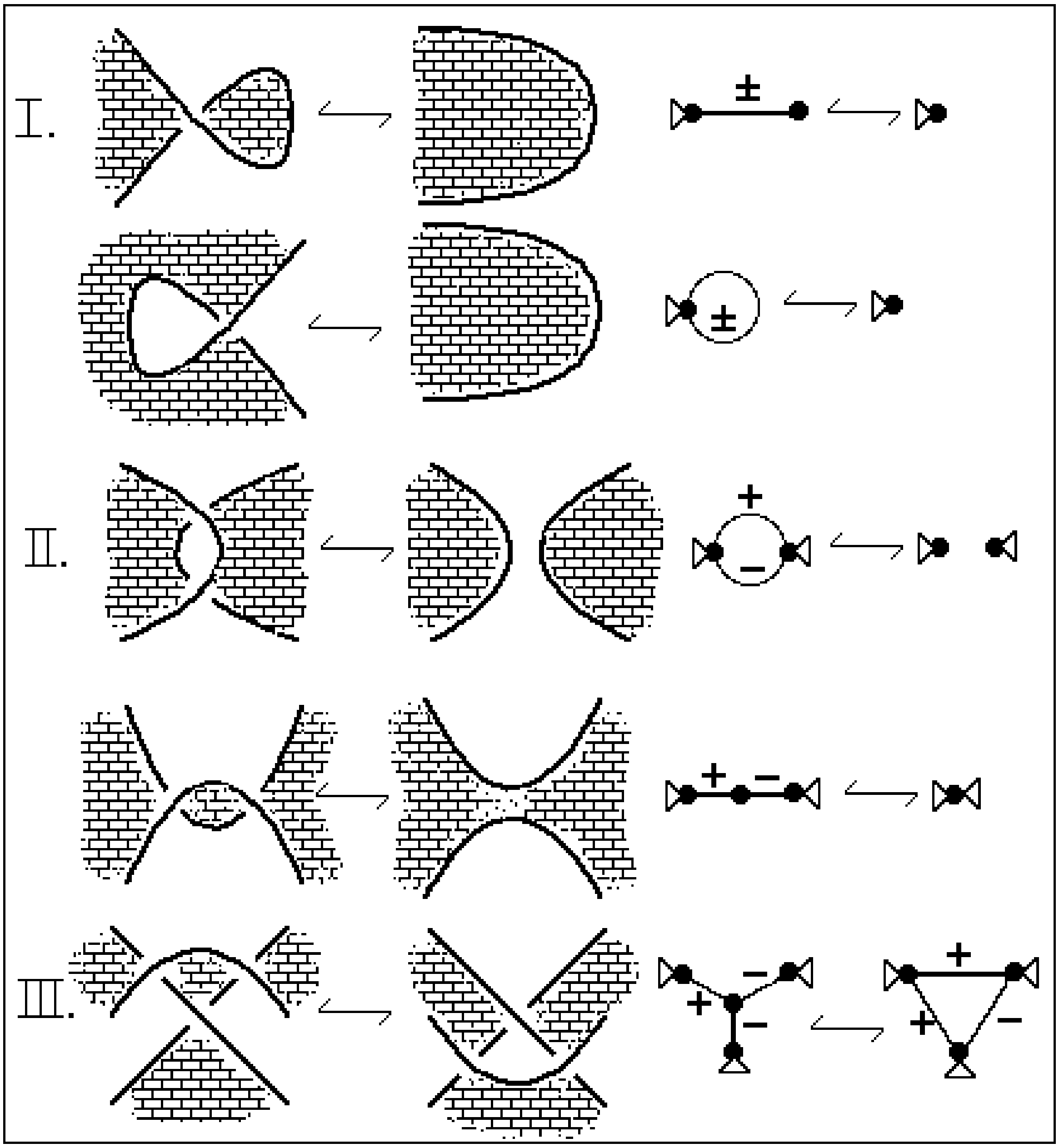

Figure 11 - Graphical Reidemeister Moves

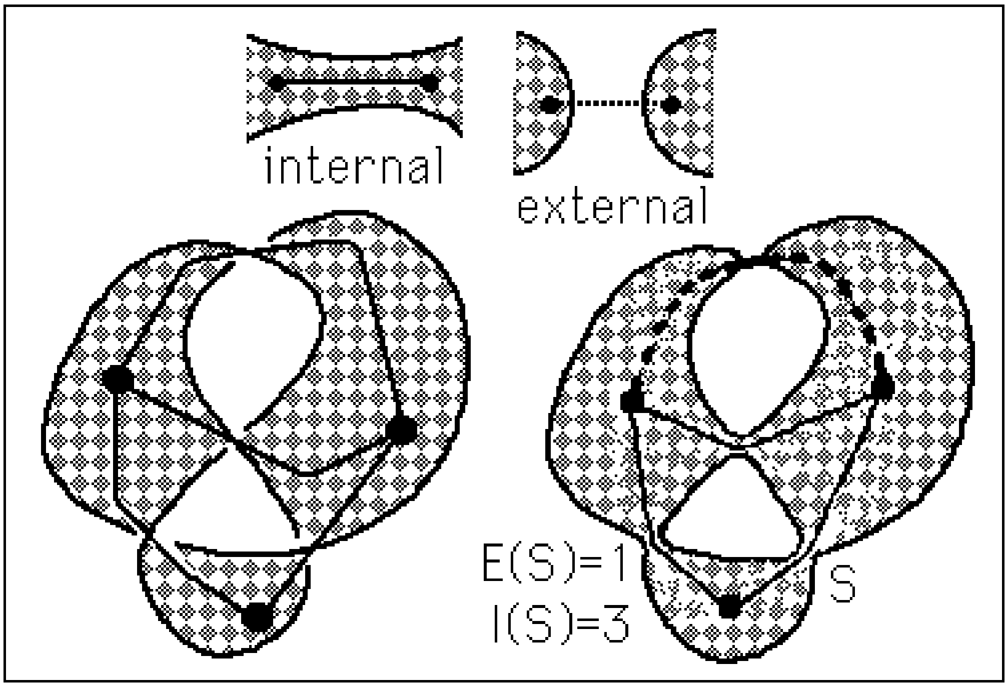

Figure 12 - Internal and External Edges

Proof. See [23]. We omit the proof here. //

See Figure 13 for an example illustrating this proposition.

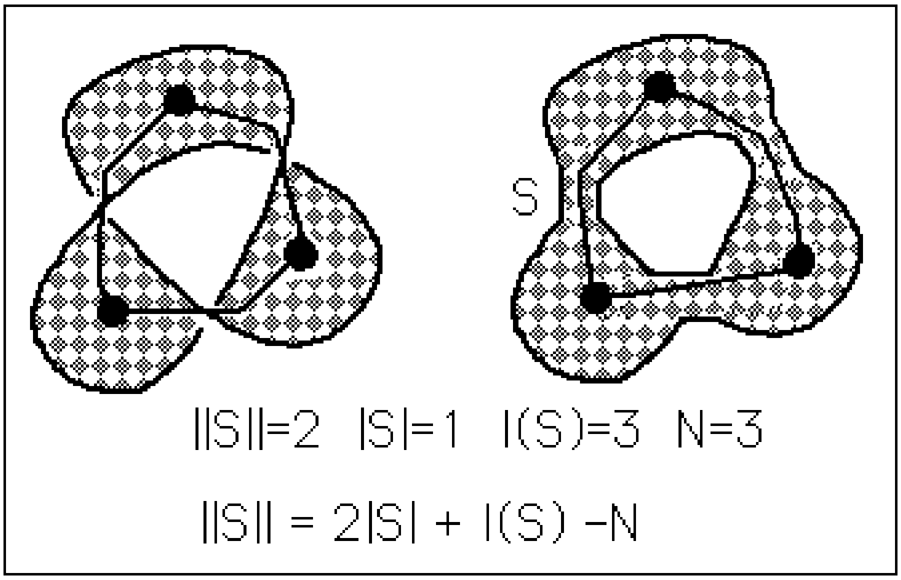

Figure 13 - Illustrating Boundary and Region Count

Let K be a link diagram. An internal edge in a state S will be said to be of type $+$ if the vertex weight is A (see section 2) and type $-$ if the vertex weight is B. An external edge in a state S will be said to be of type $+$ if the vertex weight is B and type $-$ if the vertex weight is A. Let $I_{\pm} = I_{\pm}(S)$ denote the number of internal edges in a state S of type $\pm$. Let $E_{\pm} = E_{\pm}(S)$ denote the number of external edges in a state S of type $\pm$.

