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Virtual Logic — The Flagg Resolution

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'Fire rests by changing.' — Heraclitus

I. Introduction

This is my eighth column on virtual logic. In this column we shall consider a mode of paradox resolution that I call the 'Flagg Resolution' after its inventor James M. Flagg (See [1], [2]).

II. A Classical Paradox in Logic – Epimenides' Liar Paradox

Consider the statement:

J= 'This statement is false.'

If J is true then J states that J is false.
If J is false then J states that J is true.
Thus we can write

J = Not J.

One might shun such a statement, but perhaps there is a way to deal with it. One solution that has been proposed is to say that J has a new kind of truth value, neither true nor false, and to devise a logic with more than two truth values in which the statement J can live. In this 3 (or more) — valued logic some rules of standard logic and set theory have to be sacrificed.

To see how this sacrifice comes about, consider the following statement

S =J or Not J.

If S were a statement in ordinary logic, we would say that S is true, since (P or Not P) is true for any statement P in ordinary logic. Thus

True = J or No J.

But Not J = J and so

True = J or J = J

((P or P) is logically equivalent to P for any statement P in ordinary logic).

Thus

True = J

But then

False = Not True = Not J = J = True.

So

False = True!

We have a paradox.

It is not possible to treat the statements S and J with the property J= Not J in exactly the same way that we treat the usual statements of logic and set theory.

In analyzing this paradox it has been traditionally observed that the paradox would disappear if we dropped the rule, as the intuitionists suggest,

True = P or Not P

and made a new logic that did not assume this ‘law of the excluded middle.’

This approach is quite interesting and it leads to whole new fields of logic. Nevertheless, it is possible that the law of the excluded middle is not the only rule we could drop that would keep the contradiction from occurring, and perhaps nothing need be really lost!

III. The Flagg Resolution

Around 1980 I had the good fortune to meet Flagg who told me of his original way to resolve the Liar Paradox and other paradoxes in mathematics and logic.

Flagg observes that the sentence

S = J or Not J (with J = Not J).

would not give us all this trouble if we took to heart the following rule:

THE FLAGG RESOLUTION

YOU ARE NOT ALLOWED TO EXCHANGE ‘J’ FOR ‘NOT J’ ANYWHERE UNLESS YOU DO SO EVERYWHERE.
THERE IS ONE AND ONLY ONE EXPRESSION ‘J’.
SUBSTITUTION OF ‘NOT J’ FOR ‘J’ (OR VICE VERSA) IS THEREFORE
A GLOBAL REPLACEMENT.
OTHER THAN THIS SPECIAL TREATMENT OF ‘J’ ,
NO RULES OF LOGIC ARE CHANGED FROM CLASSICAL LOGIC.

That is, if you wish to make a substitution of ‘J’ for ‘Not J’ the substitution must be done for every appearance of ‘J’.

Lets see how the Flagg Resolution applies to our particular paradox of the previous section.

True = J or Not J.

If, in this expression, we wish to replace J by Not J we can do it to both J’s and conclude

True = Not J or Not Not J.

We can write

Not Not J = J

since this is a rule of ordinary logic and finally conclude

True = Not J or J.

This is the same as

True = J or Not J

since (A or B) and (B or A) are equivalent in ordinary logic. No paradox appears.

The beauty of the Flagg Resolution is that we can accommodate new logical values and self-reference (like J) into the reasoning process without making artificial adjustments to the logic itself.

Historical Remark.

The Resolution originally came about as the result of a conflict in point-of-view between F. J. Varela [10] and G. Spencer-Brown [8]. Varela wrote a paper [10] showing how to replace the Calculus of Indications of G. Spencer-Brown (see previous columns in this series) with a 'Calculus for Self-Reference' that embodied the patterns of three-valued logic with the third value interpreted as autonomy or self-reference. This constituted Kleene's three-valued logic interpreted for self-reference. (See [7] for the metamathematical point of view.) The Varela calculus is excellent, but not necessary to patch any paradox in Spencer-Brown. This is the lesson of the Flagg Resolution. For a discussion of the relationship of the Varela calculus with the Flagg Resolution, see my paper [1] 'Imaginary Values in Mathematical Logic'. The Flagg Resolution put the very beautiful Calculus of Indications back on its stated ground — contrary to Varela. In the course of that discussion, the Resolution was discovered to have wider applications.