Proposition 6. With the terminology of the previous paragraph the (three variable) bracket polynomial of a link diagram K can be expressed by the formula

$$[K] = d^{-N(K)} \sum_S (Ad)^{I_+}(Bd)^{I_-}B^{E_+}A^{E_-}(d^2)^{|S|}.$$

Proof.

$$[K] = \sum_S A^{I_+ + E_-} B^{I_- + E_+} d^{||S||}$$

$$= \sum_S A^{I_+ + E_-} B^{I_- + E_+} d^{2|S| + I(S) - N(K)}$$

$$= d^{-N(K)} \sum_S (Ad)^{I_+} (Bd)^{I_-} B^{E_+} A^{E_-}.$$

since $I(S) = I_+ + I_-$. This completes the proof. //

A little thought now shows that this reformulation of the bracket state sum in terms of the signed graph can be expressed by the formulas

$$[G|+] = Ad[G'] + B[G'']$$

$$[G|-] = Bd[G'] + A[G'']$$

$$[p \sqcup G] = d^2[G]$$

where $[G|\pm]$ denotes G with a selected signed edge with sign $\pm$, G' denotes the signed graph obtained from G by contracting the special edge and G'' denotes the signed graph obtained from G by deleting the special edge. See Figure 14 for a diagrammatic notation for this recursion.

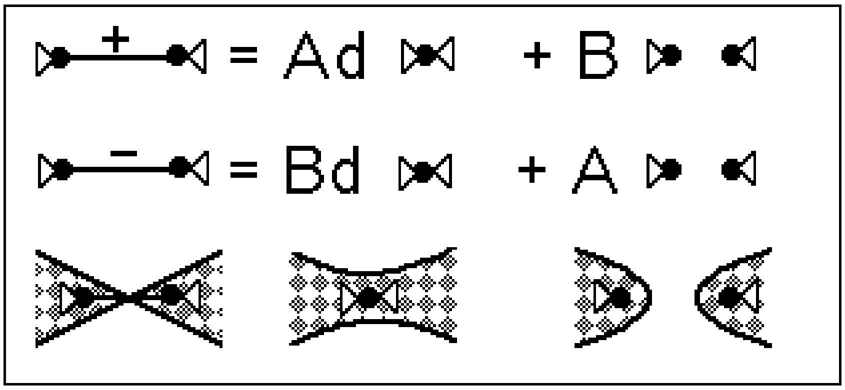

Figure 14 - Graphical Recursion

We now further translate this graphical bracket state sum into a *spin model* (a generalization of the Potts model in statistical mechanics) on $G(K)$. In the course of this translation we shall see the full definition of the spin model emerge consistently with its description in [21]. We shall henceforth refer to $G(K)$ by the letter G alone.

First choose a "spin" set $\{1, 2, 3, ..., n\}$ for some positive integer n. Assign spins from the set $\{1, 2, 3, ..., n\}$ to the vertices of G. Assign weights $w_{\pm}(\alpha, \beta)$ to each edge of G that is labelled with the spins α and β at its ends. See Figure 15. The weights are in an appropriate commutative ring, often the complex numbers.

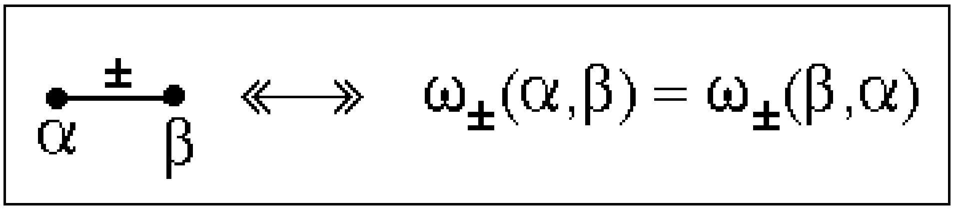

Figure 15 - Edge Spins and Vertex Weights

Now form the *partition function*

$$Z_G = (\sqrt{n})^{-N} \sum_{\sigma} \prod_{edges} w_{\pm}(\alpha, \beta).$$

Here σ denotes a spin assignment to the vertices of G.

It is then easy to see that with $d = \sqrt{n}$ and $w_{\pm}(\alpha, \beta)$ defined as shown below, that

$$Z_{G(K)} = [K]$$

for the three variable bracket polynomial $[K](A, B, d)$.

$$w_+(\alpha, \beta) = Ad + B, \alpha = \beta$$

$$w_+(\alpha,\beta) = B, \alpha \neq \beta.$$

$$w_-(\alpha,\beta) = A + Bd, \alpha = \beta$$
$$w_-(\alpha,\beta) = A, \alpha \neq \beta.$$

In the topological case, we have $B = A^{-1}$ and $d = \sqrt{n} = -A^2 - A^{-2}$. Thus, for a given choice of n, the A is a specific complex number. The weights are then given by the formulas:

$$w_+(\alpha,\beta) = -A^3, \alpha = \beta$$
$$w_+(\alpha,\beta) = A^{-1}, \alpha \neq \beta.$$

$$w_-(\alpha,\beta) = -A^{-3}, \alpha = \beta$$
$$w_-(\alpha,\beta) = A, \alpha \neq \beta.$$

Since the bracket polynomial in the topological case is invariant under the second and third Reidemeister moves, it follows that Z_G will be invariant under the graphical counterparts to the second and third Reidemeister moves. These invariances lead in turn to a set of conditions on the vertex weights in the spin model. This is illustrated in Figure 16 and the corresponding equations are given in the text below. The behaviour under type I moves is illustrated in Figure 16. These behaviours lead to equations 1.1 and 1.2. Equation 0 just expresses the symmetry of this model (an assumption that can be dropped.)