The reader will by now have appreciated that the Flagg Resolution raises numerous questions about logical and mathematical practice. Is there a precedent for this treatment of paradox? Will it apply to apparently deeper paradoxes such as the Russell paradox? What is the meaning of this global nature of the entity J? If J is an exemplar of a 'cybernetic unity', what is the meaning of this global nature? We will attempt to answer some of these questions in the following sections of the column. (Since this is a column in a cybernetic journal, we will not attempt here to define a cybernetic unity. Let that be a question for later columns!)

IV. Waveforms and Time Series

There is a model for the Flagg Resolution that is based on a model of temporality. Think of J as a temporal sequence or alternation of True (T) and False (F):

J = ... TFTFTFTFTFTFTFTFTFTFTFTFTFTFTFTFTF .. . .

You can also think of such a sequence as a time series J{n} (= T or F) where n is any integer (we proceed from time in the infinite past to time in the infinite future). The series Not J is defined by the equation

(Not J){n} = Not J{n}

so that Not J is false exactly when J is true and vice versa.

If you look at J and Not J together at any given time then certainly

J or Not J = T

because one or the other of them is true at any given time.

On the other hand, if you look at J alone or Not J alone as patterns then you are tempted to identify them since they are just shifted by one time step:

J = ... TFTFTFTFTFTFTFTFTFTFTFTFTFTFTFTFTF .. . .
Not J = ... FTFTFTFTFTFTFTFTFTFTFTFTFTFTFTFTFT .. . .

Thus in this model, the statement J=Not J becomes a meta-statement about the structure of the time series J, while the Flagg Resolution's law of substitution becomes a way of maintaining synchronicity by making either no phase shift, or simultaneous phase shifts in the course of calculation. In this model it would be ludicrous to consider

J or Not J as reducible to J!

Instead of the time series we can model J as an ordered pair

J= [T,F]

where this constitutes two steps in the order of the time series.
Then

Not J = [F,T]

if we regard the negation of the series as the new series obtained by negating each of its temporal values, and

J = Not J

only in the sense of confusing the order of these two-step sequences.

In fact it was exactly because of this interpretation of extra-logical values in terms of waveforms that Flagg and I immediately realized our rapport on this issue. I had, when Flagg and I first met, just finished writing some papers on a waveform interpretation where the context was a four valued logic that did not allow the law of the excluded middle! (See [1], [4], [5], [6].)

What I had done was say 'Alright, let us change the definition of Not so that Not J = J and let us retain the time series.' This is done by saying that the 'Not' of a time series includes negating every term and shifting the whole series by one step! This is a really trenchant idea. It means that one is including a delay of one time step in the operation of negation. In this model we have

Not [A,B] = [Not B, Not A].

Then Not [T,F] = [Not F, Not T] = [T,F].

I called

I = [F,T]

and

J = [T,F].

I have two distinct wave forms I and J, each satisfying its own negation, each looking like the other in isolation, and together we have

IJ = T.

In the formalism of the ordered pairs this is expressed as

[A,B] or [C,D] = [A or C, B or D].

and T is identified with the pair [T,T]
while F is identified with the pair [F,F].

Remember that the ordered pairs stand for time series and that [T,T] just stands for the series

.. . . TTTTTTTTTTTTTTTTTTTT .. . .

that states True at all times.

In my waveform model for four valued logic I had everything nicely in place and now you can see what happened to the law of the excluded middle!

In my model it was just plain that

I or Not I

was simply not equal to T.

Well, I leave it to the reader to judge interest in that. It is really a topic for another column. An exposition of the waveform algebra and corresponding dynamics of forms can also be found in Chapter 12 of Varela's book 'Principles of Biological Autonomy' [11]. This chapter is an essentially verbatim rendering of our [Kauffman and Varela] joint paper 'Form Dynamics' [4].

Multiple valued logics are ways to encapsulate dilemmas in controlled contexts that admit calculations. In mathematics there are many parallels to this, not the least being the fact that every knot (yes a knot in a rope!) is a kind of paradox and has associated with it its very own multiple valued logic with the number of values a topological invariant of the knot. See [3], but consider:

When is a knot a knot?
When it is in the plane it is not.
When it is not in the plane, then it is a knot.

You can see however, that it was not necessary to do anything as drastic as the four-valued logic to accommodate waveforms. The Flagg Resolution provides the abstract background for an atemporal view of this temporal resolution.

G. Spencer-Brown in 'Laws of Form' [8] Chapter 11 said 'Since we do not wish, if we can avoid it, to leave the form, the state we envisage is not in space but in time.'. Originally one would have interpreted this transition into time as directly arising from the recursive series implicit in any given re-entering or self-referential form (such as J = Not J). In other words a paradox creates a buzzer when you implement it (off implies on and on implies off); yet the implemented circuit is a process and is therefore not a paradox. With the Flagg Resolution we see that the modification of a simple law of substitution in logic/algebra is sufficient to create an algebraic notion of simultaneity. This simultaneity is not in a state of time (or space), but it is the precursor, or *archetype* of timespace.