0. $w_\pm(\alpha,\beta) = w_\pm(\beta,\alpha)$

1.1 The value a is independent of the choice of α.

$$w_+(\alpha,\alpha) = a,$$

$$w_+(\alpha,\alpha) = a^{-1}.$$

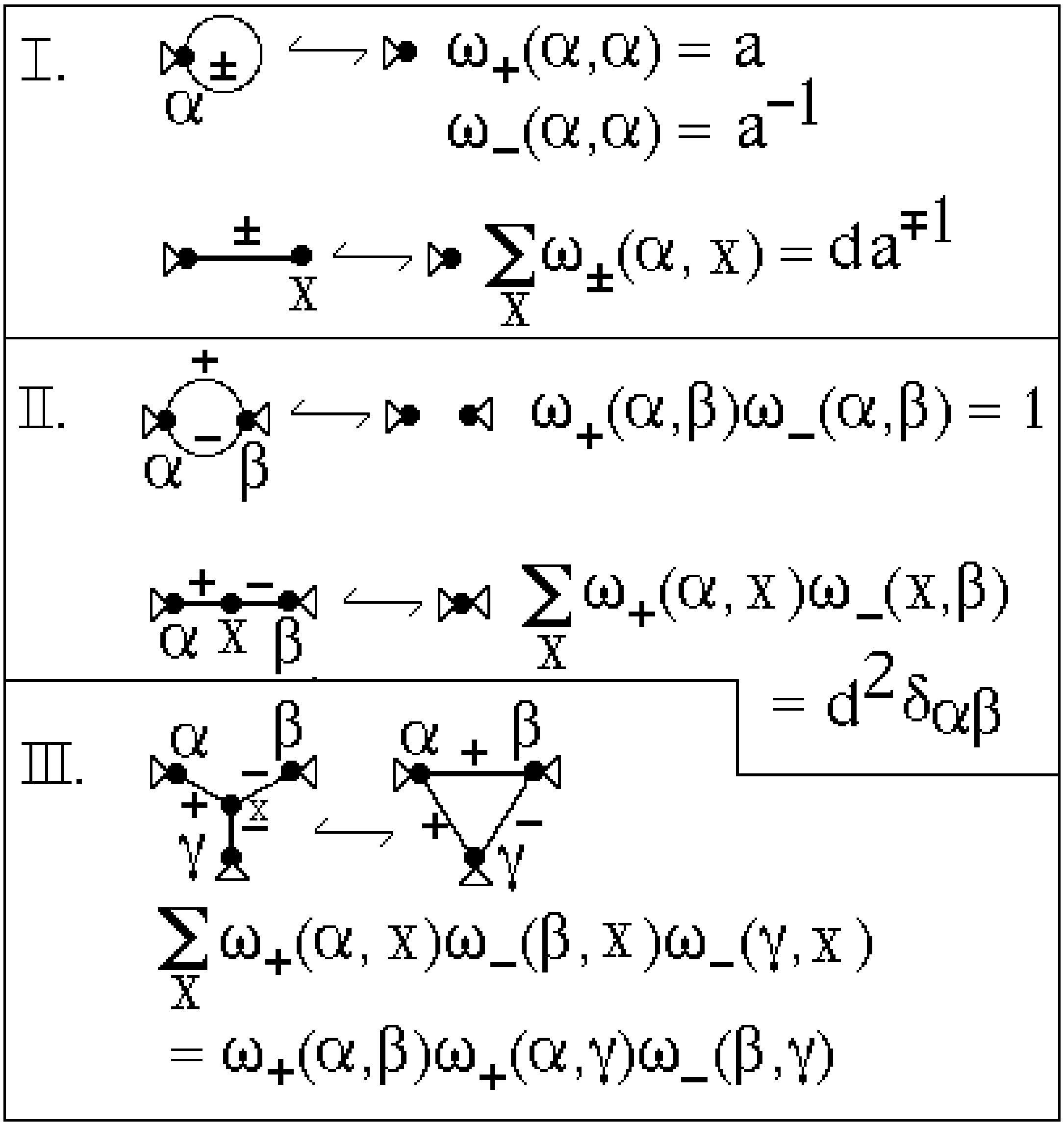

Figure 16 - Vertex Weights and Reidemeister Moves

1.2

$$\sum_x w_+(\alpha, x) = da^{-1},$$

$$\sum_x w_-(\alpha, x) = da.$$

2.1.

$$w_+(\alpha, \beta) w_-(\alpha, \beta) = 1$$

2.2.

$$\sum_x w_+(\alpha, x) w_-(x, \beta) = n\delta_{\alpha\beta},$$

where $\delta_{\alpha\beta}$ denotes the Kronecker delta that is equal to one when α and β are equal and is equal to zero when they are unequal.

3.

$$\sum_x w_+(\alpha, x) w_-(\beta, x) w_-(\gamma, x) = \sqrt{n} w_+(\alpha, \beta) w_+(\gamma, \alpha) w_-(\beta, \gamma).$$

Each of these conditions can be expressed more concisely in matrix algebra as follows

0.

$$W_\pm = W_\pm^t$$

where M^t denotes the transpose of the matrix M and $(W_\pm)_{\alpha\beta} = w_\pm(\alpha, \beta)$.

1.1.

$$I \circ W_+ = aI$$

and

$$I \circ W_- = a^{-1}I.$$

Here $M \circ N$ is the *Hadamard product* of the matrices M and N given by the formula

$$(M \circ N)_{ij} = M_{ij}N_{ij}.$$

1.2.

$$JW_+ = da^{-1}J$$

and

$$JW_{-} = daJ$$

where J is an n x n matrix all of whose entries are equal to 1. Note that J is the identity matrix for the Hadamard product.

2.1.

$$W_{+} \circ W_{-} = J$$

2.2.

$$W_{+}W_{-} = d^2 I$$

3.

$$W_{-}V(\alpha,\gamma) = dw_{+}(\alpha,\gamma)V(\alpha,\gamma).$$

Here $V(\alpha,\gamma)$ is the column vector with entries

$$V(\alpha,\gamma)_x = w_{+}(\alpha,x)w_{-}(x,\gamma).$$

This matrix reformulation of the necessary properties of the spin model is very suggestive. It suggests that an appropriate context for studying spin models would be in a matrix algebra that is closed under ordinary matrix product and under the Hadamard product. In fact Jaeger discovered that the right context is the Bose-Mesner algebra of an association scheme, as we shall see in the next section.

6 Association Schemes and Bose-Mesner Algebra

An *association scheme* consists in a set X and a collection of relations

R_0, R_1, ..., R_n on X with

1.

$$R_0 = \Delta(X) = \{(x,x)|x \in X\}.$$

2.

$$R_0 \cup R_1 \cup ... \cup R_n = X \times X.$$

3.

$$R_i \cap R_j = \phi$$

when $i \neq j$.