V. Mathematics, Structure and Time

Sometimes I think that Mathematics is an enterprise self-designed to do away with time. No matter that you started with a temporal process or a time series, by the time (sic) you have formulated your data and the problem in mathematical terms, *time* has disappeared into structure. The essential quality of time that opens up a future is not formalized in mathematics and yet it is lingering there in the surprises of a recursion and the unimagined consequences of a given set of assumptions.

In the Flagg Resolution we give a structural linkage between J and Not J that corresponds or is isomorphic to their relative phase shifts in a temporal interpretation. It is certainly not necessary to take this interpretation. Without the interpretation, we have an entity whose structural self-description (J = Not J) is linked with its appearance in the formal text of the mathematics. (Hence it cannot be exchanged locally for its negation.)

A text of Mathematics obeying the rule of substitution inherent in the Flagg Resolution is no longer an objective, or classical description. This text is dependent upon the relative positioning of functions of J within that text. The values of the text are functions of its own structure. There is no separating the text and what that text describes.

The statement (a small text)

S = J or Not J

has value True not through the description of an external state of affairs about J (We have abandoned the temporal interpretation for this discussion.), but through the extension of a known logical relation beyond its normal call of duty. From the truth of 'J or Not J'

I cannot deduce that either J is true or that Not J is true. These *factors*, like strange puzzle pieces are neither True nor False and yet, in combination, a definite truth value emerges.

It is here that we reach the essential circularity and inseparability of the issue, the cybernetic hub of the matter. The ground of our reason is neither true nor false, but structural and global. In this ground the system and its observer are neither separate, nor coincident. Taken singly, system and observer have no logical values, nor any determinate form. Taken together they are one and true in the sense of the integrity that stands behind the very notion of Truth.

We divide and propagate dichotomies: system/observer, body/mind, object/process leading to object, self/other, and more. Only in the unity of such dichotomies are there classical truth values. In the sides of the division or distinction there are valid imaginary truth values exemplifying the strong non-locality of the Flagg Resolution. The world is one, and when it is broken into distinct parts these parts can partake of classical logic only in the non-local structure of the whole.

Does this sound like a discussion of non-locality in quantum physics? I believe that indeed it does. In splitting the world, in making a distinction, we can continue to use classical logic only in a context of non-local or global relations. There is too little space here to follow this trail, but we will return to it in later columns.

VI. The Russell Paradox

The Russell paradox shook the foundations of logic and mathematics at the turn of the century. It is simple and profound.

Let AB denote the statement that 'B is a member of A'. With this notation we can define sets (collections) by equations. For example, the equation

Ax = xx

states that 'x is a member of A exactly when x is a member of itself' and the equation

Rx = Not(xx)

states that x is a member of R exactly when x is NOT a member of itself. *R is the Russell set.* The paradox occurs when we ask whether R can be a member of R. For if

Rx = Not(xx)

then, substituting R for x, we have

RR = Not(RR).

R is a member of R exactly when R is not a member of R. We are in the domain of the Liar Paradox. The Flagg Resolution takes care of the Russell paradox in exactly the same way it takes care of the Liar Paradox. The statement that

'R is a member of R exactly when R is not a member of R'

is interpreted as a statement of substitution

'R is a member of R' = 'R is not a member of R.'

'RR = Not(RR).'

Note the implicit use of the equals sign as a transliteration of the words 'exactly when'. In fact it is by adopting these substitutions that we created the paradox in the first place. The Flagg resolution is an algebraic way get out of the train of difficulties engendered by a profligate use of the rule of substitution.

If you wish to replace R's self-membership by R's non-self membership anywhere then you must do so everywhere! It is not a local matter. The entire mathematical universe (or the universe delineated by a given text) waxes and wanes as a whole with respect to this statement. We can begin to see here the philosophical and mathematical import of the Resolution. The Resolution allows the Russell set to be a member of itself. The Resolution also allows the Russell set to NOT be a member of itself. The Resolution tells us that we must take these statements globally, as they are the roots (or factors) of a distinction.

Yes, *concepts do define sets*. The sets that concepts define bifurcate into sets that allow local substitution and behave in a 'classical' manner, and sets like the Russell set that insist on an examination of the whole context before substitution is allowed. The Resolution forces us to examine the fundamental act of substitution in Mathematics and assures us, after examination, that Logic and Reason continue to function in these domains.