4. For all i, $R_i^t = R_k$ for some k. Here R^t denotes the *transpose* of the relation R: xR^ty if and only if yRx.

5. If $(x, y) \in R_k$, let p_{ij}^k denote the number of z such that $(x, z) \in R_i$ and $(z, y) \in R_j$. It is given that p_{ij}^k is *independent* of the choice of x and y.

Association schemes arise in many combinatorial contexts [30]. There is a natural matrix algebra that describes any given association scheme. This is called the *Bose-Mesner algebra* and it is constructed as follows:

Let A_i be the adjacency matrix of the relation R_i. That is, $(A_i)_{xy}$ equals 1 if $(x, y) \in R_i$ and is 0 otherwise. Then

1.

$$A_i A_j = \sum_{k=0}^{d} p_{ij}^k A_k$$

2.

$$A_i \circ A_j = \delta_{ij} A_i$$

3.

$$\sum_{i=1}^{n} A_i = J$$

where J is the matrix all of whose entries of ones.

These statements in matrix algebra encapsulate the properties of an association scheme.

Now note that the weight matrices for the spin model corresponding to the bracket polynomial are given by the formulas

$$W_+ = (Ad + B)I + B(J - I)$$
$$W_- = (Bd + A)I + A(J - I).$$

Here I and J are n x n matrices for the spin models with $d = \sqrt{n}$. The specialization $d = -(A^2 + A^{-2})$, $B = A^{-1}$ gives the topological model satisfying the corresponding spin model equations for the second and third Reidemeister moves. However the decomposition of the general weight matrix into combinations of I and $J - I$ corresponds directly to the bracket expansion into two smoothings and to the contraction deletion formula that we gave for the graphical version of this model.

This example shows that the weight matrices for the spin model of the bracket polynomial are expressed as linear combinations ot the basis matrices for the simplest Bose-Mesner algebra. These matrices correspond directly to the relations "same" and "different" that underlie the combinatorics of this model. In fact we have shown this result for the full bracket polynomial whence for a version of the Tutte polynomial for signed graphs [26]. State summations for more complex relations will be captured by other Bose-Mesner algebras.

It was with this idea in hand that Jaeger went searching for other link invariants that could be modelled using the Bose-Mesner algebra of an association scheme. To this end one can set

$$W_+ = \sum_{k=0}^{n} x_k A_k$$

$$W_- = \sum_{k=0}^{n} x_k^{-1} A_k$$

and it follows that

$$W_+ \circ W_- = J$$

since the equations $A_i \circ A_j = \delta_{ij} A_i$ and $\sum_{i=1}^{n} A_i = J$ hold in the Bose-Mesner algebra.

The other relations for a topological spin model are harder to come by and this is the beginning of a long and complex story. Rather than tell it here we refer the reader to Jaeger's excellent papers and to other papers that grew out of his work. In particular, Jaeger eventually proved [18] that any spin model naturally gives rise to a Bose-Mesner algebra for an association scheme such that the weight matrices are elements of this algebra. Another proof of this result was recently given by Nomura [32].

We finish by indicating Jaeger's most striking result on spin models [10]. He began his study by going to the first step beyond the Bose-Mesner algebra generated by I and $J - I$. In this step one takes a new matrix with entries zero or one, A, such that the generators of the Bose-Mesner algebra are I, A and $B = J - I - A$ (so that $I + A + B = J$). Letting G be the graph whose adjacency matrix is A (i.e. the vertices of G are in one-to-one correspondence with the spin set for the model and two vertices i and j are connected by an edge in G exactly when the matrix entry A_{ij} is equal to one). Then it follows from the axioms of the Bose-Mesner algebra that

$$A^2 + (\mu - \lambda)A + (\mu - k)I = \mu J$$

and G is a *strongly regular graph* of type (n, k, λ, μ) where

1. n is the number of vertices in G,
2. each vertex in G has exactly k neighbors,
3. two vertices joined by an edge in G have exactly λ neighbors,
4. two distinct vertices not joined by an edge in G have exactly μ neighbors.

(These properties constitute the definition of strong regularity of G.)

One wants to build a spin model with such a Bose-Mesner algebra using weight matrices of the form

$$W_+ = rI + sA + t(J - I - A),$$

$$W_- = r^{-1}I + s^{-1}A + t^{-1}(J - I - A).$$

Jaeger discovered [10] that such a spin model exists for

$$(n, k, \lambda, \mu) = (100, 22, 0, 6)$$

with G the *Higman-Sims graph*, a strongly regular graph of 100 vertices whose group (the Higman-Sims group) of automorphisms is one of the important simple finite groups. Jaeger's model gives a specialization of the Kauffman polynomial at a root of unity [24]. This remarkable example remains a puzzle to this day. One suspects that Jaeger's Higman-Sims example is the tip of an iceberg of yet to be discovered new mathematics interrelating combinatorics, statistical mechanics, group theory and topology.

7 A Graphical Approach to Bose-Mesner Algebra

This last section is really a remark to the effect that a "quantum network" generalization of the electrical circuit analogy for series and parallel graphs leads naturally to a combination of ordinary and Hadamard matrix multiplication and to most of the properties of a Bose-Mesner algebra. By a *quantum network* I mean a graph that has been equipped with generalized matrices (tensors) at its nodes so that a coloring of the edges from an appropriate index set yields vertex weights for each node and a partition function that is the sum of the products of these vertex weights as the edges receive all possible colors. Such a partition function generalizes the spin model and will be referred to as the *amplitude* of the network.

In the electrical case and in the graphical version of the bracket polynomial we have the analogs of the open and closed switches in the form of edges that correspond to contraction (closed switch) and deletion (open switch). The identity matrix I corresponds to the closed switch. I shall give an interpretation where the matrix J (a square matrix all of whose entries are equal to one) corresponds to the open switch.

First note the symbolism in Figure 17. Working first with the electrical analogy, we see that a series connection of an input-output circuit A with the closed switch yields A, so that the closed switch is the identity for a series connection. On the other hand a parallel connection with the open switch shows that the open switch is the identity for parallel connection.

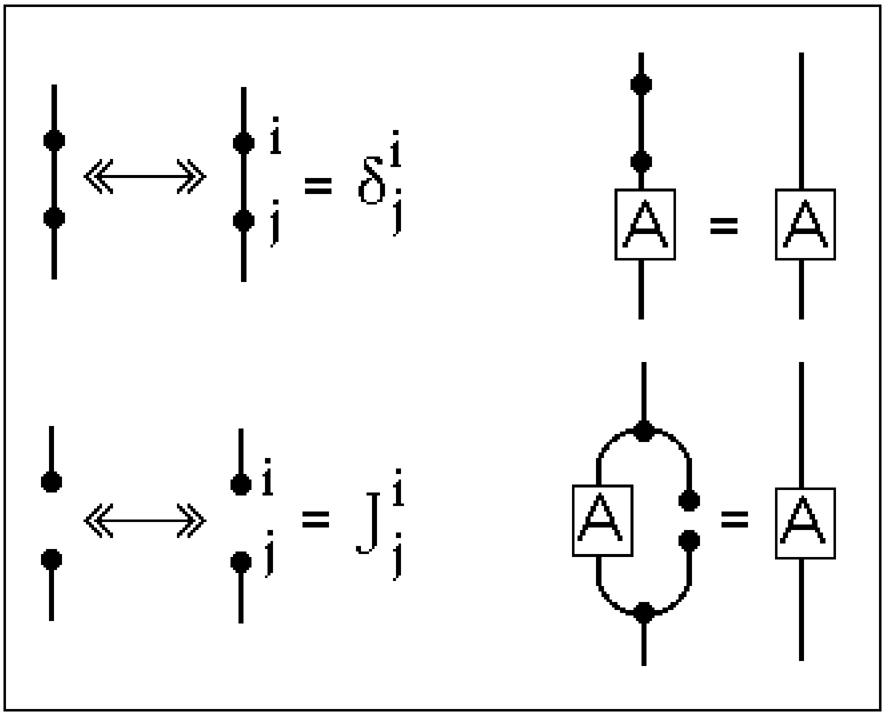

Figure 17 - Open and Closed Switches

Going to the quantum network, we replace the input-output network with a matrix or tensor A^i_j where i and j denote the possible states or colorings on the input and output lines. Each choice of such colors yields a specific amplitude for A. The identity for series connection is obviously the identity matrix δ^i_j in the form of the Kronecker delta. We need to *define* a parallel connection of tensors A and B (A^i_j and B^i_j). View Figure 18. In this figure we have assigned special trivalent vertices E^i_{pq} and E^{rs}_j so that