VI. And Logic Itself

The preceding discussion has led us to understand that there is nothing special about the 'dichotomy' J / Not J with J = Not J.

This is the character of any distinction. *Yes*, it is distinct. *No*, there is no distinction 'really'. It is a play we play to imagine that the difference is maintained. Classical logic, set theory and Cantor's paradise can maintain their own facade under the fundamental dictum of the Flagg Resolution. Read this paper again, and this time start not with J/ Not J but with A / B with 'True = A or B' and the equation 'A = B.' Let Flagg tell you that you can substitute A for B, or B for A only everywhere or nowhere. See logic conspire to retain its distinction in the face of the void.

VII. Afterword

In this paper we have illustrated the Flagg Resolution in relation to the Liar Paradox and the Russell Paradox. The Resolution itself is quite general, applying to all the classical paradoxes of logic and mathematics. How can this be? It is necessary to step back and examine the notion of truth. Listen to Flagg himself [9]:

> What Godel's theorems [on the incompleteness of formal systems [see column number 7 — L.K.] teach us is that there is one and only one truth-value. The idea that there could be *another value* and that it figures somehow in our reasoning is purely imaginary, or at best only conceptual, an intellectual means, an *als ob* , as it were, that we use toward some definite destination in our reasoning. We show what is indicated, or more narrowly, what is true, only by pretending that what is could be otherwise . . . What is important about every proof by *reductio* [proof by contradiction — L.K.] is that each must be thorough-going, or global in its implications, *salva veritate*. Every proof by *contradiction* implies the integrity of mathematics, as if the whole of mathematics had been written by a single hand. The mere possibility of a 'false theorem' is global enough — in its pathology — to infect the whole of mathematics. Language cannot convey the absoluteness of this illness, but every practicing mathematician has a working intuition of its import. This is to say, there are indeed only two values: *indication*, which cuts deeper, and *truth*. We may equate them. Outside this, there is only the void. One's choices are indication or truth, or none at all. This inherent limitation of mathematics is mind-boggling, but only because it is neither psychological nor cognitive at its root, but *an ultimate matter of fact* that is, at its foundation, therapeutic. We may throw off a host of illnesses and problems peculiar to our time such as the Theory of Types, Grothendieck Universes, Intuitionism and the like, and restore mathematics' original beauty. We can only speak about indication, and this is how we go about constructing proofs. The virtue of the Resolution is that it shows this state of affairs clearly and quite globally: that this is all that we can do, there was never a paradox in the first place, we only supposed there was.

VIII. Last Word

Virtual logic stands outside the logic of true and false and points to that domain from which Reason springs forth. Our method of paradox resolution is a direct descendant of Ludwig Wittgenstein's injunction to solve problems as they arise. '. . . it is vitally important to see that a contradiction is not a germ which shows general illness.' [12]. In the Flagg Resolution we give a method whereby one can take a seemingly contradictory entity at face value, modify the mode by which we indicate that entity (the way we engage in substitution and reference to the entity), and retain standard logical discourse.

The practical consequence of this move is of great importance for the foundations of information science. On the one hand the Resolution is a formal move that allows the discourse to continue to flow forward. On the other hand the issue of deeper understanding is placed squarely in the hands of the person or observer in the system. The responsibility is in the hands of the observer. The mathematics is in the hands of the mathematician. Formal systems and computer systems are aids in the flow of communication, but the responsibility goes to the observer. It is the observer who initiates the creative process by distinguishing the space and indicating its contents. It is the observer who stands to gain or lose in the dialogue about consistency and continuation of the conversation.

In Mathematics a theorem is proved when it becomes evident. The proof is not a matter of true and false; it is a matter of coherent indication. What is indicated is a collection of (from the stance of the mathematician) indisputable distinctions that cohere into the conclusion that is the theorem. Reason and logic are seen within this coherence. The theorem is not proved by logic or reason. The distinctions that constitute the proof are not a matter of dispute. They are agreed upon by the community of mathematicians, and are each reproducible by every individual in that group. It is exactly because proof goes beyond true and false that the gambit of proof by contradiction can be taken. The mathematical universe is not fragile. It will not collapse in the face of an apparent contradiction any more than the discovery of Russell's paradox caused bridges to fall.

We stand on the other side of the watershed of Godel's Theorem. No single machine, no single formal system can encompass the acts of Reason. We must act responsibly in an endless creation of such systems and the patterns that can be generated with them.

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