$$(A \circ B)^i_j = E^i_{pq} A^p_r A^q_s E^{rs}_j$$

represents the parallel connection of A and B. We define

$$E^i_{pq} = \delta^i_p \delta^i_q$$

and

$$E^{rs}_j = \delta^r_j \delta^s_j$$

so that the triple vertices demand that all their legs have identical labels.

It is then easy to see that $A \circ B$ as defined above is exactly the Hadamard product of matrices and so the quantum analog of a parallel connection is Hadamard multiplication, with J as the identity. At this point it is clear that an algebra of matrices closed under both ordinary and Hadamard product would be appropriate for studying quantum networks with both series and parallel connections. The Bose-Mesner algebra with its particularly simple closure under the Hadamard product is one example of such an algebra.

8 Notes About the References

In the bibliography to this paper the following references are a list of Francois Jaeger's papers that related directly to knot theory: [2], [3], [4], [5], [6], [7], [8], [9], [10], [11], [12], [13], [14], [15], [16], [17], [18], [19].

The papers [29], [30], [31], [32], [33] are survey or research papers related to spin models and association schemes. This bibliography is by no means complete in terms of literature of authors other than Francois Jaeger. The subject is an area of active research.

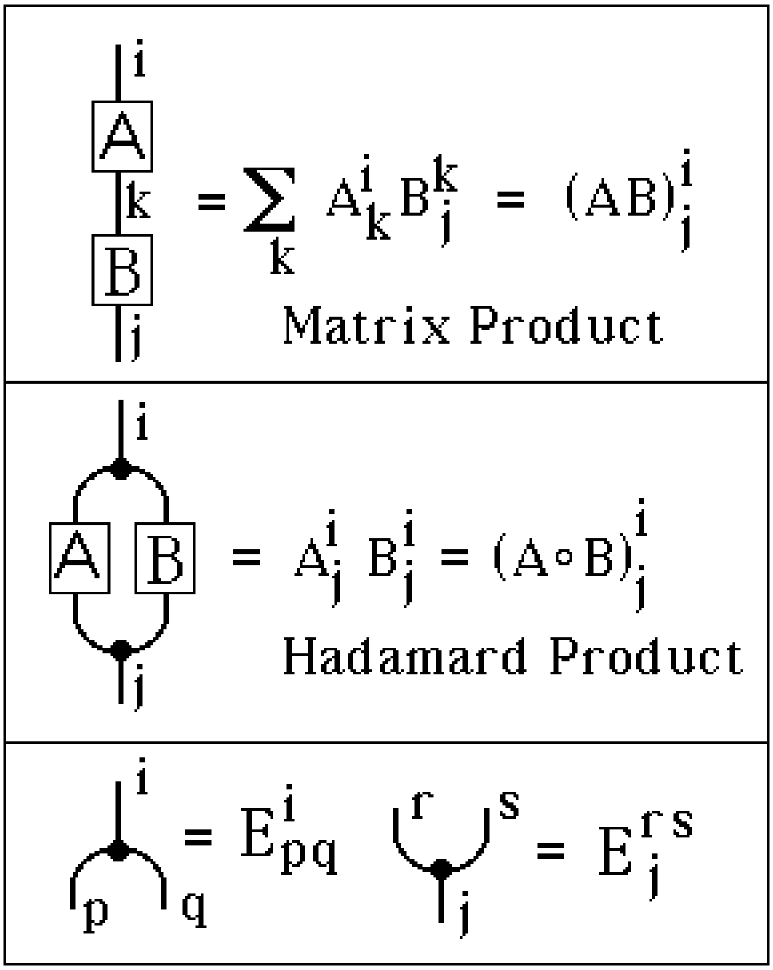

Figure 18 - Parallel and Series Connections in a Quantum Network

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